Math 2177 recitation: Review

TA: Yu Zhang

November 20 2018

(You can find all my recitation handouts and their solutions on my homepage http://u.osu.edu/yuzhang/teaching/)

Exercise 1. Consider the function $f(x,y) = 4x^2 + 10y^2$

(a) Find critical points of the given $f(x,y)$ and classify them. Compute the values of f at the critical points.

(b) Use the method of Lagrange multipliers to find the maximum and the minimum values of the given $f(x,y)$ on the circle $x^2 + y^2 = 4$.

(c) Find the absolute maximum and the absolute minimum values of the given $f(x,y)$ on the disk $x^2 + y^2 \leqslant 4$. Use parts (a) and (b).

Solution 1. (a) $0 = f_x = 8x$, $0 = f_y = 20y$ implies $x = y = 0$. The only critical point is (0, 0).

Classify the type: We compute $f_{xx}(x,y) = 8$, $f_{yy}(x,y) = 20$, $f_{xy}(x,y) = f_{yx}(x,y) = 0$. Therefore, $D(x,y) = f_{xx}(x,y)f_{yy}(x,y) - (f_{xy}(x,y))^2 = 160$. In particular, $D(0,0) = 160 > 0$. Moreover, $f_{xx}(0,0) = 8 > 0$. Therefore, f has a local minimum at (0, 0).

We have $f(0,0) = 0$.

(b) Let $g(x,y) = x^2 + y^2 - 4$. We need to find the maximum and the minimum values of the given $f(x,y)$ when $g(x,y) = 0$. We solve the following system

$$\begin{cases} f_x(x,y) = \lambda g_x(x,y) \\ f_y(x,y) = \lambda g_y(x,y) \\ g(x,y) = 0 \end{cases} \rightarrow \begin{cases} 8x = 2\lambda x \\ 20y = 2\lambda y \\ x^2 + y^2 = 4 \end{cases}$$

From the first equation we get $2x(4 - \lambda) = 0$. So $x = 0$ or $\lambda = 4$.

Case 1: $x = 0$, then $y = \pm 2$.

Case 2: $\lambda = 4$, then $y = 0$. Therefore $x = \pm 2$.

Now we compute the values of $f(x,y)$ at these points

$$f(0,2) = 40 = f(0,-2),\ f(2,0) = 16 = f(-2,0)$$

Therefore, the minimum values of f on $x^2 + y^2 = 4$ is $f(2,0) = f(-2,0) = 16$ and the maximum value of f on $x^2 + y^2 = 4$ is $f(0,2) = f(0,-2) = 40$.

(c) By (a) and (b), the absolute minimum of $f(x,y)$ on the disk $x^2 + y^2 \leqslant 4$ is $f(0,0) = 0$ and the absolute maximum of $f(x,y)$ on the disk $x^2 + y^2 \leqslant 4$ is $f(0,2) = f(0,-2) = 40$.

Exercise 2. Evaluate the following integral by first converting to polar coordinates.

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{0} cos(x^2+y^2)dydx$$

Solution 2.

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{0} cos(x^2+y^2)dydx = \int_{\pi}^{2\pi}\int_{0}^{1} rcos(r^2)drd\theta = \int_{\pi}^{2\pi} \frac{1}{2}sin(1)d\theta = \frac{\pi}{2}sin(1)$$

Exercise 3. Determine if the following vector fields are conservative and find a potential function for the vector field if it is conservative.

$$\bar{F} = (2x^3y^4+x)\bar{i} + (2x^4y^3+y)\bar{j}$$

Solution 3. Let $P = 2x^3y^4 + x$, $Q = 2x^4y^3 + y$. Then $P_y = 8x^3y^3 = Q_x$ So, the vector field is conservative.

Now let's find the potential function. We want a function $f(x,y)$ such that $f_x = P = 2x^3y^4 + x$ and $f_y = Q = 2x^4y^3 + y$. Integrating P with respect to x, we get $f(x,y) = \frac{1}{2}x^4y^4 + \frac{1}{2}x^2 + h(y)$. Differentiating with respect to y gives $2x^4y^3 + h'(y) = Q = 2x^4y^3 + y$ so $h'(y) = y$. Thus, $h(y) = \frac{1}{2}y^2 + C$ and we can take it to be $\frac{1}{2}y^2$ as we are just looking for one potential.

We get $f(x,y) = \frac{1}{2}x^4y^4 + \frac{1}{2}x^2 + \frac{1}{2}y^2$.

Exercise 4. $\mathbf{A} = \begin{bmatrix} 2 & 3 & -1 & -9 \\ 0 & 1 & 1 & 1 \\ -1 & 2 & 3 & 4 \end{bmatrix}$.

(1) Find all solutions to $\mathbf{A}\overline{x} = 0$

(2) Find all solutions to $\mathbf{A}\overline{x} = \overline{b}$ given that $\overline{p} = \begin{bmatrix} 3 \\ -5 \\ 7 \\ 0 \end{bmatrix}$ is a solution to $\mathbf{A}\overline{x} = \overline{b}$.

Describe the solutions in parametric vector form, and give a geometric description of the solution sets.

Solution 4. (1) $\begin{bmatrix} 2 & 3 & -1 & -9 \\ 0 & 1 & 1 & 1 \\ -1 & 2 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} -1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 2 & 3 & -1 & -9 \end{bmatrix} \sim \begin{bmatrix} -1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 7 & 5 & -1 \end{bmatrix} \sim$
$\begin{bmatrix} -1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -2 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -3 & -4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & 8 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$

Solutions to $\mathbf{A}\overline{x} = 0$ are $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2t \\ 3t \\ -4t \\ t \end{bmatrix}$, where t is any real number.

(2)Notice $\overline{p}$ is a particular solution. All solutions to $\mathbf{A}\overline{x} = \overline{b}$ are
$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = t\begin{bmatrix} -2 \\ 3 \\ -4 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ -5 \\ 7 \\ 0 \end{bmatrix}$, where t is any real number.

Geometrically, it is a line in R^4 passing through $\begin{bmatrix} 3 \\ -5 \\ 7 \\ 0 \end{bmatrix}$.

Exercise 5. (1) Let $v_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 4 \end{bmatrix}$, $v_2 = \begin{bmatrix} 3 \\ 2 \\ -2 \\ 1 \end{bmatrix}$, $w = \begin{bmatrix} 5 \\ 8 \\ -12 \\ -5 \end{bmatrix}$. Determine whether w is a linear combination of v_1 and v_2.

(2) Determine whether v_1, v_2 and w are linearly dependent.

Solution 5. (1) Consider the equation $x_1 v_1 + x_2 v_2 = w$. This equation has corresponding augmented matrix $\begin{bmatrix} 2 & 3 & 5 \\ -1 & 2 & 8 \\ 3 & -2 & -12 \\ 4 & 1 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. This shows $w = -2v_1 + 3v_2$ is a linear combination of v_1 and v_2.

(2) From (1), we know $w = -2v_1 + 3v_2$. Therefore $-2v_1 + 3v_2 - w = w - w = 0$ is a nonzero solution to $x_1 v_1 + x_2 v_2 + x_3 w = 0$. v_1, v_2 and w are linearly dependent.