# Math 2177 recitation: PDE 2

TA: Yu Zhang

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(You can find all my recitation handouts and their solutions on my homepage http://u.osu.edu/yuzhang/teaching/)

## 1 Solving heat equation

As an example, we look at the following partial differential equation (PDE):

$$\begin{cases} u_t = \beta u_{xx}, \quad 0 < x < L, \quad t > 0 & (\textbf{PDE}) \\ u(0,t) = u(L,t) = 0, \quad t > 0 & (\textbf{Boundary Condition}) \\ u(x,0) = f(x), \quad 0 < x < L & (\textbf{Initial Condition}) \end{cases}$$

By separating variables, we can solve this PDE in 4 steps:

**Step 1. Write $u(x,t) = X(x)T(t)$ to turn the PDE into two ordinary differential equations (with boundary conditions)**

Let $u(x,t) = X(x)T(t)$, we obtain the boundary value problem

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = X(L) = 0 \end{cases} \quad \text{and } T'(t) = -\lambda\beta T(t)$$

**Step 2. Find all eigenvalues $\lambda_n$ and their corresponding eigenfunctions $X_n$ of the boundary value problem in step 1.**

Depending on the value of $\lambda$, the boundary value problem

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = X(L) = 0 \end{cases}$$

may only have zero solution $X(x) \equiv 0$. We want to determine those values of $\lambda$ for which the boundary value problem has nontrivial solutions. These nontrivial solutions are called the eigenfunctions of the problem, the eigenvalues are those corresponding values of $\lambda$.

By computations we conclude eigenvalues are $\lambda_n = (\frac{n\pi}{L})^2$ and corresponding eigenfunctions are $X_n(x) = \sin(\frac{n\pi}{L}x)$.

**Step 3. Use $\lambda_n$ to find corresponding $T_n$. Then find the general solution $u(x,t) = \sum_{n=1}^{\infty} c_n X_n(x) T_n(t)$ satisfying both the PDE and boundary condition.**

For $\lambda = (\frac{n\pi}{L})^2$, general solution of $T'(t) = -\beta\lambda T(t)$ is

$$T_n(t) = ce^{-\beta\lambda t} = ce^{-\beta(\frac{n\pi}{L})^2 t}$$

Now general solution for $u(x,t)$ is

$$u(x,t) = \sum_{n=1}^{\infty} c_n X_n(x) T_n(t) = \sum_{n=1}^{\infty} c_n \sin(\frac{n\pi}{L}x) e^{-\beta(\frac{n\pi}{L})^2 t}$$

**Step 4. Use the initial condition to determine the coefficients $c_n$ then get final answer $u(x,t)$.**

From the initial condition $u(x,0) = f(x)$ we know $\sum_{n=1}^{\infty} c_n \sin(\frac{n\pi}{L}x) = f(x)$. When $f(x)$ is already a linear combination of $\sin(\frac{n\pi}{L}x)$, we can directly read the coefficients $c_n$ and get final answer $u(x,t)$

**Exercise 1.** Find the solution to the heat flow problem

$$\begin{cases} u_t = 7u_{xx}, \quad 0 < x < \pi, \quad t > 0 \\ u(0,t) = u(\pi,t) = 0, \quad t > 0 \\ u(x,0) = 3sin(2x) - 6sin(5x), \quad 0 < x < \pi \end{cases}$$

**Solution 1.** In this case, $\beta = 7$, $L = \pi$. Hence general solution is

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin(nx) e^{-7n^2 t}$$

Initial condition implies $\sum_{n=1}^{\infty} c_n \sin(nx) = 3sin(2x) - 6sin(5x)$. Hence $c_2 = 3$, $c_5 = -6$. All other coefficients vanish. Therefore

$$u(x,t) = 3\sin(2x)e^{-28t} - 6\sin(5x)e^{-175t}$$

**Exercise 2.** Find the solution to the heat flow problem

$$\begin{cases} u_t = 2u_{xx}, \quad 0 < x < 1, \quad t > 0 \\ u(0,t) = u(1,t) = 0, \quad t > 0 \\ u(x,0) = 3sin(3\pi x) + 5sin(5\pi x) + sin(9\pi x), \quad 0 < x < 1 \end{cases}$$

**Solution 2.** In this case, $\beta = 2$, $L = 1$. Hence general solution is

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin(n\pi x) e^{-2n^2\pi^2 t}$$

Initial condition implies $\sum_{n=1}^{\infty} c_n \sin(n\pi x) = 3sin(3\pi x) + 5sin(5\pi x) + sin(9\pi x)$. Hence $c_3 = 3$, $c_5 = 5$, $c_9 = 1$. All other coefficients vanish. Therefore

$$u(x,t) = 3\sin(3\pi x)e^{-18\pi^2 t} + 5sin(5\pi x)e^{-50\pi^2 t} + sin(9\pi x)e^{-162\pi^2 t}$$

# 2 Fourier series

Let $f(x)$ be a continuous periodic function with period $2L$. Then $f(x)$ has a Fourier series

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n cos(\frac{n\pi x}{L}) + b_n sin(\frac{n\pi x}{L}))$$

where

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)cos(\frac{n\pi x}{L})dx$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)sin(\frac{n\pi x}{L})dx$$

Moreover when $f(x)$ and $f'(x)$ are piecewise continuous, $F(x) = f(x)$ for all $x$.

Special cases:

(1) If $f(x)$ is an even function meaning $f(x) = f(-x)$, then all $b_n = 0$.

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos(\frac{n\pi x}{L})$$

(2) If $f(x)$ is an odd function meaning $f(x) = -f(-x)$, then all $a_n = 0$.

$$F(x) = \sum_{n=1}^{\infty} b_n sin(\frac{n\pi x}{L})$$

**Exercise 3.** Consider the following function

$$f(x) = \pi^2 - x^2, -\pi \leqslant x \leqslant \pi, f(x + 2\pi) = f(x)$$

(a) Is $f(x)$ even, odd, or neither?

(b) Find the Fourier series $F(x)$ of the given $f(x)$ with period $T = 2\pi$. You may use the information you obtain in (a).

(c) What is $F(\pi)$ and $F(4\pi)$?

(d) Evaluate $\sum_{n=1}^{\infty} \frac{1}{n^2}$ using parts (b) and (c), i.e., the expression of $F(\pi)$ as a series when $x = \pi$ and the value of $F(\pi)$ from the convergence theorem.

**Solution 3.** (a) Notice that $f(x) = \pi^2 - x^2$ on $-\pi \leqslant x \leqslant \pi$ and $-\pi \leqslant x \leqslant \pi$ is an interval symmetric with respect to 0. On $-\pi \leqslant x \leqslant \pi$, we have $f(-x) = \pi^2 - (-x)^2 = \pi^2 - x^2 = f(x)$, i.e., $f(x)$ is even on $-\pi \leqslant x \leqslant \pi$. Also, $f(x)$ is $2\pi$-periodic, therefore, it is even.

(b) In part (a) we saw that f(x) is an even function, therefore, the sine coefficients $b_n$ in its Fourier series are 0 for all $n$. We now compute the other coefficients. We have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi^2 - x^2)dx = \frac{1}{\pi}(\pi^2 x - \frac{x^3}{3})|_{x=-\pi}^{x=\pi} = \frac{4}{3}\pi^2$$

By using integration by parts

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi^2 - x^2)cos(nx)dx = \frac{1}{\pi}((\pi^2 - x^2)\frac{sin(nx)}{n} - 2x\frac{cos(nx)}{n^2} + 2\frac{sin(nx)}{n^3})|_{x=-\pi}^{x=\pi}$$

$$= \frac{1}{\pi}(-4\pi\frac{cos(n\pi)}{n^2}) = -\frac{4}{n^2}cos(n\pi) = -\frac{4}{n^2}(-1)^n = (-1)^{n+1}\frac{4}{n^2}$$

So the Fourier series of $f(x)$ is

$$F(x) = \frac{2}{3}\pi^2 + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{4}{n^2}cos(nx)$$

(c) Notice that $f(x)$ and $f'(x)$ are piecewise continuous,

$$F(\pi) = f(\pi) = \pi^2 - \pi^2 = 0$$

$$F(4\pi) = f(4\pi) = f(2\pi) = f(0) = \pi^2$$

(d) We have

$$0 = F(\pi) = \frac{2}{3}\pi^2 + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{4}{n^2}cos(n\pi) = \frac{2}{3}\pi^2 + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{4}{n^2}(-1)^n$$

$$= \frac{2}{3}\pi^2 + \sum_{n=1}^{\infty}(-1)^{2n+1}\frac{4}{n^2} = \frac{2}{3}\pi^2 - \sum_{n=1}^{\infty}\frac{4}{n^2} = \frac{2}{3}\pi^2 - 4(\sum_{n=1}^{\infty}\frac{1}{n^2})$$

It follows

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$