Math 2177 recitation: Differential equations 3

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(You can find all my recitation handouts and their solutions on my homepage http://u.osu.edu/yuzhang/teaching/)

Remark about exercise 5 from last time:
Last week we solved initial value problem

$$y'' + 3y' - 10y = 0,\ y(0) = \alpha,\ y'(0) = -2$$

and got $y = (\frac{5\alpha-2}{7})e^{2t} + (\frac{2\alpha+2}{7})e^{-5t}$. Then we need to decide the value of α that makes $\lim\limits_{t\to\infty} y(t) = 0$.

Warning: do NOT write $0 * \infty = 0$. The expression $0 * \infty = 0$ could raise confusions and it's not valid. For example, when we consider $\lim\limits_{t\to\infty}(\frac{1}{t})(2t)$, this limit has form $0 * \infty$ but we know $\lim\limits_{t\to\infty}(\frac{1}{t})(2t) = \lim\limits_{t\to\infty} 2 = 2$.

We can argue in the following way:

If $\frac{5\alpha-2}{7} \neq 0$,

$$\lim_{t\to\infty} y(t) = (\frac{5\alpha-2}{7})\lim_{t\to\infty} e^{2t} + (\frac{2\alpha+2}{7})\lim_{t\to\infty} e^{-5t} = (\frac{5\alpha-2}{7}) * \infty + (\frac{2\alpha+2}{7}) * 0 = \infty$$

If $\frac{5\alpha-2}{7} = 0$,

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty}(\frac{2\alpha+2}{7})e^{-5t} = (\frac{2\alpha+2}{7})\lim_{t\to\infty} e^{-5t} = (\frac{2\alpha+2}{7}) * 0 = 0$$

Therefore, when $\lim\limits_{t\to\infty} y(t) = 0$, $\alpha = \frac{2}{5}$.

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In this recitation we consider nonhomogeneous equations.

The process of solving nonhomogeneous equation $ay'' + by' + cy = g(t)$:

1. Find the general solution $y_h(t)$ of the corresponding homogeneous equation $ay'' + by' + cy = 0$.

2. Find appropriate form of the particular solution $Y(t)$ by using the method of Undetermined Coefficients.

3. Substitute $Y(t)$ back into $ay'' + by' + cy = g(t)$ to determine the coefficients of $Y(t)$.

4. Then the general solution of $ay'' + by' + cy = g(t)$ is $y(t) = y_h(t) + Y(t)$.

1 Form of paticular solution $Y(t)$

$Y(t)$ should have the same form as $g(t)$ but we might need to multiply by another t or t^2 to ensure no term in $Y(t)$ is a solution of $ay'' + by' + cy = 0$.

Exercise 1. Consider the nonhomogeneous second order linear equation of the form

$$y'' - 4y' + 8y = g(t)$$

(a) Find $y_h(t)$, the solution of its corresponding homogeneous equation

For each of parts (b), (c), and (d), write down the form of particular solution that you would use to solve the given equation using the Method of Undetermined Coefficients. Do not attempt to solve the coefficients
(b) $y'' - 4y' + 8y = 2e^{2t} - 5t^2 + sin(2t)$
(c) $y'' - 4y' + 8y = -e^{2t}sin(2t) + 1$
(d) $y'' - 4y' + 8y = t^2e^{-t}cos(5t)$

Solution 1. (a) The corresponding characteristic equation is $r^2 - 4r + 8 = 0$. We get $r = 2 \pm 2i$. Hence general solution is $y_h(t) = c_1e^{2t}cos(2t) + c_2e^{2t}sin(2t)$.
(b)$Y(t) = (Ae^{2t}) + (Bt^2 + Ct + D) + (Ecos(2t) + Fsin(2t))$
(c)$Y(t) = t(Ae^{2t}cos(2t) + Be^{2t}sin(2t)) + C$
(d)$Y(t) = (At^2 + Bt + C)e^{-t}cos(5t) + (Dt^2 + Et + F)e^{-t}sin(5t)$

2 Solving nonhomogeneous equation

Exercise 2. Consider the second order nonhomogeneous linear equation

$$y'' - y' - 2y = 3sin(2t)$$

(a) Find $y_h(t)$, the solution of its corresponding homogeneous equation
(b) Find $Y(t)$, a particular solution of the nonhomogeneous equation
(c) Find the general solution of the nonhomogeneous equation

Solution 2. (a) The characteristic equation is $r^2 - r - 2 = 0$. We solve the equation and get $r_1 = 2, r_2 = -1$. So the general solution is $y_h = c_1e^{2t} + c_2e^{-t}$.
(b) Let $Y(t) = Acos(2t) + Bsin(2t)$. Then $Y'(t) = -2Asin(2t) + 2Bcos(2t)$, $Y''(t) = -4Acos(2t) - 4Bsin(2t)$. Since $Y(t)$ is a particular solution of $y'' - y' - 2y = 3sin(2t)$, we get

$$\begin{aligned}
3sin(2t) &= Y'' - Y' - 2Y \\
&= (-4Acos(2t) - 4Bsin(2t)) - (-2Asin(2t) + 2Bcos(2t)) - 2(Acos(2t) + Bsin(2t)) \\
&= (-4A - 2B - 2A)cos(2t) + (-4B + 2A - 2B)sin(2t) \\
&= (-6A - 2B)cos(2t) + (2A - 6B)sin(2t)
\end{aligned}$$

By comparing coefficients, we have $-6A - 2B = 0, 2A - 6B = 3$. Then $A = \frac{3}{20}, B = -\frac{9}{20}$. So $Y(t) = \frac{3}{20}cos(2t) - \frac{9}{20}sin(2t)$
(c) General solution is $y(t) = c_1e^{2t} + c_2e^{-t} + \frac{3}{20}cos(2t) - \frac{9}{20}sin(2t)$.

Exercise 3. Consider the second order nonhomogeneous linear equation

$$y'' - 2y' + 5y = 5t^2 + 6t - 12$$

(a) Find $y_h(t)$, the solution of its corresponding homogeneous equation
(b) Find $Y(t)$, a particular solution of the nonhomogeneous equation
(c) Find the general solution of the nonhomogeneous equation

Solution 3. (a) The characteristic equation is $r^2-2r+5=0$. We solve the equation and get $r = 1 \pm 2i$. So the general solution is $y_h = c_1 e^t cos(2t) + c_2 e^t sin(2t)$.

(b) Let $Y(t) = At^2 + Bt + C$. Then $Y'(t) = 2At + B$, $Y''(t) = 2A$. Since $Y(t)$ is a particular solution of $y'' - 2y' + 5y = 5t^2 + 6t - 12$, we get

$$\begin{aligned} 5t^2 + 6t - 12 &= Y'' - 2Y' + 5Y \\ &= (2A) - 2(2At + B) + 5(At^2 + Bt + C \\ &= 5At^2 + (-4A + 5B)t + (2A - 2B + 5C) \end{aligned}$$

By comparing coefficients, we have

$$5A = 5, -4A + 5B = 6, 2A - 2B + 5C = -12$$

Then $A = 1, B = 2, C = -2$. So $Y(t) = t^2 + 2t - 2$

(c) General solution is $y(t) = c_1 e^t cos(2t) + c_2 e^t sin(2t) + t^2 + 2t - 2$.

Exercise 4. Find a second order linear equation which has

$$y(t) = C_1 e^t + C_2 e^{-2t} + 3t - 1$$

as its general solution.

Solution 4. From the general solution we know this is a solution for nonhomogeneous second order linear equation where $y_h(t) = C_1 e^t + C_2 e^{-2t}, Y(t) = 3t - 1$

From y_h we know $r_1 = 1, r_2 = -2$. Therefore characteristic equation is $(r - 1)(r + 2) = r^2 + r - 2 = 0$. The equation we're seeking is in the form of

$$y'' + y' - 2y = g(t)$$

We plug in $Y(t) = 3t-1$ to the equation. $g(t) = Y''+Y'-2Y = 0+3-2(3t-1) = -6t + 5$. So the equation we're looking for is

$$y'' + y' - 2y = -6t + 5$$

3 Bonus questions

Exercise 5. Let $y(t)$ be a solution of $y'' + y = -tg(t)y'$, where $g(t) \geqslant 0$ for all t. Prove $|y(t)|$ is bounded.

Hint: Multiply both side of the equation by $2y'$

Solution 5. Multiply both side of the equation by $2y'$. We get

$$\frac{d}{dt}((y')^2 + y^2) = 2y'y'' + 2yy' = -tg(t)(y')^2$$

Thus, $(y')^2 + y^2$ is nondecreasing for $x \leqslant 0$ and nonincreasing for $x \geqslant 0$. Hence

$$|y(t)|^2 = y^2 \leqslant (y')^2 + y^2 \leqslant (y'(0))^2 + y(0)^2$$