Math 2177 recitation: Differential equations 1

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(You can find all my recitation handouts on my homepage
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1 Terminology

Order: the largest order of derivation of y involved in the differential equation
Linear differential equations: differential equations of the form

$$a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \cdots + a_1(t)y'(t) + a_0(t)y(t) = g(t)$$

If a differential equation is not linear, it is called nonlinear.
A linear differential equation is called homogeneous if the equation has form

$$a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \cdots + a_1(t)y'(t) + a_0(t)y(t) = 0$$

, i.e. the right hand side is 0.
Otherwise the linear differential equation is called nonhomogeneous.

Exercise 1. For each of the differential equation below, (1) find its order, (2) determine if it is linear or nonlinear (3) if it is linear, then determine if it is homogeneous or nonhomogeneous.
(a)$y'' - 3y' = 0$
(b)$y''' + 2sin(t)e^{3t}y'' + cos(t)y = 3$
(c)$(y^2 + 2y - 1)y^{(9)} - y^{(12)} = 18$
(d)$y'' + 3y' + 2y = siny$
(e)$y' = \frac{t}{y}$

Solution 1. (a)order = 2, linear, homogeneous.
(b)order = 3, linear, nonhomogeneous.
(c)order = 12, nonlinear
(d)order = 2, nonlinear
(e)order = 1, nonlinear

Exercise 2. Which function is a solution of the differential equation

$$(y')^2 - 5ty = 5t^2 + 1$$

(A)$y(t) = t^2$
(B)$y(t) = e^{5t}$
(C)$y(t) = -t$
(D)$y(t) = \frac{1}{-5t}$

Solution 2. (c)

2 Second order linear homogeneous equations

Now we consider second order linear homogeneous equations, i.e. equations of the form

$$y'' + p(t)y' + q(t)y = 0$$

Theorem 1. If y_1 and y_1 are solutions of the homogeneous equation

$$g(t)y'' + p(t)y' + q(t)y = 0$$

then so is $c_1y_1 + c_2y_2$ for arbitrary constant c_1 and c_2.

General second order linear homogeneous equations are still not easy to solve. For this moment, we only consider second order linear homogeneous equation with constant coefficients:

$$ay'' + by' + cy = 0$$

To solve this type of equations, we need to consider the corresponding characteristic equation

$$ar^2 + br + c = 0$$

2.1 Case 1: $b^2 - 4ac > 0$

When $b^2 - 4ac > 0$, $ar^2 + br + c = 0$ has two distinct real roots

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a},\ r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Then the general solution for $ay'' + by' + cy = 0$ is $y = c_1e^{r_1t} + c_2e^{r_2t}$, c_1, c_2 are arbitrary constants.

Exercise 3. Which equation has $y_1 = e^{-2t}$ and $y_2 = e^{3t}$ as two solutions?
(A)$-2y' + 3y = 0$
(B)$y'' + y' - 6y = 0$
(C)$-y'' + y' + 6y = 0$
(D)$2y'' + 10y' - 12y = 0$

Solution 3. (c)

Exercise 4. Find the particular solution to

$$y'' + 3y' + 2y = 0,\ y(0) = 0,\ y'(0) = \alpha$$

For what value(s) of α is the $\lim_{t\to\infty} y(t) = 0$

Solution 4. The characteristic equation of $y'' + 3y' + 2y = 0$ is $r^2 + 3r + 2 = 0$. We get two distinct real roots $r = -1, -2$.

So the general solution is $y = c_1e^{-t} + c_2e^{-2t}$. Then $y'(t) = -c_1e^{-t} - 2c_2e^{-2t}$. By the initial conditions, we get $y(0) = 0 = c_1 + c_2$, $y'(0) = \alpha = -c_1 - 2c_2$.

Then we have $c_1 = \alpha$ and $c_2 = -\alpha$. So the particular solution is $y = \alpha e^{-t} - \alpha e^{-2t}$.

Since $\lim_{t\to\infty} e^{-t} = 0$ and $\lim_{t\to\infty} e^{-2t} = 0$, we have $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \alpha e^{-t} - \alpha e^{-2t} = \alpha(0) - \alpha(0) = 0$

So all values of α would make the limit 0.

Exercise 5. Let $y(t)$ be the solution of the initial value problem

$$y'' + 2y' - 8y = 0,\ y(0) = \alpha,\ y'(0) = -2$$

Suppose $\lim_{t\to\infty} y(t) = 0$, find the value of α.
(A)$\alpha = \frac{1}{2}$
(B)$\alpha = -1$
(C)$\alpha = -4$
(D)$\alpha = 8$

Solution 5. (A)
General solution is $y = c_1e^{2t} + c_2e^{-4t}$. Then $y'(t) = 2c_1e^{2t} - 4c_2e^{-4t}$. By the initial conditions, we get $y(0) = \alpha = c_1 + c_2$, $y'(0) = -2 = 2c_1 - 4c_2$. Then we have $c_1 = \frac{2\alpha-1}{3}$ and $c_2 = \frac{\alpha+1}{3}$.
To have $\lim_{t\to\infty} y(t) = 0$, we need $c_1 = 0$. This implies $\alpha = \frac{1}{2}$.

2.2 Case 2: $b^2 - 4ac = 0$

Discuss in next recitation...

2.3 Case 3: $b^2 - 4ac < 0$

Discuss in next recitation...

3 Bonus questions

Exercise 6. Suppose a, b, and c are all positive constants such that $b^2 - 4ac > 0$. Let $y(t)$ be a solution of the differential equation

$$ay'' + by' + cy = 0$$

Show that $\lim_{t\to\infty} y(t) = 0$.

Solution 6. Only need to show

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} < 0,\ r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} < 0$$