Math 2177 recitation: Differential equations 2

TA: Yu Zhang

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(You can find all my recitation handouts and their solutions on my homepage http://u.osu.edu/yuzhang/teaching/)

In this recitation we consider second order linear homogeneous equation with constant coefficients:

$$ay'' + by' + cy = 0$$

To solve this type of equations, we need to consider the corresponding characteristic equation

$$ar^2 + br + c = 0$$

1 Case 1: $b^2 - 4ac > 0$

When $b^2 - 4ac > 0$, $ar^2 + br + c = 0$ has two distinct real roots

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \ r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Then the general solution for $ay'' + by' + cy = 0$ is $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$, c_1, c_2 are arbitrary constants.

(This case has been discussed in last recitation.)

2 Case 2: $b^2 - 4ac = 0$

When $b^2 - 4ac = 0$, $ar^2 + br + c = 0$ has repeated real roots

$$r_1 = r_2 = r = -\frac{b}{2a}$$

Then the general solution for $ay'' + by' + cy = 0$ is $y = c_1 e^{rt} + c_2 t e^{rt}$, c_1, c_2 are arbitrary constants.

Exercise 1. Find the particular solution to

$$y'' - 8y' + 16y = 0,\ y(0) = 3,\ y'(0) = 10$$

Then decide $\lim_{t\to\infty} y(t)$

Solution 1. The characteristic equation of $y'' - 8y' + 16y = 0$ is $r^2 - 8r + 16 = 0$. We solve the equation and get $r = 4$. So the general solution is $y = c_1e^{4t} + c_2te^{4t}$. Then $y'(t) = 4c_1e^{4t} + c_2e^{4t} + 4c_2te^{4t}$.

By the initial conditions, we get $y(0) = 3 = c_1$, $y'(0) = 10 = 4c_1 + c_2$. Then we have $c_1 = 3$ and $c_2 = -2$. So the particular solution is $y = 3e^{4t} - 2te^{4t}$.

$\lim_{t\to\infty} y(t) = \lim_{t\to\infty}(3e^{4t} - 2te^{4t}) = \lim_{t\to\infty}(3 - 2t)e^{4t} = -\infty$

3 Case 3: $b^2 - 4ac < 0$

When $b^2 - 4ac < 0$, $ar^2 + br + c = 0$ has two complex roots of the form

$$r_{1,2} = \lambda \pm i\mu, \quad (r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a})$$

Then the general solution for $ay'' + by' + cy = 0$ is $y = c_1e^{\lambda t}\cos(\mu t) + c_2e^{\lambda t}\sin(\mu t)$, c_1, c_2 are arbitrary constants.

Exercise 2. Find the particular solution to initial value problem

$$y'' - 8y' + 17y = 0,\ y(0) = -4,\ y'(0) = -1$$

Solution 2. The characteristic equation of $y'' - 8y' + 17y = 0$ is $r^2 - 8r + 17 = 0$. We solve the equation and get $r_{1,2} = 4 \pm i$. So the general solution is $y = c_1e^{4t}\cos t + c_2e^{4t}\sin t$. Then $y'(t) = 4c_1e^{4t}\cos t - c_1e^{4t}\sin t + 4c_2e^{4t}\sin t + c_2e^{4t}\cos t$.

By the initial conditions, we get $y(0) = -4 = c_1$, $y'(0) = -1 = 4c_1 + c_2$. Then we have $c_1 = -4$ and $c_2 = 15$. So the particular solution is $y = -4e^{4t}\cos t + 15e^{4t}\sin t$.

Exercise 3. Find the particular solution to initial value problem

$$y'' + 16y = 0,\ y(\frac{\pi}{2}) = -10,\ y'(\frac{\pi}{2}) = 3$$

Solution 3. The characteristic equation of $y'' + 16y = 0$ is $r^2 + 16 = 0$. We solve the equation and get $r_{1,2} = \pm 4i$. So the general solution is $y = c_1\cos 4t + c_2\sin 4t$. Then $y'(t) = -4c_1\sin 4t + 4c_2\cos 4t$.

By the initial conditions, we get $y(\frac{\pi}{2}) = -10 = c_1$, $y'(\frac{\pi}{2}) = 3 = 4c_2$. Then we have $c_1 = -10$ and $c_2 = \frac{3}{4}$. So the particular solution is $y = -10\cos 4t + \frac{3}{4}\sin 4t$.

4 More exercises

Exercise 4. Construct a second order homogeneous linear equation with constant coefficients, such that $y_1 = \frac{1}{\pi}e^{-t}$ and $y_2 = -\pi e^{3t}$ are two of its solutions.

Solution 4. The characteristic equation should have -1 and 3 as two roots, so the characteristic equation should be $(r + 1)(r - 3) = 0$, which is $r^2 - 2r - 3 = 0$. Thus the differential equation should be $y'' - 2y' - 3y = 0$

Exercise 5. Let $y(t)$ be the solution of the initial value problem

$$y'' + 3y' - 10y = 0,\ y(0) = \alpha,\ y'(0) = -2$$

Suppose $\lim\limits_{t\to\infty} y(t) = 0$, find the value of α.

Solution 5. The characteristic equation is $r^2 + 3r - 10 = 0$. We solve the equation and get $r_1 = 2, r_2 = -5$. So the general solution is $y = c_1e^{2t} + c_2e^{-5t}$. Then $y'(t) = 2c_1e^{2t} - 5c_2e^{-5t}$.

By the initial conditions, we get $y(0) = \alpha = c_1 + c_2$, $y'(0) = -2 = 2c_1 - 5c_2$. Then we have $c_1 = \frac{5\alpha-2}{7}$ and $c_2 = \frac{2\alpha+2}{7}$.

To have $\lim\limits_{t\to\infty} y(t) = 0$, we need $c_1 = 0$. This implies $\alpha = \frac{2}{5}$.

5 Bonus questions

Exercise 6. (1) Show that the boundary-value problem

$$y'' + \lambda y = 0,\ y(0) = 0,\ y(1) = 0$$

has only the trivial solution $y = 0$ for the cases $\lambda = 0$ and $\lambda < 0$.

(2) For the case $\lambda > 0$, find the values of λ for which this problem has a nontrivial solution and give the corresponding solution.

Solution 6. (1) Case 1: $\lambda = 0$. The differential equation has general solutions $y = c_1 + c_2t$. $y(0) = 0,\ y(1) = 0$ implies $c_1 = c_2 = 0$. Hence $y = 0$.

Case 2: $\lambda < 0$. The differential equation has general solutions $y = c_1e^{\sqrt{-\lambda}t} + c_2e^{-\sqrt{-\lambda}t}$. $y(0) = 0,\ y(1) = 0$ implies $c_1 = c_2 = 0$. Hence $y = 0$.

(2) For the case $\lambda > 0$, the differential equation has general solutions $y = c_1\cos(\sqrt{\lambda}t) + c_2\sin(\sqrt{\lambda}t)$. $y(0) = 0,\ y(1) = 0$ implies $c_1 = 0, c_2\sin(\sqrt{\lambda}) = 0$. To obtain nontrivial solutions, we need $\sin(\sqrt{\lambda}) = 0$. Then $\sqrt{\lambda} = n\pi$ where n is an integer. Therefore $\lambda = n^2\pi^2$ and $y = c_2\sin(n\pi t)$ where n is an integer.

Exercise 7. Suppose a, b, and c are all positive constants. Let $y(t)$ be a solution of the differential equation

$$ay'' + by' + cy = 0$$

Show that $\lim\limits_{t\to\infty} y(t) = 0$.

Solution 7. Case 1: $b^2 - 4ac > 0$. Show $r_1, r_2 < 0$.

Case 2: $b^2 - 4ac = 0$. Show $r < 0$.

Case 3: $b^2 - 4ac < 0$. Show $\lambda < 0$.