Math 2177 recitation: PDE 1

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(You can find all my recitation handouts and their solutions on my homepage http://u.osu.edu/yuzhang/teaching/)

As an example, we look at the following partial differential equation (PDE):

$$\begin{cases} u_t = \beta u_{xx}, \quad 0 < x < L, \quad t > 0 & (\textbf{PDE}) \\ u(0,t) = u(L,t) = 0, \quad t > 0 & (\textbf{Boundary Condition}) \\ u(x,0) = f(x), \quad 0 < x < L & (\textbf{Initial Condition}) \end{cases}$$

By separating variables, we can solve this PDE in 4 steps:

Step 1. Write $u(x,t) = X(x)T(t)$ to turn the PDE into two ordinary differential equations (with boundary conditions)

Let $u(x,t) = X(x)T(t)$. Then $u_t = X(x)T'(t)$, $u_{xx} = X''(x)T(t)$. Plugging into $u_t = \beta u_{xx}$ we get $X(x)T'(t) = \beta X''(x)T(t)$. Therefore

$$\frac{T'(t)}{\beta T(t)} = \frac{X''(x)}{X(x)}$$

In this case, they must be the same constant function, denote by $-\lambda$. Therefore

$$X''(x) = -\lambda X(x) \text{ and } T'(t) = -\lambda \beta T(t)$$

Now $u(0,t) = u(L,t) = 0$ implies $X(0)T(t) = 0$ and $X(L)T(t) = 0$ for $t > 0$. Hence either $T(t) = 0$ for all $t > 0$, which implies $u(x,t) \equiv 0$, or $X(0) = X(L) = 0$. Ignoring the trivial solution $u(x,t) \equiv 0$ we obtain the boundary value problem

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = X(L) = 0 \end{cases}$$

and $T'(t) = -\lambda \beta T(t)$

Exercise 1. Separate the following partial differential equations into two ordinary differential equations using $u(x,t) = X(x)T(t)$

(a)$u_t = ku_{xx}$. $u_x(0,t) = u_x(L,t) = 0$

(b)$u_t = ku_{xx}$. $u(-L,t) = u(L,t), u_x(-L,t) = u_x(L,t)$

(c)$u_{tt} = c^2 u_{xx}$. $u(0,t) = u(L,t) = 0$

(d)$u_t = ku_{xx} - u$. $u(0,t) = 0, -u_x(L,t) = u(L,t)$

Solution 1. (a)

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X'(0) = X'(L) = 0 \end{cases} \quad \text{and } T'(t) = -k\lambda T(t)$$

(b)

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(-L) = X(L) \\ X'(-L) = X'(L) \end{cases} \quad \text{and } T'(t) = -k\lambda T(t)$$

(c)

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = X(L) = 0 \end{cases} \quad \text{and } T''(t) = -c^2\lambda T(t)$$

(d)

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = 0 \\ X'(L) + X(L) = 0 \end{cases} \quad \text{and } T'(t) = -(k\lambda + 1)T(t)$$

Step 2. Find all eigenvalues λ_n and their corresponding eigenfunctions X_n of the boundary value problem in step 1.

Depending on the value of λ, the boundary value problem

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = X(L) = 0 \end{cases}$$

may only have zero solution $X(x) \equiv 0$. We want to determine those values of λ for which the boundary value problem has nontrivial solutions. These solutions are called the eigenfunctions of the problem, the eigenvalues are those special values of λ.

Case 1: $\lambda < 0$. General solution is $X(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$. $X(0) = X(L) = 0$ implies $c_1 = c_2 = 0$. Hence $X(x) = 0$.

Case 2: $\lambda = 0$. General solution is $X(x) = c_1 + c_2 x$. $X(0) = X(L) = 0$ implies $c_1 = c_2 = 0$. Hence $X(x) = 0$.

Case 3: $\lambda > 0$. General solution is $X(x) = c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x)$. $X(0) = X(L) = 0$ implies $c_1 = 0, c_2\sin(\sqrt{\lambda}L) = 0$. To obtain nontrivial solutions, we need $\sin(\sqrt{\lambda}L) = 0$. Then $\sqrt{\lambda}L = n\pi$ where n is an integer. Therefore $\lambda_n = (\frac{n\pi}{L})^2$ are eigenvalues. Eigenfunctions are $X_n(x) = C\sin(\frac{n\pi}{L}x)$.

Exercise 2. Consider the second order equation

$$y'' + \lambda y = 0$$

Decide whether the following statements are True or False:

(a)For any value of λ, there is a unique solution satisfying boundary conditions $y(0) = 0$ and $y'(2\pi) = 0$

(b)For any value of λ, there is a unique solution satisfying boundary conditions $y(\pi) = 0$ and $y'(\pi) = 32$

(c)For any value of λ, there exists a solution satisfying $y(0) = 0$ and $y'(0) = 2$

(d)For any value of λ, there exists a solution satisfying $y(0) = 0$ and $y(\pi) = 2$

Solution 2. (a)F

(b)T

(c)T

(d)F

Step 3. Use λ_n to find corresponding T_n. Then find the general solution $u(x,t) = c_n X_n(x) T_n(t)$ satisfying both the PDE and boundary condition.

To be discussed next time...

Step 4. Use the initial condition to determine the coefficients c_n then get final answer $u(x,t)$.

To be discussed next time...