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Core Gödel

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In memoriam Mic Detlefsen

Abstract This study examines how the Gödel phenomena are to be treated in core logic. We show in formal detail how one can use core logic in the metalanguage to prove Gödel's incompleteness theorems for arithmetic even when classical logic is used for logical closure in the object language.

1 Introduction

This study aims to show how the core logician can furnish an adequate and natural formalization of the metalinguistic reasoning leading to Gödel's celebrated first incompleteness theorem (G1). The informally rigorous proof of (G1) that we undertake to formalize here is the one that proceeds via the representability of recursive functions in Robinson arithmetic Q.[1]

A main metatheorem about core logic $\mathbb{C}$ states that from any intuitionistic proof of φ from a set Δ of premises, one can extract a core proof either of φ or of $\perp$ from premises drawn from Δ.[2] So, if one is morally certain that one's set Δ of premises is consistent—or, equivalently, is satisfiable—then one can confidently say that to any intuitionistic proof of φ from Δ there corresponds a core proof of the very same result. For one is morally certain that the possibility of $\perp$ being provable from Δ has been ruled out.

This moral certainty is consensually attained in the case of the low-consistency-strength mathematics involved here. The usual base system RCA_0, the weakest of the 'Big Five' so-called second-order subsystems of arithmetic in reverse mathematics,

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suffices to prove the consistency of exponential function arithmetic (EFA, also known as Kalmar arithmetic). See Simpson [11, p. 45].

What this study will reveal is that the aforementioned 'corresponding' core proof—in the case of (G1)—is a tight and direct formalization of the informally rigorous, constructive meta-reasoning taking one from the rather weak axioms used, to (G1) itself as conclusion. And the natural explanation of this tight fit (of fully formalized proof with informally rigorous proof) is that mathematical reasoning is always *relevant*, which is a distinguishing feature of proofs in classical core logic[3]—hence, obviously, in core logic. We fully expect the phenomenon of tight and direct formalizability in core logic $\mathbb{C}$, or in classical core logic $\mathbb{C}^+$ (depending on whether the reasoning involved is, respectively, constructive or nonconstructive), to be a general one, across all mathematics. It would be an inexplicable irony if it were achievable only in the case of the reasoning used to establish (G1).

The question then arises: how is the sought core proof to be determined? Our answer, in the setting of this case-study, is: by *direct* formalization, in core logic, of the existing informally rigorous, constructive reasoning that we already have at hand. Core logic is entirely *natural* for the formalization, as a *natural deduction*, of any stretch of informally rigorous, constructive reasoning. And there is no better way to convince anyone skeptical of such a claim on behalf of core logic than to show how it handles one of the most celebrated and intellectually important results in mathematical logic.

Note that this achievement on the part of core logic is not, logically, already anticipated by the foundational discovery that (G1) is provable (at the metalevel) in EFA, for this discovery presupposed that EFA is closed under classical logic. Nor is core logic's achievement anticipated by the foundational discovery that Peano Arithmetic conservatively extends Heyting arithmetic (HA) on Π^0_2-sentences, among which is (G1).[4] This discovery presupposed that HA is closed under intuitionistic logic, which contains the rule ex falso quodlibet, a rule that both $\mathbb{C}$ and $\mathbb{C}^+$ eschew.

2 Rules of Inference for the Core Systems

2.1 Rules of core logic We provide below a graphic statement of introduction and elimination rules for the core systems $\mathbb{C}$ and $\mathbb{C}^+$. Major premises of eliminations stand proud, with no nontrivial proof work above them. This ensures normality of proofs. Boxes next to discharge strokes mean that vacuous discharge is not permitted. Diamonds mean that vacuous discharge *is* permitted. The rules for introducing universals and eliminating existentials have the usual restrictions on occurrences of the parameter a, which we indicate with the notation $\not{\textcircled{a}}$.

$$(\neg\mathrm{I})\qquad \dfrac{\begin{array}{c}\Box\overline{\quad}^{(i)}\\ \varphi\\ \vdots\\ \bot\end{array}}{\neg\varphi}\,(i) \qquad\qquad (\neg\mathrm{E})\qquad \dfrac{\neg\varphi \quad \begin{array}{c}\vdots\\ \varphi\end{array}}{\bot}$$

(∧I) $$\dfrac{\begin{matrix}\vdots & \vdots\\ \varphi & \psi\end{matrix}}{\varphi\wedge\psi}$$

(∧E) $$\dfrac{\varphi\wedge\psi \qquad \begin{matrix}\underbrace{\overset{(i)\,\square\,(i)}{\varphi\ ,\ \psi}}\\ \vdots\\ \theta\end{matrix}}{\theta}\,(i)$$

(∨I) $$\dfrac{\begin{matrix}\vdots\\ \varphi\end{matrix}}{\varphi\vee\psi} \quad \dfrac{\begin{matrix}\vdots\\ \psi\end{matrix}}{\varphi\vee\psi}$$

(∨E) $$\dfrac{\varphi\vee\psi \quad \begin{matrix}\overset{\square\,(i)}{\varphi}\\ \vdots\\ \theta/\bot\end{matrix} \quad \begin{matrix}\overset{\square\,(i)}{\psi}\\ \vdots\\ \theta/\bot\end{matrix}}{\theta/\bot}\,(i)$$

(→I)(a) $$\dfrac{\begin{matrix}\overset{\square\,(i)}{\varphi}\\ \vdots\\ \bot\end{matrix}}{\varphi\to\psi}\,(i)$$

(→I)(b) $$\dfrac{\begin{matrix}\overset{\diamond\,(i)}{\varphi}\\ \vdots\\ \psi\end{matrix}}{\varphi\to\psi}\,(i)$$

(→E) $$\dfrac{\varphi\to\psi \quad \begin{matrix}\vdots\\ \varphi\end{matrix} \quad \begin{matrix}\overset{\square\,(i)}{\psi}\\ \vdots\\ \theta\end{matrix}}{\theta}\,(i)$$

(∃I) $$\dfrac{\begin{matrix}\vdots\\ \varphi^x_t\end{matrix}}{\exists x\varphi}$$

(∃E) $$\dfrac{\exists x\varphi^{\cancel{a}} \quad \begin{matrix}\underbrace{\overset{\square\,(i)}{\cancel{a}\ \ldots\varphi^x_a\ldots\ \cancel{a}}}\\ \vdots\\ \psi^{\cancel{a}}\end{matrix}}{\psi}\,(i)$$

(∀I) $$\dfrac{\begin{matrix}\cancel{a}\\ \vdots\\ \varphi\end{matrix}}{\forall x\varphi^a_x}$$

(∀E) $$\dfrac{\forall x\varphi \quad \begin{matrix}\underbrace{\overset{(i)\ \cdots\square\cdots\ (i)}{\varphi^x_{t_1}\ ,\ \ldots\ ,\ \varphi^x_{t_n}}}\\ \vdots\\ \theta\end{matrix}}{\theta}\,(i)$$

Deducibility in core logic $\mathbb{C}$ will be represented by the subscripted single turnstile $\vdash_{\mathbb{C}}$.

2.2 Rules of classical core logic $\mathbb{C}^+$ In order to obtain classical core logic from core logic, it suffices to add either classical reductio or dilemma. These two strictly classical rules are interderivable in core logic. In each of these rules, it is the sentence φ that is its 'classical focus.' This is because the reasoner who applies the rule is presuming that φ is determinately true, or false, even though it might not be known (or indeed, even knowable) which is the case. (See Tennant [15] for extended philosophical discussion of this point.)

(CR) $$\begin{array}{c} \Box\underline{\quad}^{(i)} \\ \neg\varphi \\ \vdots \\ \underline{\quad\bot\quad}{(i)} \\ \varphi \end{array}$$

(Dil) $$\begin{array}{c} \begin{array}{cc} \Box\underline{\quad}^{(i)} & \Box\underline{\quad}^{(i)} \\ \varphi & \neg\varphi \\ \vdots & \vdots \\ \psi & \psi/\bot \end{array} \\ \hline \psi \end{array}{(i)}$$

Deducibility in classical core logic $\mathbb{C}^+$ will be represented by the subscripted single turnstile $\vdash_{\mathbb{C}^+}$.

3 Formal Arithmetic

3.1 Formal language

Definition 1 The *language $\mathcal{L}$ of arithmetic* is the first-order language with identity, based on the familiar extralogical expressions 0, s, $+$, and $\times$, with an unlimited supply of parameters.

For present purposes, we take the logic of the language $\mathcal{L}$ to be standard in the respect that it is *unfree*. Thus it is tacitly assumed that a name like 0 does denote and that functions are total, that is, defined on all possible inputs. (Thus everything has a successor, any two things have a sum, etc.) It is also assumed that one's theory T in $\mathcal{L}$ aims to talk about all *and only* numbers (i.e., things that have successors, things that can be added together, etc.).

It is very important to understand this in advance since these tacit assumptions do *not* hold within the wider setting that would be considered by a logicist. This is because the logicist takes the task to be that of characterizing the mathematical structure of the natural numbers, to be sure, but *also* to be to explain how those very numbers can be involved in counting the (finite) extensions of predicates that may very well be satisfied by objects *other than* numbers. The logicist, therefore, would task himself further with characterizing a predicate N (for 'is a natural number'), and recover the content of T by deriving 'N-restricted' versions of the T-axioms within his wider logicist theory, whose extralogical vocabulary can in general properly exceed that of $\mathcal{L}$. In this study, however, we are concerned only with theories of 'pure arithmetic,' whose domains of discourse are intended to be (just) the natural numbers.

Let $\exists!t$ be short for $\exists x\, x = t$. If one were to base one's theory T on a *free* logic (such as the one provided in Tennant [13, Chapter 7]), then one would routinely add to one's list of axioms the following:

$$\begin{array}{l} \exists!0, \\ \forall x \exists! sx, \\ \forall x \forall y \exists!(x+y), \\ \forall x \forall y \exists!(x\times y). \end{array}$$

We shall pursue the standard treatment, however, which involves using an unfree logic.

3.2 Some axioms We shall be considering only $\mathcal{L}$-theories T that contain the following selection from among the well-known Peano–Dedekind axioms or rules governing the extra-logical expressions 0, s, $+$, and $\times$:

$$\begin{aligned}
&\forall x \neg sx = 0,\\
&\forall x \forall y (s(x) = s(y) \to x = y),\\
&\forall x\, x + 0 = x,\\
&\forall x \forall y\, x + s(y) = s(x + y),\\
&\forall x\, x \times 0 = 0,\\
&\forall x \forall y\, x \times s(y) = (x \times y) + x.
\end{aligned}$$

The last four of these six axioms are commonly called the *recursion equations* for $+$ and $\times$.

Note that we do not necessarily include in T any instances of the axiom scheme of mathematical induction, which is a crucial source of theoretical power for Peano–Dedekind arithmetic. That scheme is

$$\big(\Phi(0) \wedge \forall x\big(\Phi(x) \to \Phi(sx)\big)\big) \to \forall y \Phi(y).$$

We may also express it as a rule of inference:[5]

$$\dfrac{\begin{matrix} \\ \\ \vdots \\ \Phi(0)\end{matrix} \qquad \begin{matrix} \diamond\overline{\qquad}^{(i)} \\ \Phi(a) \\ \vdots \\ \Phi(sa)\end{matrix}}{\Phi(t)}\,(i)$$

where the parameter a occurs only where indicated,

and t may be any closed term

It is important to mark the dischargeable assumption $\Phi(a)$ (the so-called *inductive hypothesis*) as 'permissible' and not 'obligatory' (which is what the diamond does). That is, we must permit *vacuous* discharge of $\Phi(a)$—instances where the inductive step is carried out *without* recourse to the inductive hypothesis. This is because the inferencing required in order to consummate an application of the relevant axiomatic instance of the axiom scheme of mathematical induction would run as follows:

$$\dfrac{(\Phi(0)\wedge\forall x(\Phi(x)\to\Phi(sx)))\to\forall y\Phi(y) \qquad \dfrac{\begin{matrix}\vdots\\ \Phi(0)\end{matrix} \qquad \dfrac{\dfrac{\begin{matrix}\diamond\overline{\qquad}^{(i)}\\ \Phi(a)\\ \vdots \\ \Phi(sa)\end{matrix}}{\Phi(a)\to\Phi(sa)}\,(i)}{\forall x(\Phi(x)\to\Phi(sx))}}{\Phi(0)\wedge\forall x\big(\Phi(x)\to\Phi(sx)\big)}}{\forall y\Phi(y)}$$

The rule that is being applied at the step marked (i) is that of $\to$-I; in core logic, as in minimal, intuitionistic, and classical logic, vacuous discharge is permitted with $\to$-I when the consequent of the conditional being proved is the conclusion of the subordinate proof. This is (part of) the well-known rule of conditional proof. What core logic has by way of addition (for introducing $\to$) is the extra rule part that allows one to infer a conditional $\varphi \to \psi$ upon reducing its antecedent φ to absurdity. This part of $\to$I in core logic is stated as follows:

$$\dfrac{\begin{matrix}\square^{(i)} \\ \varphi \\ \vdots \\ \bot\end{matrix}}{\varphi \to \psi}\,(i)$$

where now the box indicates *obligatory* or *nonvacuous* discharge—the reductio assumption φ must really have been used in deriving $\bot$.

One easy result provable by induction is that every number is either zero or a successor.

Observation 1 *PA proves* $\forall y(y=0 \vee \exists x\, y=sx)$.

Proof

$$\dfrac{\dfrac{\dfrac{}{0=0}}{0=0 \vee \exists x\, 0=sx} \qquad \dfrac{\dfrac{\dfrac{}{sa=sa}}{\exists x\, sa=sx}}{sa=0 \vee \exists x\, sa=sx}}{\dfrac{b=0 \vee \exists x\, b=sx}{\forall y(y=0 \vee \exists x\, y=sx)}}\,(1)$$

□

In the formal core proof just given, the parameter b takes the place of the term t in the statement of the rule of mathematical induction. Note that in the inductive step no use is made of the inductive hypothesis, which would have been $a=0 \vee \exists x\, a=sx$, had there been any call for it. The result just proved, namely,

$$\forall y(y=0 \vee \exists x\, y=sx),$$

is important, because it is to be taken as an axiom when we state the theory Q below. Q *lacks* the axiom scheme of mathematical induction. It has only finitely many axioms.

A primitive recursive formula in the language of arithmetic is one that has no unbounded quantifications. Every provable primitive recursive sentence is true in the standard model $\mathbb{N}$. Conversely, every true primitive recursive sentence that is true in $\mathbb{N}$ is provable. The six axioms that we stated at the outset *decide* every primitive recursive sentence. (They get the 'atomic diagram' of $\mathbb{N}$ right.) This means that for primitive recursive sentences φ, we have the equivalence of the following:

$$T \nvdash \varphi \quad ; \qquad T \vdash \neg\varphi \quad ; \qquad T, \varphi \vdash \bot.$$

We shall call these, for convenience, the *p.r. equivalences*.

We now use 'L' as a sortal variable over first-order languages in general. Bear in mind that our chosen name for the first-order language of arithmetic is '$\mathscr{L}$.'

Definition 2 Suppose Δ is a set of sentences in some language L. Then Δ is *complete* if and only if for every sentence φ in L, either $\Delta \vdash \varphi$ or $\Delta, \varphi \vdash \bot$.

Definition 3 '$M \Vdash \varphi$' is short for 'φ is true in the model M.'

Definition 4 Let M be any model for a language L. Th_L(M) is the set of all sentences in L that are true in M. In symbols:

$$\mathrm{Th}_L(M) = \{\varphi \in L \mid M \Vdash \varphi\}.$$

We shall write '$\mathrm{Th}(M)$' for '$\mathrm{Th}_L(M)$' whenever we can.

Observation 2 *For any model M of a language L, Th(M) is consistent and complete.*

All terms, formulae, and sentences are to be understood from now on as in the first-order language $\mathcal{L}$ of arithmetic. For present purposes, one particular model will hold our attention: the so-called *standard model for arithmetic*.

Definition 5 The standard model for arithmetic, denoted $\mathbb{N}$, has for its domain exactly the natural numbers, and interprets the name (constant) 0, the successor symbol s, the addition function symbol $+$, and the multiplication function symbol $\times$ in the obviously correct way.

Definition 6 As a special case of Definition 4:

$$Th(\mathbb{N}) = \{\varphi \in \mathcal{L} \mid \mathbb{N} \Vdash \varphi\}.$$

Definition 7 For any natural number n, the *numeral* for n is the term

$$s \dots s0,$$

with n occurrences of the successor-function symbol s. We abbreviate the numeral for n as

$$\underline{n}.$$

Example: $\underline{3}$ is $sss0$.

Observation 3 *Because the symbol '0' is used in both the object-language and the metalanguage, it is true by definition (in the metalanguage) that*

$$0 = \underline{0}.$$

Likewise, whenever we use 's' for successor in the metalanguage, it is true by definition (in the metalanguage) that

$$s\underline{n} = \underline{sn}.$$

Definition 8 The following set of axioms defines the theory of arithmetic known as Q:

1. $\forall x \forall y\, (sx = sy \rightarrow x = y)$;
2. $\forall x\, \neg\, sx = 0$;
3. $\forall x \forall y\; x + sy = s(x + y)$;
4. $\forall x\; x + 0 = x$;
5. $\forall x\; x \times 0 = 0$;
6. $\forall x \forall y\; x \times sy = (x \times y) + x$;
7. $\forall x\, (x = 0 \vee \exists y\; x = sy)$.

Definition 9 The following set of axioms defines the familiar and standard theory of arithmetic known as Peano arithmetic (PA):

1. $\forall x \forall y\, (sx = sy \rightarrow x = y)$;
2. $\forall x\, \neg\, sx = 0$;
3. $\forall x \forall y\; x + sy = s(x + y)$;
4. $\forall x\; x + 0 = x$;
5. $\forall x\; x \times 0 = 0$;
6. $\forall x \forall y\; x \times sy = (x \times y) + x$;

7. all instances of the axiom scheme of mathematical induction:
$(\Phi(0) \wedge \forall x(\Phi(x) \rightarrow \Phi(sx))) \rightarrow \forall y \Phi(y)$.

Note that even with recourse to its axiom scheme of mathematical induction, PA meets a crucially important requirement on any axiomatization of a theory T about any given subject matter. This is the requirement that the set of axioms (which, here, would include all *instances* of the scheme) be *effectively decidable*. That is, there should be an effective method that correctly decides, when given any sentence of the language (here, $\mathcal{L}$), whether it counts as an axiom of the theory in question.

This requirement often goes unremarked. One reason for this, perhaps, is that axiom sets for various other interesting theories are often finite, and the effective decidability of any finite set of sentences hardly merits remarking. But the reason that the requirement of effective decidability of the set of axioms is so important *in general* is this: without it, we would not be able to *tell*, effectively, whether a purported proof of a theorem of the theory is indeed a proof! The requirement is epistemologically basic. Along with the requirement that the relation

Π is a proof (in system $\mathcal{S}$) of the conclusion φ from the (finite) set Δ of premises

be effectively decidable, the requirement of the effective decidability of the set of axioms of a theory T is what ensures that T (i.e., the set of theorems $\mathcal{S}$-provable from those axioms) is effectively *enumerable* (even if not itself effectively *decidable*). Such effective enumerability is what is meant by *axiomatizability*.

Observation 4 $\mathsf{Q} \subseteq PA \subseteq Th(\mathbb{N})$.

Observation 5 *Observation 4 is by inspection of axioms, and Observation 1. In fact, we have* $\mathsf{Q} \subset PA \subset Th(\mathbb{N})$*, but we do not officially 'know' this yet.*

The proper inclusion $\mathsf{Q} \subset$ PA follows from the provable unprovability-in-Q of various theorems of PA (indeed, some of them so 'obviously' true-in-$\mathbb{N}$ that their unprovability-in-Q comes as quite a surprise). Boolos and Jeffrey [1, pp. 171–72] *infra* list the following ten examples (the asterisked ones can be found in Tarski, Mostowski, and Robinson [12, p. 55]):[6]

$$
\begin{aligned}
&{}^*\forall x\, \neg x = sx,\\
&\forall x\, \forall y\; x+(y+z) = (x+y)+z,\\
&\forall x\, \forall y\; x+y = y+x,\\
&{}^*\forall x\; 0+x = x,\\
&\forall x\; x < sx,\\
&\forall x\, \forall y\, \neg(x < y \wedge y < x),\\
&\forall x\, \forall y\, \forall z\; x\times(y\times z) = (x\times y)\times z,\\
&\forall x\, \forall y\; x\times y = y\times x,\\
&{}^*\forall x\; 0\times x = 0 \text{ (Boolos and Jeffrey actually wrote } \forall x\, 0\times x = x,\\
&\qquad\qquad \text{which must have been in error.)},\\
&\forall x\, \forall y\, \forall z\; x\times(y+z) = (x\times y)+(x\times z).
\end{aligned}
$$

They provide a simple countermodel to show that they are not logical consequences of Q. They supplement the natural numbers with two rogue elements a and b and define mappings (to interpret s, $+$, and $\times$ on $\mathbb{N} \cup \{a, b\}$) that extend their standard interpretations on $\mathbb{N}$. This model satisfies Q but falsifies each of the foregoing ten examples.

Burgess [2, p. 56] reports an observation by Kripke that a model for Q can be given in the cardinal numbers (finite and infinite), with the successor function interpreted as identity on the infinite cardinals. This highlights how very little of what is going on among the naturals is distilled out by the axioms of Q—as is clear from the list above of surprising nontheorems of Q.

A more advanced way to establish the same proper inclusion ($\mathsf{Q} \subset \mathrm{PA}$) is to point out that PA is known not to be finitely axiomatizable (Ryll-Nardzewski [9]), whereas Q obviously is.

The proper inclusion $\mathrm{PA} \subset \mathrm{Th}(\mathbb{N})$ is an immediate consequence of Gödel's first incompleteness theorem. It will be established as Theorem 7, using only core logic at the metalevel.

The system Q is due to Raphael Robinson (see Robinson [8]). As we have just seen in the foregoing definitions, the official set of Q-axioms results from the well-known axioms for Peano arithmetic by replacing the latter's axiom *scheme* of mathematical induction (which of course has infinitely many instances) by the *single* axiom

$$\forall x\,(x = 0 \vee \exists y\; x = sy).$$

The latter axiom ensures only that the successor function is *onto* all elements of the domain except for 0 (which the first axiom says is initial, i.e., not a successor of anything). By contrast, the principle of mathematical induction, especially in its second-order version

$$\forall_2 \Phi\big(\big(\Phi(0) \wedge \forall x\big(\Phi(x) \to \Phi(sx)\big)\big) \to \forall y \Phi(y)\big),$$

is an attempt to express the thought that the standard natural numbers *exhaust the domain* of $\mathbb{N}$.

The corresponding attempt at first order, via the axiom scheme, is of course Pyrrhic: there are nonstandard countable models of first-order $\mathrm{Th}(\mathbb{N})$. These models fail to be isomorphic to $\mathbb{N}$ because of 'alien intruders' in their domains—so-called *nonstandard* natural numbers creep in. This problem bedevils any attempt at first order to specify 'up to isomorphism' an intended model, every one of whose individuals is denoted by some closed term of the language. (See Tennant [16].) It is little solace to know (see MacDowell and Specker [7]) that every nonstandard model of $\mathrm{Th}(\mathbb{N})$ is an elementary end-extension of $\mathbb{N}$ (and indeed that $\mathbb{N}$ is the only model of $\mathrm{Th}(\mathbb{N})$ of which every *other* model of $\mathrm{Th}(\mathbb{N})$ is an elementary end-extension). For the first problem is: how do we get our hands on $\mathrm{Th}(\mathbb{N})$? And the second problem is that all that this elementary end-extension result tells us is that if $\exists x \varphi(\underline{\vec{n}}, x)$ is true in such an extension M, then for some standard natural number m we have that $\varphi(\underline{\vec{n}}, \underline{m})$ is true in M. That is to say, $\mathbb{N}$ provides all the 'witnessing' required for the truth of existentials in M. We cannot be forced to go rummaging for witnesses within the nonstandard part of a nonstandard model M. Nor, however, could we force a Martian interlocutor to 'stay within $\mathbb{N}$' when building up a mental picture of what sort of abstract world it might be that makes all of $\mathrm{Th}(\mathbb{N})$ true. We cannot in general make our Martian deaf to a chorus of alien witnessing within some nonstandard M.

The important thing about Q is that it is finitely axiomatizable and hence is the logical closure of a single sentence (the conjunction of its finitely many axioms). This is the feature that does all the work for the metamathematical purposes served by the choice of Q. Therefore, with an eye just to these purposes, there is no great value in insisting that one's (finite) set of axioms for Q be *irredundant*. All that matters is that the chosen axioms 'for' Q be finite in number and have among them at least the ‘official’ ones stated above. One can tolerate as ‘axioms’ any (finite) number of theorems that are derivable from the official few axioms for Q.

The proof of the representability theorem for Q is greatly facilitated by choosing two such theorems to add to the official list of axioms—namely,

$$\text{(i)}\quad \forall x \forall y (x+y=0 \rightarrow x=0)$$

and

$$\text{(ii)}\quad \forall x \forall y \left(\forall z(\neg x+sz=y \wedge \neg y+sz=x) \rightarrow x=y\right).$$

This was the expository route adopted by Tennant in [13]. Here, however, we shall adopt the more ‘purist’ route and actually derive, within official Q, the extra two results just mentioned.[7] In fact, we shall derive them as *rules*. This is for two reasons: using them as rules makes the proof of representability more direct, and the derivations themselves afford another opportunity to see core logic at work in faithfully formalizing the mathematical reasoning involved.

3.3 Giving Q by means of rules rather than axioms Here we restate the foregoing axioms for Q as atomic rules of inference:

$$\frac{st=su}{t=u} \qquad \frac{st=0}{\bot}$$

Any intersubstitutions of
$t+0$ with t
$t+su$ with $s(t+u)$
$t.0$ with $\mathbf{0}$
$t.su$ with $(t.u)+t$

$$\dfrac{\begin{array}{cc} \Box\,\overline{}^{\,(i)} & \Box\,\overline{}^{\,(i)} \\ t=0 & t=sa \\ \vdots & \vdots \\ \varphi/\bot & \varphi/\bot \end{array}}{\varphi/\bot}\,(i)$$

where a does not occur in t or in any undischarged assumption, other than $t=sa$, in the right-hand subproof

We shall also use the following rules, to be derived within Q:

Any intersubstitutions of
$st+u$ with $s(t+u)$

$$\dfrac{\begin{array}{cc} \Box\,\overline{}^{\,(i)} & \Box\,\overline{}^{\,(i)} \\ t+sa=u & u+sa=t \\ \vdots & \vdots \\ \bot & \bot \end{array}}{t=u}\,(i)$$

where a is parametric

4 Some Deducibilities in Q

In this section, we shall concentrate on making use of the following five axioms of Q, conveniently labeled:

Q1. $\forall x \forall y\, (sx=sy \rightarrow x=y)$.
Q2. $\forall x\, \neg\, sx=0$.
Q3. $\forall x\, (x=0 \vee \exists y\, x=sy)$.
Q4. $\forall x\, x+0=x$.
Q5. $\forall x \forall y\, x+sy=s(x+y)$.

The rule versions of these five axioms have already been given in Section 3.3.

4.1 Some atomic deducibilities

Lemma 1 *We have*

$$\forall m \forall n \; m \neq n \;\; \rightarrow \;\; \mathsf{Q}, \underline{m} = \underline{n} \vdash_{\mathbb{C}} \bot.$$

Proof Use $|m-n|$ applications of the rule Q1

$$\frac{st = su}{t = u},$$

followed by an application of the rule Q2

$$\frac{st = 0}{\bot}.$$

□

Lemma 2 *The rule*

$$\frac{a+b=0}{a=0}$$

is core-derivable in Q. *(Indeed, it is* atomically *derivable.)*

Proof

$$\dfrac{\dfrac{\overline{a=0}^{(1)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\overline{b=0}^{(2)} \quad a+b=0}{a+0=0} \quad \overline{a=sc}^{(1)}}{sc+0=0} \qquad \mathsf{Q4}: \; sc+0=sc}{sc=0}}{\bot}\,\mathsf{Q2}}{a=0}\,(1)\;\mathsf{Q3} \qquad \dfrac{\dfrac{\dfrac{\overline{b=sd}^{(2)} \quad a+b=0}{a+sd=0}}{s(a+d)=0}\,\mathsf{Q5}}{\bot}\,\mathsf{Q2}}{a=0}\,(2)\;\mathsf{Q3}$$

Note that the steps marked (1) and (2) that discharge case assumptions are applications of the rule corresponding to the axiom Q3:

$$\forall x (x=0 \vee \exists y \; x = sy).$$

We are taking advantage here of core logic's liberalized rule of $\vee$-elimination (proof by cases), which allows one to bring down as the main conclusion the conclusion of either one of the two case proofs if the conclusion of the other case proof is absurdity.

□

4.2 Some nonatomic deducibilities

Lemma 3 *We have*

$$\forall n > 0 \;\; \mathsf{Q}, \{\varphi(\underline{k}) \mid k < n\}, sb+a = \underline{n} \vdash_{\mathbb{C}} \varphi(a).$$

Proof Note that in the statement of this result, the symbol '<' is being used in the metalanguage and has its usual arithmetical meaning.

The result is proved by a course-of-values induction on n, carried out at the meta-level. For the basis step of $n=1$, we need to establish

$$\mathsf{Q}\,,\; \varphi(0)\,,\; sb+a = s0 \;\vdash_{\mathbb{C}}\; \varphi(a).$$

This is done as follows:

$$\dfrac{\varphi(0) \qquad \dfrac{\overline{a=0}^{\,(1)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{sb+a=s0 \quad \overline{a=sc}^{\,(1)}}{sb+sc=s0}}{s(sb+c)=s0}\,\text{Q5}}{sb+c=0}\,\text{Q1}}{sb=0}\,L2}{\perp}\,\text{Q2}}{a=0}\,(1)\ \text{Q3}}{\varphi(a)}$$

The inductive hypothesis in our course-of-values induction is to the effect that, no matter what formula $\varphi(x)$ one chooses, there is a Q-proof Σ of the following form:

$$\begin{array}{c}\underbrace{\{\varphi(\underline{k}) \mid k < n\}\ ,\ sb+a=\underline{n}} \\ \Sigma \\ \varphi(a)\end{array}$$

Here, a is parametric—that is, a does not occur in the chosen formula $\varphi(x)$. So we may replace a with any other parameter that meets the same condition.

For our purposes in carrying out the inductive step, we choose the following disjunction as a particular instance for $\varphi(x)$:

$$\vee_{i=0}^{n-1} x=\underline{i},$$

that is,

$$x=0 \vee x=\underline{1} \vee \ldots \vee x=\underline{n-1}.$$

Relative to this choice for $\varphi(x)$, and using the parameter c, the proof Σ whose existence may be assumed by inductive hypothesis takes the more specific form:

$$\begin{array}{c}\underbrace{\{\vee_{i=0}^{n-1}\underline{k}=\underline{i} \mid k < n\}\ ,\ sb+c=\underline{n}} \\ \Sigma \\ \vee_{i=0}^{n-1} c=\underline{i}\end{array}$$

Note that for each value of k less than n, one of the identity disjuncts is a self-identity, and hence a logical theorem; whence, by multiple $\vee$-introductions, the disjunctive sentence $\vee_{i=0}^{n-1}\underline{k}=\underline{i}$ is also a logical theorem. So Σ really takes the form

$$\begin{array}{c}\underbrace{\{\overline{\vee_{i=0}^{n-1}\underline{k}=\underline{i}} \mid k < n\}\ ,\ sb+c=\underline{n}} \\ \Sigma \\ \vee_{i=0}^{n-1} c=\underline{i}\end{array}$$

where the overlining indicates the logical theoremhood of those particular premises of Σ.

The inductive step is now accomplished by remarking the existence of the following φ-based proof-schema in Q, exploiting Σ as an embedded subproof therein as indicated:

$$\dfrac{\dfrac{\dfrac{}{a=0}{\scriptstyle(2)} \quad \varphi(0)}{\varphi(a)} \qquad \dfrac{\begin{array}{c}\underbrace{\{\overline{\bigvee_{i=0}^{n-1}\underline{k}=\underline{i}} \mid k<n\}\ ,\ \dfrac{\dfrac{\dfrac{sb+a=s\underline{n} \quad \dfrac{}{a=sc}{\scriptstyle(2)}}{sb+sc=s\underline{n}}{\scriptstyle\mathsf{Q5}}}{s(sb+c)=s\underline{n}}{\scriptstyle\mathsf{Q1}}}{sb+c=\underline{n}}}_{\Sigma}\\ \bigvee_{i=0}^{n-1}c=\underline{i}\end{array} \quad \left\{\dfrac{\varphi(s\underline{i}) \quad \dfrac{\dfrac{}{c=\underline{i}}{\scriptstyle(1)} \quad \dfrac{}{a=sc}{\scriptstyle(2)}}{a=s\underline{i}}}{\varphi(a)}\right\}_{i<n-1} \quad \dfrac{\varphi(\underline{n}) \quad \dfrac{\dfrac{}{c=\underline{n-1}}{\scriptstyle(1)} \quad \dfrac{}{a=sc}{\scriptstyle(2)}}{a=\underline{n}}}{\varphi(a)}}{\varphi(a)}{\scriptstyle(1)\,\vee\mathrm{E}}}{\varphi(a)}{\scriptstyle(2)\,\mathsf{Q3}} \qquad \square$$

Let us refer to the proofs of Lemmas 1, 2, and 3 as Π_1, Π_2, and Π_3, respectively. Thus far we have the following trees of dependencies:

$$\begin{array}{c}\underbrace{\mathsf{Q1},\mathsf{Q2}}_{\Pi_1}\\ \mathrm{L}1\end{array} \qquad\qquad \begin{array}{c}\underbrace{\mathsf{Q1},\mathsf{Q2},\mathsf{Q3},\mathsf{Q5},\begin{array}{c}\underbrace{\mathsf{Q2},\mathsf{Q3},\mathsf{Q4},\mathsf{Q5}}_{\Pi_2}\\ \mathrm{L}2\end{array}}_{\Pi_3}\\ \mathrm{L}3\end{array}$$

Each of the leaf-node citations of an axiom $\mathsf{Q}i$ is shorthand for a metalinguistic statement of the form '$\mathsf{Q}i$ is an axiom of Q.' Remember also that the various promised proofs Π_i (including further ones to be given below) are *meta*proofs; they are proofs *about* the theory Q, not proofs constructed *within* Q. Naturally, though, they can advert to inferential moves or whole chunks of proof 'within Q' in order to track deducibility within Q.

We shall proceed further to prove Lemmas (4)–(10) below by means of proofs Π_4–Π_{10}, respectively, as indicated in the following proof-tree:

$$\begin{array}{c}\underbrace{\mathsf{Q3}\ ,\ \mathsf{Q4}\ ,\ \mathsf{Q5}\ ,\ \begin{array}{c}\underbrace{\mathsf{Q4},\mathsf{Q5}}_{\Pi_5}\\ \mathrm{L}5\end{array}\ ,\ \begin{array}{c}\underbrace{\mathsf{Q4},\mathsf{Q5}}_{\Pi_6}\\ \mathrm{L}6\end{array}\ ,\ \begin{array}{c}\underbrace{\mathsf{Q1},\mathsf{Q2},\mathsf{Q3},\mathsf{Q4},\mathsf{Q5}}_{\Pi_7}\\ \mathrm{L}7\end{array}\ ,\ \begin{array}{c}\underbrace{\mathsf{Q3}\ ,\ \begin{array}{c}\underbrace{\mathsf{Q4},\mathsf{Q5}}_{\Pi_4}\\ \mathrm{L}4\end{array}\ ,\ \begin{array}{c}\underbrace{\mathsf{Q4},\mathsf{Q5}}_{\Pi_5}\\ \mathrm{L}5\end{array}}_{\Pi_8}\\ \mathrm{L}8\end{array}}_{\Pi_9}\\ \mathrm{L}9\\ \Pi_{10}\\ \mathrm{L}10\end{array}$$

In order to minimize sideways spread, we shall adopt the step

$$\dfrac{t=u}{st=su}$$

as an abbreviation of

$$\dfrac{\dfrac{}{st=st} \quad t=u}{st=su}\ .$$

To the same end, we shall use *serial* $\forall$-elimination on occasion:

$$\dfrac{\forall x\,\psi(x)}{\psi(t)}$$

instead of the ‘official’ parallelized rule of $\forall$-elimination in core logic.
Assume as given a metaproof Π of the following simple transition:

$$\begin{array}{c} m+sn=k \\ \Pi \\ \exists y(m+n=y \wedge k=sy). \end{array}$$

It would not matter if this inference were to fail to be derivable in Q; all that matters is that it is arithmetically valid. Certainly, it is derivable in PA. Bear in mind here that we are concerned with derivability at the *meta*level.

We can now reason at the metalevel as in the following proof of Lemma 4, using mathematical induction at the step marked (6). We stress once again that this metalevel reasoning is conducted within core logic.

Lemma 4 *We have:*

$$\dfrac{m+n=k}{\mathsf{Q}\vdash \underline{m}+\underline{n}=\underline{k}}.$$

Alternatively, expressed as a metalinguistic sentence:

$$\forall x\forall z(m+x=z \rightarrow \mathsf{Q}\vdash \underline{m}+\underline{x}=\underline{z}).$$

Proof The promised proof Π_4 is as follows.

$$\Pi_4:\quad \dfrac{\dfrac{\dfrac{\dfrac{\mathsf{Q4}:\ \underline{m}+\underline{0}=\underline{m} \qquad \dfrac{\dfrac{}{m=m+0} \qquad \dfrac{}{m+0=k}{}^{(5)}}{m=k}}{\mathsf{Q}\vdash \underline{m}+\underline{0}=\underline{k}}}{m+0=k \rightarrow \mathsf{Q}\vdash \underline{m}+\underline{0}=\underline{k}}{}^{(5)}}{\forall z(m+0=z \rightarrow \mathsf{Q}\vdash \underline{m}+\underline{0}=\underline{z})} \qquad \dfrac{\dfrac{\dfrac{\begin{array}{c}\overset{(3)}{\overline{m+sn=k}} \\ \Pi \\ \exists y(m+n=y\wedge k=sy)\end{array} \qquad \dfrac{\overset{(2)}{\overline{m+n=r\wedge k=sr}} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\overset{(1)}{\overline{m+n=r}} \qquad \dfrac{\dfrac{}{\forall z(m+n=z\rightarrow \mathsf{Q}\vdash \underline{m}+\underline{n}=\underline{z})}{}^{(6)}}{m+n=r\rightarrow \mathsf{Q}\vdash \underline{m}+\underline{n}=\underline{r}}}{\mathsf{Q}\vdash \underline{m}+\underline{n}=\underline{r}}}{\mathsf{Q}\vdash s(\underline{m}+\underline{n})=\underline{sr}} \qquad \mathsf{Q5}:\ \underline{m}+\underline{sn}=s(\underline{m}+\underline{n})}{\mathsf{Q}\vdash \underline{m}+\underline{sn}=\underline{sr}} \qquad \dfrac{}{k=sr}{}^{(1)}}{\mathsf{Q}\vdash \underline{m}+\underline{sn}=\underline{k}}}{\mathsf{Q}\vdash \underline{m}+\underline{sn}=\underline{k}}{}^{(1)}}{\mathsf{Q}\vdash \underline{m}+\underline{sn}=\underline{k}}{}^{(2)}}{m+sn=k \rightarrow \mathsf{Q}\vdash \underline{m}+\underline{sn}=\underline{k})}{}^{(3)}}{\forall z(m+sn=z\rightarrow \mathsf{Q}\vdash \underline{m}+\underline{sn}=\underline{z})}{}^{(6)}}{\forall x\forall z(m+x=z \rightarrow \mathsf{Q}\vdash \underline{m}+\underline{x}=\underline{z})}$$

□

Note that putting Π above the occurrence of $\exists y(m+n=y\wedge k=sy)$ that stands as the major premise of an $\exists$-elimination strictly speaking sins against the requirement in core logic that MPEs should stand proud, with no nontrivial proof-work above them. But our rendering in this way the subproof terminating on the conclusion of the step marked (2) (the application of $\exists$-E in question) is a mere convenience. In deference to the requirement just mentioned, one could represent that subproof more captiously (using the cut-elimination notation of Tennant [17]) as

$$\left[\begin{array}{c} m+sn=k \\ \Pi \\ \exists y(m+n=y\wedge k=sy) \end{array}, \quad \dfrac{\exists y(m+n=y\wedge k=sy) \qquad \dfrac{\overset{(2)}{\overline{m+n=r\wedge k=sr}} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\overset{(1)}{\overline{m+n=r}} \qquad \dfrac{\forall z(m+n=z\rightarrow \mathsf{Q}\vdash \underline{m}+\underline{n}=\underline{z})}{m+n=r\rightarrow \mathsf{Q}\vdash \underline{m}+\underline{n}=\underline{r}}}{\mathsf{Q}\vdash \underline{m}+\underline{n}=\underline{r}}}{\mathsf{Q}\vdash s(\underline{m}+\underline{n})=\underline{sr}} \qquad \mathsf{Q5}:\ \underline{m}+\underline{sn}=s(\underline{m}+\underline{n})}{\mathsf{Q}\vdash \underline{m}+\underline{sn}=\underline{sr}} \qquad \dfrac{}{k=sr}{}^{(1)}}{\mathsf{Q}\vdash \underline{m}+\underline{sn}=\underline{k}}}{\mathsf{Q}\vdash \underline{m}+\underline{sn}=\underline{k}}{}^{(1)}}{\mathsf{Q}\vdash \underline{m}+\underline{sn}=\underline{k}}{}^{(2)}\right]$$

Lemma 5 *We have:*

$$\forall n\ \mathsf{Q}\vdash \forall x\, sx+\underline{n}=x+\underline{sn}.$$

Proof

Π_5 :

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\text{by Q4 :}\\ \mathsf{Q}\vdash sa+0=sa\end{matrix}\quad\begin{matrix}\text{by Q4 :}\\ \mathsf{Q}\vdash a+0=a\end{matrix}}{\mathsf{Q}\vdash sa+0=s(a+0)}\quad\begin{matrix}\text{by Q5 :}\\ \mathsf{Q}\vdash a+s0=s(a+0)\end{matrix}}{\mathsf{Q}\vdash sa+0=a+s0}}{\mathsf{Q}\vdash\forall x\,sx+0=x+s0}\qquad\dfrac{\dfrac{\dfrac{\begin{matrix}\text{by Q5 :}\\ \mathsf{Q}\vdash sa+\underline{sm}=s(sa+\underline{m})\end{matrix}\quad\dfrac{\dfrac{}{\mathsf{Q}\vdash\forall x\,sx+\underline{m}=x+\underline{sm}}{\scriptstyle(1)}}{\mathsf{Q}\vdash sa+\underline{m}=a+\underline{sm}}}{\mathsf{Q}\vdash sa+\underline{sm}=s(a+\underline{sm})}\quad\begin{matrix}\text{by Q5 :}\\ \mathsf{Q}\vdash a+\underline{ssm}=s(a+\underline{sm})\end{matrix}}{\mathsf{Q}\vdash sa+\underline{sm}=a+\underline{ssm}}}{\mathsf{Q}\vdash\forall x\,sx+\underline{sm}=x+\underline{ssm}}}{\mathsf{Q}\vdash sb+\underline{n}=b+\underline{sn}}{\scriptstyle(1)}}{\mathsf{Q}\vdash\forall x\,sx+\underline{n}=x+\underline{sn}}$$

□

Until now we have used '$<$' only in the metalanguage. The time has come to define the notion of ordering so that it has a precise sense *within the theory* Q. We shall use the same symbol; context will easily let us work out whether it is expressing the metalinguistic notion or the object-linguistic one, which we now define.

Definition 10 $\quad t<u \ \equiv_{\text{df}} \ \exists y\, sy+t=u.$

This Q-specific notion of 'less than' can also be captured, in inferentialist fashion, by laying down the following rules for introducing and eliminating the symbol '$<$' in atomic sentences of the object-language:

$$(<\text{-I})\quad \dfrac{sv+t=u}{t<u}\qquad\qquad (<\text{-E})\quad \dfrac{t<u\qquad \begin{matrix}\overline{sa+t=u}^{\,(i)}\\ \vdots\\ \theta\end{matrix}}{\theta}{\scriptstyle(i)}\quad a \text{ parametric}$$

Lemma 6 $\quad \forall k \ \ \mathsf{Q}\vdash s0+\underline{k}=s\underline{k}$. *Hence* $\mathsf{Q}\vdash\exists y\, sy+\underline{k}=s\underline{k}$*; that is,* $\mathsf{Q}\vdash\underline{k}<s\underline{k}$.

Proof The proof is by induction on k. The basis is immediate by Q4:

$$s0+0 \,=\, s0.$$

The inductive hypothesis is to the effect that there is a Q-proof Π, say, of

$$s0+\underline{m}=s\underline{m}.$$

In the inductive step, we need to show that there is a Q-proof of

$$s0+s\underline{m}=ss\underline{m}.$$

So we simply extend Π as follows:

$$\dfrac{\dfrac{\begin{matrix}\Pi\\ s0+\underline{m}=s\underline{m}\end{matrix}}{s(s0+\underline{m})=ss\underline{m}}{\scriptstyle\mathsf{Q5}}}{s0+s\underline{m}=ss\underline{m}}$$

□

Lemma 7 *We have:*

$$\forall n\in\mathbb{N}(n=0\ \vee\ \ \mathsf{Q}\vdash\forall x\big(x<\underline{n}\to(x=0\vee\ldots\vee x=\underline{n-1})\big).$$

Proof The proof Π_7 is by induction on n. The basis step of Π_7 is obvious. For the inductive step of Π_7, we construct the following proof within Q in the object-language:

$$\dfrac{\begin{array}{c}\color{blue}{a<si}\\ \text{i.e., } \exists w\, sw+a=si\end{array} \qquad \dfrac{\dfrac{\overline{a=0}^{\,(2)}}{a=0\vee\ldots\vee a=i} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{(3)}{\overline{sb+a=si}} \quad \overline{a=sc}^{\,(2)}}{sb+sc=si} \qquad \mathsf{Q5}:\ sb+sc=s(sb+c)}{si=s(sb+c)}}{i=sb+c}\,(\mathsf{Q1})}{\text{i.e. } c<i}}{c=0\vee\ldots\vee c=i-1}\,(\text{IH}) \qquad \dfrac{\dfrac{\overset{(1)}{\overline{c=0}} \quad \overline{a=sc}^{\,(2)}}{a=s0}}{a=s0\vee\ldots\vee a=i} \quad \cdots \quad \dfrac{\dfrac{\dfrac{\overset{(1)}{\overline{c=i-1}} \quad \overline{a=sc}^{\,(2)}}{a=s(i-1)} \qquad \overline{s(i-1)=i}}{a=i}}{a=s0\vee\ldots\vee a=i}}{\dfrac{a=s0\vee\ldots\vee a=i}{a=0\vee a=s0\vee\ldots\vee a=i}}\,(1)}{a=0\vee\ldots\vee a=i}\,(2)}{\begin{array}{c}a=0\vee\ldots\vee a=i\\ \text{i.e., } \color{blue}{a=0\vee\ldots\vee a=si-1}\end{array}}\,(3)$$

□

(The step labeled (1) is a multiple $\vee$-elimination.)

Lemma 8 *We have:*

$$\mathsf{Q}\,,\ \underline{n}<a\ \vdash\ a=s\underline{n}\vee s\underline{n}<a\,.$$

Proof

$$\Pi_8:\quad \dfrac{\begin{array}{c}\underline{n}<a,\ \text{i.e.}\\ \exists w\, sw+\underline{n}=a\end{array} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\overset{(2)}{\overline{b=0}} \quad \overline{sb+\underline{n}=a}^{\,(3)}}{s0+\underline{n}=a} \qquad \mathrm{L}4:\ 0+s\underline{n}=s\underline{n}}{a=s\underline{n}}}{a=s\underline{n}\vee s\underline{n}<a} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{(2)}{\overline{b=sc}} \quad \overline{sb+\underline{n}=a}^{\,(3)}}{ssc+\underline{n}=a} \qquad \mathrm{L}5:\ ssc+\underline{n}=sc+s\underline{n}}{sc+s\underline{n}=a}}{\exists y\, sy+s\underline{n}=a\ ,\ \text{i.e.}}}{s\underline{n}<a}}{a=s\underline{n}\vee s\underline{n}<a}}{a=s\underline{n}\vee s\underline{n}<a}\,(2)\ \mathsf{Q3}}{a=s\underline{n}\vee s\underline{n}<a}\,(3)$$

□

Lemma 9 *We have:*

$$\forall n\ \mathsf{Q}\ \vdash\ \forall x(\underline{n}<x\vee x=\underline{n}\vee x<\underline{n}).$$

Proof The proof Π_9 is by induction on n. For the basis of Π_9, we prove

$$\forall x(0<x\vee x=0\vee x<0)$$

as follows:

$$\dfrac{\dfrac{\dfrac{\overline{a=0}^{\,(1)}}{0<a\vee a=0\vee a<0} \qquad \dfrac{\dfrac{\overset{(1)}{\overline{\exists y\, a=sy}} \qquad \dfrac{\dfrac{\overset{(2)}{\overline{a=sb}} \quad \mathsf{Q4}:\ sb=sb+0}{a=sb+0}}{\begin{array}{c}\exists y\, a=sy+0\ ,\ \text{i.e.}\\ 0<a\end{array}}}{\dfrac{0<a}{0<a\vee a=0\vee a<0}}\,(2)}{0<a\vee a=0\vee a<0}\,(1)\ \mathsf{Q3}}{\forall x(0<x\vee x=0\vee x<0)}$$

The inductive hypothesis (IH) is

$$\forall x(\underline{k}<x\vee x=\underline{k}\vee x<\underline{k}),$$

and from (IH) the inductive step of Π_9 proves

$$\forall x(s\underline{k}<x\vee x=s\underline{k}\vee x<s\underline{k})$$

as follows:

$$\dfrac{\dfrac{\forall x(\underline{k}<x\vee x=\underline{k}\vee x<\underline{k})}{\underline{k}<a\vee a=\underline{k}\vee a<\underline{k}} \quad \dfrac{\dfrac{\dfrac{}{\underline{k}<a}{\scriptstyle(1)}}{a=s\underline{k}\vee s\underline{k}<a}{\scriptstyle(\mathrm{L}8)}}{s\underline{k}<a\vee a=s\underline{k}\vee a<s\underline{k}} \quad \dfrac{\dfrac{\overset{\mathrm{L}6:}{\underline{k}<s\underline{k}} \quad \dfrac{}{a=\underline{k}}{\scriptstyle(1)}}{a<s\underline{k}}}{s\underline{k}<a\vee a=s\underline{k}\vee a<s\underline{k}} \quad \dfrac{\begin{matrix}\dfrac{}{a<\underline{k}}{\scriptstyle(1)}\\ \Sigma\\ a<s\underline{k}\end{matrix}}{s\underline{k}<a\vee a=s\underline{k}\vee a<s\underline{k}}}{\dfrac{s\underline{k}<a\vee a=s\underline{k}\vee a<s\underline{k}}{\forall x(s\underline{k}<x\vee x=s\underline{k}\vee x<s\underline{k})}}{\scriptstyle(1)}\,\vee\mathrm{E}$$

where the embedded subproof Σ is

$$\dfrac{\begin{matrix}a<\underline{k},\text{ i.e.}\\ \exists y\, sy+a=\underline{k}\end{matrix} \quad \dfrac{\dfrac{a<\underline{k} \quad \overset{\mathrm{L}7:}{a<\underline{k}\to(a=0\vee\ldots\vee a=\underline{k-1})}}{a=0\vee\ldots\vee a=\underline{k-1}} \quad \left\{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{a=\underline{i}}{\scriptstyle(1)}\quad\dfrac{}{sb+a=\underline{k}}{\scriptstyle(2)}}{sb+\underline{i}=\underline{k}}}{s(sb+\underline{i})=s\underline{k}} \quad \dfrac{\overset{\mathrm{L}5:}{ssb+\underline{i}=sb+s\underline{i}}\quad \overset{\mathrm{Q}5:}{s(sb+\underline{i})=sb+s\underline{i}}}{s(sb+\underline{i})=ssb+\underline{i}}}{ssb+\underline{i}=s\underline{k}} \quad \dfrac{}{a=\underline{i}}{\scriptstyle(1)}}{\dfrac{ssb+a=s\underline{k}}{a<s\underline{k}}}\right\}_{i<k}}{a<s\underline{k}}{\scriptstyle(1)}}{a<s\underline{k}}{\scriptstyle(2)}$$

$\Box$

Lemma 10 *The following rule is core-derivable in* Q:

$$\dfrac{\begin{matrix}\Box\dfrac{}{sa+\underline{n}=t}{\scriptstyle(i)}\\ \vdots\\ \bot\end{matrix} \qquad \begin{matrix}\Box\dfrac{}{sa+t=\underline{n}}{\scriptstyle(i)}\\ \vdots\\ \bot\end{matrix}}{t=\underline{n}}{\scriptstyle(i)} \qquad \textit{where } a \textit{ is parametric}$$

Proof

$$\Pi_{10}:\quad \dfrac{\overset{\text{By L9}:}{\underline{n}<t\vee t=\underline{n}\vee t<\underline{n}} \quad \dfrac{{\scriptstyle(3)}\dfrac{}{\underline{n}<t} \quad \begin{matrix}\dfrac{}{sa+\underline{n}=t}{\scriptstyle(1)}\\ \vdots\\ \bot\end{matrix}}{\bot}{\scriptstyle(1)<\mathrm{E}} \quad {\scriptstyle(3)}\dfrac{}{t=\underline{n}} \quad \dfrac{{\scriptstyle(3)}\dfrac{}{t<\underline{n}} \quad \begin{matrix}\dfrac{}{sa+t=\underline{n}}{\scriptstyle(2)}\\ \vdots\\ \bot\end{matrix}}{\bot}{\scriptstyle(2)<\mathrm{E}}}{t=\underline{n}}{\scriptstyle(3)}$$

Note how the step marked (3) is a multiple $\vee$-elimination (proof by cases), in which cases are allowed to simply 'close off' with $\bot$. This is permitted by the rule of $\vee$E in core logic. It allows for multiple disjunctive syllogism. $\Box$

Lemma 11 *We have:*

$$\forall m\forall n\ \mathsf{Q}\vdash_{\mathbb{C}} \underline{m}+\underline{n}=\underline{m+n}\ .$$

(Note that this is a metalinguistic claim. It will be proved using only core logic in the metalanguage.)

Proof Let m be arbitrary. We shall establish

$$\forall n\ \mathsf{Q}\vdash_{\mathbb{C}} \underline{m}+\underline{n}=\underline{m+n}$$

by induction on n.

For the basis, we reason in the metalanguage as follows:

$$\dfrac{\dfrac{\mathsf{Q}\vdash_{\mathbb{C}} \underline{m}+0=\underline{m} \qquad 0=\underline{0}}{\mathsf{Q}\vdash_{\mathbb{C}} \underline{m}+\underline{0}=\underline{m}} \qquad m=m+0}{\mathsf{Q}\vdash_{\mathbb{C}} \underline{m}+0=\underline{m+0}}$$

For the inductive step, we reason in the metalanguage as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathsf{Q}\vdash_{\mathbb{C}} \underline{m}+s\underline{n}=s(\underline{m}+\underline{n}) \quad \overset{\text{By IH:}}{\mathsf{Q}\vdash_{\mathbb{C}} \underline{m}+\underline{n}=\underline{m+n}}}{\mathsf{Q}\vdash_{\mathbb{C}} \underline{m}+s\underline{n}=s(\underline{m+n})} \quad s(\underline{m+n})=\underline{s(m+n)}}{\mathsf{Q}\vdash_{\mathbb{C}} \underline{m}+s\underline{n}=\underline{s(m+n)}} \quad s(m+n)=m+sn}{\mathsf{Q}\vdash_{\mathbb{C}} \underline{m}+s\underline{n}=\underline{m+sn}} \quad s\underline{n}=\underline{sn}}{\mathsf{Q}\vdash_{\mathbb{C}} \underline{m}+\underline{sn}=\underline{m+sn}} \qquad \square$$

Lemma 12 *We have:*

$$\forall m \forall n\ \mathsf{Q} \vdash_{\mathbb{C}} \underline{m}\times\underline{n} = \underline{m\times n}\ .$$

(Note that this is a metalinguistic claim. It will be proved using only core logic in the metalanguage.)

Proof Let m be arbitrary. We shall establish

$$\forall n\ \mathsf{Q} \vdash_{\mathbb{C}} \underline{m}\times\underline{n} = \underline{m\times n}$$

by induction on n.
For the basis, we reason in the metalanguage as follows:

$$\dfrac{\dfrac{\mathsf{Q}\vdash_{\mathbb{C}} \underline{m}\times 0 = 0 \qquad 0=\underline{0}}{\mathsf{Q}\vdash_{\mathbb{C}} \underline{m}\times\underline{0} = \underline{0}} \qquad 0 = m\times 0}{\mathsf{Q}\vdash_{\mathbb{C}} \underline{m}\times\underline{0} = \underline{m\times 0}}$$

For the inductive step, we reason in the metalanguage as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathsf{Q}\vdash_{\mathbb{C}} \underline{m}\times s\underline{n}=(\underline{m}\times\underline{n})+\underline{m} \quad \overset{\text{By IH:}}{\mathsf{Q}\vdash_{\mathbb{C}} \underline{m}\times\underline{n}=\underline{m\times n}}}{\mathsf{Q}\vdash_{\mathbb{C}} \underline{m}\times s\underline{n}=\underline{m\times n}+\underline{m}} \quad \overset{\text{Lemma 11:}}{\underline{m\times n}+\underline{m}=\underline{(m\times n)+m}}}{\mathsf{Q}\vdash_{\mathbb{C}} \underline{m}\times s\underline{n}=\underline{(m\times n)+m}} \quad m\times sn=(m\times n)+m}{\mathsf{Q}\vdash_{\mathbb{C}} \underline{m}\times s\underline{n}=\underline{m\times sn}} \quad s\underline{n}=\underline{sn}}{\mathsf{Q}\vdash_{\mathbb{C}} \underline{m}\times\underline{sn}=\underline{m\times sn}} \qquad \square$$

5 Recursive Functions

Here we explain Gödel's 'general recursive' functions, which we shall simply call *recursive* functions when no confusion can arise. We shall not follow Gödel's own definition, but rather that due to Kleene [5], of what Boolos and Jeffrey [1, p. 162] call 'Recursive' functions (with uppercase 'R'). Kleene proved Recursive functions to be exactly the recursive functions that Gödel defined. The great advantage of the inductive definition of Recursive functions is that only *two* operations are involved in building them up from the initial, or basic, functions. These two operations are composition and minimization of regular functions. Because of a suitable choice of basic functions, one can dispense with the operation of primitive recursion.

It is quite ironical that this succinct definition of recursive functions eschews the function-forming operation of primitive recursion. But the hidden 'need' for recursion is met by the availability of addition and multiplication as basic functions. The reader should not lose sight of the fact that the Peano axioms for arithmetic include the so-called *recursion equations* for addition and multiplication. It is gratifying to know that *that* much recursion is all one really needs for the construction of recursive functions in general.

Definition 11 The characteristic function $c_=$ of identity is defined as follows:

$$c_=(m,n) = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

Definition 12 f is a *regular* $(n+1)$-place function just in case for all $k_1, \ldots, k_n$ there exists some k such that $f(k_1, \ldots, k_n, k) = 0$.

Definition 13 Suppose some number satisfies $\varphi(x)$. Then $\lambda x[\varphi(x)]$ is the least number to do so.

Definition 14 The projection function id_k^m yields the kth member of any m-tuple $(1 \leq k \leq m)$:

$$\mathrm{id}_k^m(n_1, \ldots, n_m) = n_k \ .$$

Definition 15 (Recursive functions)
The basic recursive functions are $+$ (addition), $\times$ (multiplication), the characteristic function $c_=$ of identity, and the projection functions id_k^m.
All other recursive functions are built up by means of the following two operations:

1. Composition: if f is an n-place recursive function and $g_1(\vec{k}), \ldots, g_n(\vec{k})$ are m-place recursive functions, then $f(g_1(\vec{k}), \ldots, g_n(\vec{k}))$ is an m-place recursive function.
2. Minimization of regular functions: if f is a regular $(n+1)$-place recursive function, then h is an n-place recursive function, where

$$h(\vec{k}) \ = \ \lambda p\big[f(\vec{k}, p) = 0\big].$$

Gödel's definition of (general) recursive functions provided a precise formal explication of the informal notion of computable functions of natural numbers. So too—and rather more naturally and convincingly—did Turing's definition of functions computable by Turing machines. It is intuitively obvious that Gödel's and Turing's functions are computable. Moreover, their two definitions are *provably coextensive* (they define the *same class* of functions on the natural numbers). Indeed, *every* formal explication ever offered for the notion of computable functions has proved to be coextensive with the explications of Gödel and of Turing. This is what underlies the strong consensus behind Church's thesis.

> CHURCH'S THESIS: Any function from $\mathbb{N}$ into $\mathbb{N}$ (partial or total) is computable if *and only if* it is recursive.

Gödel's definition is particularly useful for the metamathematician since it affords a method of proof by mathematical induction of general results about recursive functions. Theorem 1 below is one of the most important such results.

6 The Gödel Phenomena

6.1 Some terminological preliminaries

Definition 16 The *syntactic items* of $\mathcal{L}$ are its terms and formulae, and proofs made up out of its sentences. We shall call the set of all such items Σ.

Definition 17 A *Gödel-coding* is an effective, one-one assignment

$$\sharp : \Sigma \rightarrow \mathbb{N}$$

to each syntactic item E in Σ (be it a term, a formula, or a proof) of a unique code number $\sharp E \in \mathbb{N}$.

$\sharp$ is so devised that the *converse* (one-one) assignment

$$\sharp^{-1} : \mathbb{N} \rightarrow \Sigma$$

is also effective, in the following sense:

For each $n \in \mathbb{N}$ the effectively determined output $\sharp^{-1}(n)$ is of one of two forms: either the comment 'n is not the code number of any syntactic item in Σ'; or the syntactic item E such that $\sharp E = n$.[8]

Observation 6 *It follows from our notational conventions that for any syntactic item E, $\underline{\sharp E}$ is the numeral for the code number of E.*

Definition 18 Let $\Psi(x_1, \ldots, x_n)$ be an n-place relation among natural numbers. Let $\psi(x_1, \ldots, x_n)$ be an n-ary formula in the language $\mathcal{L}$ of arithmetic. Let T be any consistent theory in $\mathcal{L}$. We say that in the theory T the formula ψ *numeralwise represents the numerical relation* Ψ just in case the following two implications hold for all natural numbers $k_1, \ldots, k_n$:

if $\Psi(k_1, \ldots, k_n)$, then $T \vdash \psi(\underline{k_1}, \ldots, \underline{k_n})$; and
if not-$\Psi(k_1, \ldots, k_n)$, then $T, \psi(\underline{k_1}, \ldots, \underline{k_n}) \vdash \bot$.

Definition 19 Let $f(x_1, \ldots, x_n)$ be an n-place function whose arguments and values are natural numbers. Let $\varphi(x_1, \ldots, x_n, y)$ be an $(n+1)$-ary formula in the language $\mathcal{L}$ of arithmetic. Let T be any consistent theory in $\mathcal{L}$. We say that in the theory T the formula φ *numeralwise represents the numerical function* f just in case the following holds for all natural numbers $k_1, \ldots, k_n, k$:

if $f(k_1, \ldots, k_n) = k$, then $T \vdash \forall y(\varphi(\underline{k_1}, \ldots, \underline{k_n}, y) \leftrightarrow y = \underline{k})$.

Equivalently:

if $f(k_1, \ldots, k_n) = k$, then $\left\{\begin{array}{l} T, \varphi(\underline{k_1}, \ldots, \underline{k_n}, a) \vdash a = \underline{k} \\ T, a = \underline{k} \vdash \varphi(\underline{k_1}, \ldots, \underline{k_n}, a) \end{array}\right\}$,

where a is parametric.

Observation 7 *Under the assumptions of Definition 19, we have*

$$T \vdash \varphi\big(\underline{k_1}, \ldots, \underline{k_n}, \underline{f(k_1, \ldots, k_n)}\big).$$

Observation 8 *Suppose that in the theory T the formula φ numeralwise represents the numerical function f. Suppose moreover that for any two distinct numbers i, j, we have $T, \underline{i} = \underline{j} \vdash \bot$. Suppose $f(k_1, \ldots, k_n) = k$. Let m be any number distinct from k. It follows that*

$$T, \varphi(\underline{k_1}, \ldots, \underline{k_n}, \underline{m}) \vdash \bot.$$

6.2 Gödel's blessing and Gödel's curse The Gödel phenomena in the theory of arithmetic underscore the stark contrast between *logical* completeness and *theory* completeness. Despite classical first-order logic's soundness:

$$\Delta \vdash \varphi \rightarrow \Delta \models \varphi$$

and completeness:

$$\Delta \models \varphi \rightarrow \Delta \vdash \varphi$$

—the latter known as 'Gödel's blessing'—there is also, alas, 'Gödel's curse'. It consists of his famous First Incompleteness Theorem (G1), and Second Incompleteness Theorem (G2), which can be stated as follows:

For any decidable and sufficiently strong set $\mathcal{A}$ of axioms in the first-order language of arithmetic, (G1) there is a Π^0_1-sentence γ (called 'the Gödel-sentence' for $\mathcal{A}$) such that

$$\mathcal{A} \nvdash \bot \;\rightarrow\; \mathcal{A} \nvdash \gamma$$

(i.e., if $\mathcal{A}$ is consistent, then $\mathcal{A}$ does not prove γ) ,

and

$$\left[\forall \text{ p.r. } \psi (\forall n\, \mathcal{A} \vdash \psi \underline{n} \;\rightarrow\; \mathcal{A}, \forall x \psi \nvdash \bot)\right] \;\rightarrow\; \mathcal{A}, \gamma \nvdash \bot$$

(i.e., if $\mathcal{A}$ is 1-consistent, then $\mathcal{A}$ does not refute γ) ;

and

(G2) where $\mathrm{Con}_{\mathcal{A}}$ is the Π^0_1-sentence codifying the consistency of $\mathcal{A}$:

$$\mathcal{A} \nvdash \bot \;\rightarrow\; \mathcal{A} \nvdash \mathrm{Con}_{\mathcal{A}}\,.$$

We shall call a proof whose premises are in $\mathcal{A}$ an $\mathcal{A}$-proof. The sentence γ of (G1) is the sentence that 'says,' via the Gödel-coding, that there is no $\mathcal{A}$-proof of γ. γ is a Π^0_1-sentence because it is so constructed as to be interdeducible with a sentence of the form

$$\forall x \neg \mathcal{P}_{\mathcal{A}}(x, \underline{\sharp\gamma}),$$

where the primitive recursive predicate $\mathcal{P}_{\mathcal{A}}(x, y)$ numeralwise represents the (effectively decidable) relation 'x is the code number of an $\mathcal{A}$-proof of the sentence with code number y' in that the following two implications hold for all numbers n, m:

if m *does* code an $\mathcal{A}$-proof of the sentence with code number n,
then $\mathcal{A} \vdash \mathcal{P}_{\mathcal{A}}(\underline{m}, \underline{n})$; and
if m *does not* code an $\mathcal{A}$-proof of the sentence with code number n,
then $\mathcal{A}, \mathcal{P}_{\mathcal{A}}(\underline{m}, \underline{n}) \vdash \bot$.

More informally, γ can be understood as 'saying'

'for every natural number n, n is not the code-number of an $\mathcal{A}$-proof of γ.'

Likewise $\mathrm{Con}_{\mathcal{A}}$ is a Π^0_1-sentence because it 'says'

'for every natural number n, n is not the code-number of an $\mathcal{A}$-proof of $\bot$.'

We shall show how to construct such a Gödel-sentence γ in Section 11.3.

7 The Role of Core Logic in the Incompleteness Theorems

Core logic $\mathbb{C}$ is included within intuitionistic logic $\mathbf{I}$ and is distinguished by its eschewal of ex falso quodlibet (EFQ). Likewise, classical core logic $\mathbb{C}^+$ is included within classical logic $\mathbf{C}$ and eschews EFQ. In any logic that contains it, the effect of EFQ is to make it the case that in any language governed by that logic there is only one inconsistent theory—namely, the whole language itself. The main question of interest to the core logician is whether this feature is essential for the derivation of Gödel's celebrated incompleteness theorems. The main question splits into two.

(i) Do we need to have inconsistent theories in the *object language* 'exploding' via EFQ to the whole language, in order to derive (G1) and/or (G2)?
(ii) Do we need to have inconsistent theories in the *metalanguage* 'exploding' via EFQ to the whole language, in order to derive (G1) and/or (G2)?

Our answer to (i) is that all of the formal proofs furnished for the preceding lemmas on the way to the representability theorem have been core proofs. Representability is an entirely constructive result, and core logic suffices for constructive (intuitionistic) mathematics.

Our answer to (ii) will be negative, as we shall see in our subsequent metalinguistic reasoning to obtain (G1) and related results. All such reasoning to follow is clearly constructive and free of any appeals to EFQ—indeed, so obviously so that we shall take the liberty of leaving intact the more informal (but rigorous) reasoning at the metalevel whose strict formalization as core proofs would be but a routine chore.

Recall that we have been discussing any decidable and sufficiently strong set $\mathcal{A}$ of axioms in the first-order language of arithmetic. Gödel's incompleteness theorems concern *classical* theories in the first-order language $\mathcal{L}$ of arithmetic (with 0, s, $+$, and $\times$ as extralogical primitives). This is because the aim is to show that even if one uses the strongest permissible logic—orthodox classical logic **C**—for closure of $\mathcal{A}$ at the object-level, arithmetical incompleteness is still inevitable. Classical core logic $\mathbb{C}^+$, however, could be used in place of **C**, without any loss of deductive reach in the case where $\mathcal{A}$ is *consistent*. In the case where $\mathcal{A}$ is *in*consistent, $\mathbb{C}^+$ will reveal that fact. These claims are justified by the following metatheorems about the core systems.

1. If Δ logically implies φ in classical logic **C**, then in classical core logic $\mathbb{C}^+$ there is a proof either of φ or of $\bot$ whose premises lie in Δ:

$$\Delta \vdash_{\mathbf{C}} \varphi \;\rightarrow\; (\Delta \vdash_{\mathbb{C}^+} \varphi \;\text{ or }\; \Delta \vdash_{\mathbb{C}^+} \bot).$$

2. Every theorem of classical logic **C** is a theorem of classical core logic $\mathbb{C}^+$:

$$\vdash_{\mathbf{C}} \varphi \;\rightarrow\; \vdash_{\mathbb{C}^+} \varphi.$$

3. Every set of sentences that is inconsistent in classical logic **C** is inconsistent in classical core logic $\mathbb{C}^+$:

$$\Delta \vdash_{\mathbf{C}} \bot \;\rightarrow\; \Delta \vdash_{\mathbb{C}^+} \bot.$$

4. If φ is a classical consequence of a classically consistent set Δ of sentences, then φ is deducible in classical core logic $\mathbb{C}^+$ from premises lying in Δ:

$$(\Delta \nvdash_{\mathbf{C}} \bot \;\text{ and }\; \Delta \vdash_{\mathbf{C}} \varphi) \;\rightarrow\; \Delta \vdash_{\mathbb{C}^+} \varphi.$$

Even when *classical* closure of $\mathcal{A}$ (at the object-level) is at issue, the incompleteness theorems can be established *constructively* at the metalevel. Corresponding to the foregoing results concerning ordinary classical logic **C** and classical core logic $\mathbb{C}^+$, we have the following results in the intuitionistic or constructive case. They are of interest here because of their application at the *meta*level in the proof of Gödel's incompleteness theorems.

1′. If Δ logically implies φ in intuitionistic logic **I**, then in core logic $\mathbb{C}$ there is a proof either of φ or of $\bot$ whose premises lie in Δ:

$$\Delta \vdash_{\mathbf{I}} \varphi \;\rightarrow\; (\Delta \vdash_{\mathbb{C}} \varphi \;\text{ or }\; \Delta \vdash_{\mathbb{C}} \bot).$$

2′. Every theorem of intuitionistic logic **I** is a theorem of core logic $\mathbb{C}$:

$$\vdash_{\mathbf{I}} \varphi \;\rightarrow\; \vdash_{\mathbb{C}} \varphi.$$

3′. Every set of sentences that is inconsistent in intuitionistic logic **I** is inconsistent in core logic $\mathbb{C}$:

$$\Delta \vdash_{\mathbf{I}} \bot \;\rightarrow\; \Delta \vdash_{\mathbb{C}} \bot.$$

4′. If φ is an intuitionistic consequence of an intuitionistically consistent set Δ of sentences, then φ is deducible in core logic $\mathbb{C}$ from premises lying in Δ:

$$(\Delta \nvdash_{\mathbf{I}} \bot \;\text{ and }\; \Delta \vdash_{\mathbf{I}} \varphi) \;\rightarrow\; \Delta \vdash_{\mathbb{C}} \varphi.$$

We shall examine here how the proofs of the incompleteness theorems go through when the logic of the object language is classical core logic $\mathbb{C}^+$ and the logic of the metalanguage is core logic $\mathbb{C}$.

Let us be granted the widely shared assumption that our (meta)mathematical axioms are consistent. Then result (4) above tells us that every classical (meta)mathematical result that follows from those axioms can be proved using only classical core logic $\mathbb{C}^+$ at the metalevel. And result (4′)—more attuned to the specifics of the proofs of the incompleteness theorems—tells us that every (meta)mathematical result that follows *constructively* from those axioms can be proved using only core logic $\mathbb{C}$. We therefore have an answer to the second of our two earlier questions:

> Do we need to have inconsistent theories in the *metalanguage* 'exploding' via EFQ to the whole language, in order to derive (G1) and/or (G2)?

The answer to the second question is negative: we do *not* need EFQ in our metalogic in order to derive (G1) and (G2)—for the simple reason that one does not need EFQ in order to derive *any* mathematical theorem from a consistent set of mathematical axioms. If (G1) and (G2) enjoy constructive proofs, then they have core proofs. But if their proofs involve necessary recourse to strictly classical rules of inference (such as classical reductio ad absurdum, or dilemma), then they at least have classical core proofs. As it happens, (G1) and (G2) are constructively provable; hence core logic $\mathbb{C}$ suffices for their proof at the metalevel.

We can therefore focus our inquiry now on the effect at the *object*-level of eschewing EFQ. We use the notation $[\Delta]_{\mathcal{S}}$ for the deductive closure of the set Δ in the logical system $\mathcal{S}$. So

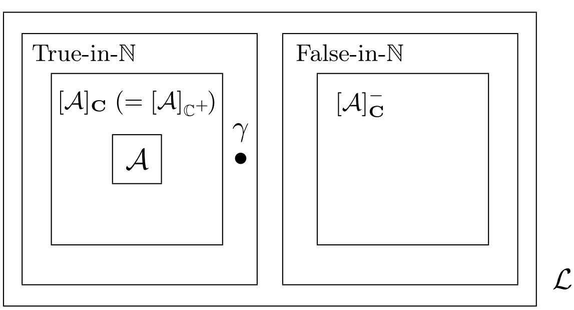
$$[\Delta]_{\mathcal{S}} \;=_{df}\; \{\varphi \in \mathcal{L} \mid \Delta \vdash_{\mathcal{S}} \varphi\}.$$

We shall analogously define

$$[\Delta]^{-}_{\mathcal{S}} \;=_{df}\; \{\varphi \in \mathcal{L} \mid \Delta, \varphi \vdash_{\mathcal{S}} \bot\}.$$

Could Gödel—had he but known of core logic—have shown that the *classical core logical* closure of any decidable, consistent and sufficiently strong set $\mathcal{A}$ of arithmetical axioms is incomplete? And that it cannot contain $\mathrm{Con}_{\mathcal{A}}$?

Looked at one way, the answer is obviously affirmative. The following picture is worth a thousand of anyone's words, including mine. Recall that γ is the sentence (for (G1)) that 'says,' via the Gödel-coding, that there is no $\mathcal{A}$-proof of γ.



If the two boxes labeled True-in-$\mathbb{N}$ and False-in-$\mathbb{N}$ *partition* the language $\mathcal{L}$, then we have the realist's picture. For the realist, the principle of bivalence is true: every sentence of $\mathcal{L}$ is determinately true or false in the structure $\mathbb{N}$ of the natural numbers.

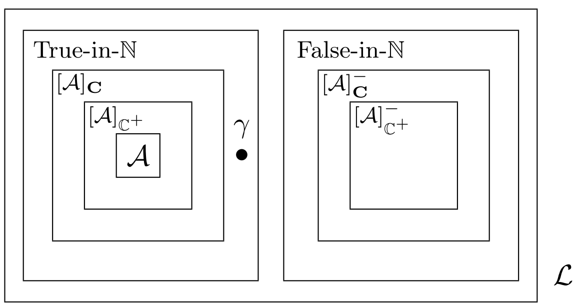
If, however, the two boxes in question fail to partition $\mathcal{L}$, then the situation is somewhat like that just depicted—the sort of picture the moderate antirealist has in mind, even if he has to exercise extreme caution in the way he describes it.[9]

Whichever way we decide the debate between the realist and the antirealist, an important aspect of the picture above remains invariant: the independent Gödel-sentence γ will lie outside the logical closure of $\mathcal{A}$. This will be the case no matter whether we use full classical logic $\mathbf{C}$ or only classical core logic $\mathbb{C}^+$ for such closure. (Remember, this is logical closure within the *object-language*.) And this is owing to the 'overkill' reason that the two closures just mentioned are identical, *on the assumption that* $\mathcal{A}$ *is consistent*. Moreover, the Gödel-sentence γ is *true*-in-$\mathbb{N}$, even though not logically implied by $\mathcal{A}$ (whence its negation $\neg\gamma$ is consistent with $\mathcal{A}$), on the assumption that $\mathcal{A}$ is consistent.

Why call this an 'overkill' reason? Because an even *weaker* reason already suffices for the absence of γ from $[\mathcal{A}]_{\mathbb{C}^+}$, given its known absence (courtesy of Gödel) from $[\mathcal{A}]_{\mathbf{C}}$. This is the simple fact, from inspection of the rules of inference of $\mathbb{C}^+$ and $\mathbf{C}$ respectively, that

$$\vdash_{\mathbb{C}^+} \subseteq \vdash_{\mathbf{C}} .$$

So we could have begun, *a fortiori*, with the picture



in which it is left open whether

$$[\mathcal{A}]^-_{\mathbb{C}^+} = [\mathcal{A}]^-_{\mathbf{C}}.$$

The mere inclusion

$$[\mathcal{A}]^-_{\mathbb{C}^+} \subseteq [\mathcal{A}]^-_{\mathbf{C}}$$

shows that Gödel's reasoning is just as fatal for $[\mathcal{A}]_{\mathbb{C}^+}$ as it was for $[\mathcal{A}]_{\mathbf{C}}$. The sentence γ lies beyond reach from the axioms $\mathcal{A}$, no matter whether it is ordinary classical logic $\mathbf{C}$ that is used for closure, or classical core logic $\mathbb{C}^+$. One could go so far as to say that the Gödel phenomena are so robust that they manifest themselves *despite* result (4) above (to the effect that the classical core closure of Δ is identical to the classical closure of Δ, provided only that Δ is classically consistent).

As is becoming clear, there are two levels at which one needs to consider logical closure: the object-level and the metalevel. In the standard presentation of the Gödel phenomena, one considers the closure of the axioms $\mathcal{A}$ under classical logic $\mathbf{C}$, but uses only *intuitionistic* logic $\mathbf{I}$ at the metalevel. The core logician seeks to emulate this achievement by using only core logic $\mathbb{C}$ at the metalevel. Like the intuitionistic logician, the core logician can reason constructively at the metalevel about classical proofs and classical deducibility *in the object-language*. (Formal proofs are after all finitary, inductively definable objects; one can reason *about* them constructively, even though the object-linguistic reasoning that such proofs formalize is not itself constructive in general.) We emphasize again: the Gödel phenomena are most emphatically established by allowing for full classical closure of the axioms $\mathcal{A}$ at the object-level, and showing (constructively, at the metalevel) that the Gödel sentence γ is *still* unprovable from $\mathcal{A}$.

In this study we are contemplating using either classical logic **C** or classical core logic $\mathbb{C}^+$ at the object-level, and using either core logic $\mathbb{C}$ or classical core logic $\mathbb{C}^+$ at the metalevel. We need a succinct notation for easy reference to various theoretical constellations. We shall use the prefix

$$\left[\frac{\mathcal{S}_1}{\mathcal{S}_2}\right]\text{-}$$

to indicate that we are reasoning at the metalevel using logical system $\mathcal{S}_1$, about a theory at the object-level that is closed under deducibility in logical system $\mathcal{S}_2$. We therefore have in prospect

$$\left[\frac{\mathbb{C}}{\mathbf{C}}\right]\text{-},\ \left[\frac{\mathbb{C}}{\mathbb{C}^+}\right]\text{-},\ \left[\frac{\mathbb{C}^+}{\mathbf{C}}\right]\text{-, and } \left[\frac{\mathbb{C}^+}{\mathbb{C}^+}\right]\text{-type}$$

investigations.

The first—the $\left[\frac{\mathbb{C}}{\mathbf{C}}\right]$-type—would be the most impressive, if it can be carried out. It would be the core logician's emulation of (and improvement upon) the most impressive achievement to date, which, as indicated above, has been a Gödel's investigation of type $\left[\frac{\mathbf{I}}{\mathbf{C}}\right]$. In order to emulate (and improve upon) it, the core logician needs to conduct an investigation of type $\left[\frac{\mathbb{C}}{\mathbf{C}}\right]$, with core logic $\mathbb{C}$ taking over from intuitionistic logic **I** at the metalevel.

The second—the $\left[\frac{\mathbb{C}}{\mathbb{C}^+}\right]$-type—is what one would expect as the Gödelian 'metachallenge' had mathematics, in both its constructive and classical varieties, only ever been formalized according to the core logician's canons of relevance.

The third and fourth types are easily comprehensible variations on the first two, and we shall not consider them in any further detail here. They represent the 'fall-back' approaches available to the core logician, should the Gödelian project elude constructive capture.

But it will not. This study is a completed investigation of type $\left[\frac{\mathbb{C}}{\mathbf{C}}\right]$.

8 Consistency, ω-Consistency, and 1-Consistency

8.1 Definitions of the notions A unary formula is a formula with exactly one free variable; a binary formula is one with exactly two.

We shall use ψ as a sortal variable over unary formulae, and we shall take their sole free variable to be x. We shall write ψt for ψ^x_t, and also sometimes write ψx for ψ to remind the reader of its free variable. We abbreviate "primitive recursive" as 'p.r.'

In the metalanguage we shall be quantifying on some occasions over the natural numbers, using the sortal variable n ($\forall n$, $\exists n$) and on other occasions over formulae, using the sortal variable ψ ($\forall\psi$, $\exists\psi$). Whichever sortal variable α we use, note that the following equivalence (interdeducibility) holds constructively:

$$\neg\exists\alpha\Phi\alpha \dashv\vdash \forall\alpha\neg\Phi\alpha. \tag{i}$$

So too does this one:

$$\neg(P \wedge Q) \dashv\vdash (P \rightarrow \neg Q). \tag{ii}$$

Noting these constructive interdeducibilities will make it easier to appreciate the claims of constructive equivalence (in the metatheory) to follow.

Definition 20 T is ω-inconsistent $\equiv_{\mathrm{df}}$ $\exists\psi(\forall n\, T \vdash \psi\underline{n} \wedge T \vdash \exists x \neg\psi x)$.

Definition 21 T is 1-inconsistent $\equiv_{\mathrm{df}}$ $\exists$ p.r. $\psi(\forall n\, T \vdash \psi\underline{n} \wedge T \vdash \exists x \neg\psi x)$.

Definition 22 T is inconsistent $\equiv_{\mathrm{df}}$ $T \vdash \bot$.

Corresponding to each of these notions of inconsistency is a notion of *consistency*—its straightforward contradictory. Thus we have

Definition 23 T is ω-consistent $\equiv_{\mathrm{df}}$ $\neg\exists\psi(\forall n\, T \vdash \psi\underline{n} \wedge T \vdash \exists x \neg\psi x)$.

Definition 24 T is 1-consistent $\equiv_{\mathrm{df}}$ $\neg\exists$ p.r. $\psi(\forall n\, T \vdash \psi\underline{n} \wedge T \vdash \exists x \neg\psi x)$.

Definition 25 T is consistent $\equiv_{\mathrm{df}}$ $T \nvdash \bot$.

Observation 9 *If T is ω-consistent, then T is 1-consistent.*

Now let us look at some (meta)logically equivalent ways of framing the notions of ω-consistency and 1-consistency. Of particular interest are variations that are constructively equivalent to the 'master' Definitions 23 and 24.

8.2 Equivalent formulations of ω-consistency The *definiens* is constructively equivalent to

$$\forall\psi(\forall n\, T \vdash \psi\underline{n} \rightarrow T \nvdash \exists x \neg\psi x).$$

Moreover, on the assumption that the embedded turnstile $\vdash$ is classical, this in turn is constructively equivalent both to

$$\forall\psi(\forall n\, T \vdash \psi\underline{n} \rightarrow T, \forall x \psi x \nvdash \bot).$$

and to

$$\forall\psi(\forall n\, T \vdash \psi\underline{n} \rightarrow T \nvdash \neg\forall x \psi x)$$

8.3 Equivalent formulations of 1-consistency The observations of Section 8.2 carry over to yield similar metalinguistic constructive equivalents of the definiens in the definition of 1-consistency. All that is different in this case is the restriction to p.r. formulae. But this restriction allows us to draw on the p.r. equivalences noted above. Consequently, the following are just some of the constructive equivalents of the *definiens*

$$\neg\exists \text{ p.r. } \psi(\forall n\, T \vdash \psi\underline{n} \wedge T \vdash \exists x \neg\psi x)$$

in Definition 24:

$$\forall \text{ p.r. } \psi(T \vdash \exists x \psi x \rightarrow \exists n\, T \vdash \psi\underline{n}),$$
$$\forall \text{ p.r. } \psi(\neg\exists n\, T \vdash \psi\underline{n} \rightarrow T \nvdash \exists x \psi x),$$
$$\forall \text{ p.r. } \psi(\forall n\, T \vdash \psi\underline{n} \rightarrow T \nvdash \neg\forall x \psi x),$$
$$\forall \text{ p.r. } \psi(\forall n\, T \nvdash \psi\underline{n} \rightarrow T, \neg\exists x \psi x \nvdash \bot),$$
$$\forall \text{ p.r. } \psi(\forall n\, T \vdash \neg\psi\underline{n} \rightarrow T, \forall x \neg\psi x \nvdash \bot),$$
$$\forall \text{ p.r. } \psi(\forall n\, T \vdash \neg\psi\underline{n} \rightarrow T \nvdash \neg\forall x \neg\psi x),$$
$$\forall \text{ p.r. } \psi(\forall n\, T \vdash \neg\psi\underline{n} \rightarrow T \nvdash \exists x \psi x).$$

Given that these are all (constructively) equivalent, one may pick and choose from among them according to one's convenience in order to meet certain metatheoretical goals.

9 Do the Proofs of the Incompleteness Theorems Require the Use of Ex Falso Quodlibet?

Our main concern in this study is to show that the core logician is in a position to prove the incompleteness theorems. That is, the core logician can carry out the metalogical reasoning that shows that even the use of full classical logic **C** (*with* EFQ) at the object-linguistic level fails to deliver all the first-order truths of arithmetic, provided only that the set of axioms used is consistent, sufficiently strong, and effectively decidable.

One quick and easy way to show this is simply to adduce the metatheorem that assures us that all classical consequences of a consistent set of sentences can be derived from it using only classical core logic $\mathbb{C}^+$. In the constructive case, likewise, we have the metatheorem that all constructive consequences of a consistent set of sentences can be derived from it using only core logic $\mathbb{C}$.

Opponents for whom the matter is not so directly disposed of, however, will try to raise difficulties for the core logician by making out that there are to be found, in the well-known proofs of the incompleteness theorems, what from their point of view appear to be indispensable appeals to EFQ. These appeals are either explicit, or, if merely implicit, nevertheless obvious. I agree that such appeals can be found; the crucial question for us to consider, however, is whether they are *indispensable*. They might have been made by formal logicians well versed in a tradition that has reconciled its followers to the use of EFQ, but which has failed to instruct them as to how its use can be avoided altogether. It is also important to be clear about whether the appeals to EFQ take place *at the metalogical level* or whether it is a matter of EFQ only being taken to be part of the logic of the object-language.

Consider, for example, how the exposition runs in Kleene [6], who is working within a framework of type $\left[\frac{\mathbf{I}}{\mathbf{C}}\right]$, in the passage of reasoning that shows that every ω-consistent theory is (straightforwardly) consistent. He is working with the definiens

$$\forall\psi(\forall n\, T \vdash \psi\underline{n} \rightarrow T \nvdash \neg\forall x\,\psi x)$$

for ω-consistency of T. With slight changes made in the interests of uniformity of notation, here is Kleene's reasoning ([6, p. 207]):

> ... ω-consistency implies simple consistency. For if φ be any provable formula containing no free variables, writing it as "$\varphi(x)$" where x is a variable, all of $\varphi(0), \varphi(1), \varphi(2), \ldots$ are provable (under our substitution notation ... each of these is simply φ itself); and hence if the system is ω-consistent, $\neg\forall x\varphi(x)$ is an example of an unprovable formula ... .

Kleene is working here on the background assumption that if T were *in*consistent, then we would have, courtesy of EFQ, that T proved $\neg\forall x\varphi(x)$, and this would complete his intended *reductio* of the assumption that T is simply inconsistent.

Now there are those (like the present author) who disapprove of a grammatical convention that allows a quantifier-prefix $\forall x$ to be appended to a formula with no free occurrences of the variable x. In order to mollify them, Kleene could instead have written as follows:

> ... ω-consistency implies simple consistency. For if θ be any theorem and $\xi(x)$ any unary formula, all of
>
> $$\theta\vee\xi(0),\ \theta\vee\xi(1),\ \theta\vee\xi(2),\ldots$$
>
> are provable, and hence if the system is ω-consistent,
>
> $$\neg\forall x\big(\theta\vee\xi(x)\big)$$
>
> is an example of an unprovable formula ... .

As already pointed out, Kleene is working with the formulation of the ω-consistency of T that runs

$$\forall\psi(\forall n\, T\vdash\psi\underline{n}\rightarrow T\nvdash\neg\forall x\psi x).$$

The reasoning on my suggested rewrite using the formula $\theta\vee\xi(x)$ is consummated as before, of course, by appeal to the logical fact (for the intuitionist, and hence also for the classicist) that *if* the theory T in question proves $\perp$, *then* (by EFQ) T proves *any* sentence, including $\neg\forall x(\theta\vee\xi(x))$. But this contradicts the earlier conclusion that $\neg\forall x(\theta\vee\xi(x))$ is unprovable in T. So the theory T does *not* prove $\perp$.

In this (reformulated) passage of reasoning from Kleene, it is clear that the tacit appeal to EFQ is a matter only of EFQ's being taken to be part of the logic of the object-language; Kleene is not himself applying EFQ in his metalogical reasoning.

The question now arises: is this reasoning, with its explicit appeal to EFQ at the object-level ('within' the theory T) able to sustain the methodological claim that EFQ is indispensable *within the logic for theory-closure at the object-level*? (Indispensable, that is, for the sought proof that a theory's ω-consistency implies its simple consistency.) How might Kleene have been able to reason had he been confined to a framework of type $\left[\frac{\mathbb{C}}{\mathbb{C}+}\right]$ rather than $\left[\frac{\mathbf{I}}{\mathbf{C}}\right]$? If T is closed only under the nonexplosive $\mathbb{C}^+$ rather than under $\mathbf{C}$, won't this block the Kleene-style reasoning that has just been expounded?

The answer is affirmative, but there is a surprisingly easy work-around when confronted with this blockage. It involves a subtle but reasonable reconceptualization of the single turnstile $\vdash$. The usual reading of $\Delta\vdash_{\mathcal{S}}\varphi$ is the following:

> there is a proof (in the system $\mathcal{S}$) whose conclusion is φ and whose undischarged assumptions are in Δ.

This allows for the possibility of proofs that do not appeal to every member of Δ. Indeed, this would have to be the case whenever Δ is infinite. So, putting it another way, the usual reading of $\Delta\vdash_{\mathcal{S}}\varphi$ is as follows:

> there is a proof Π (in the system $\mathcal{S}$) such that:
> (i) the conclusion of Π is φ; and
> (ii) $p\Pi$ (the set of premises of Π) is a subset of Δ.

Let us undertake just one more round of reformulation. Let us talk of the *sequent* $s\Pi$ that is established by a proof Π. $s\Pi$ will take the form $\Gamma:\Phi$, where Γ is a finite set of sentences (the set of premises of the proof), and Φ is at most a singleton. If Φ is empty, then Π establishes the conclusion $\perp$ from the premises Γ, and if Φ is $\{\varphi\}$, then Π establishes the conclusion φ from the premises Γ.

We now define the *subsequent* relation as follows.

Definition 26 $\quad\Delta_1:\Theta_1\sqsubseteq\Delta_2:\Theta_2\ \equiv_{\mathrm{df}}\ \Delta_1\subseteq\Delta_2$ and $\Theta_1\subseteq\Theta_2$.

Now, when we make a 'core deducibility' claim (for Θ at most a singleton) that

$$\Delta\vdash_{\mathcal{S}}\Theta$$

(note the reddening—the reddened and subscripted single turnstile is not the same as the equiform one in black!) this is to be understood as follows:

some $\mathcal{S}$-proof proves some *subsequent of* $\Delta : \Theta$.

More succinctly still:

there is an $\mathcal{S}$-proof Π such that $s\Pi \sqsubseteq \Delta:\Theta$.

Logicians have, after all, been happy to take 'subsetting down on the left' in their stride when framing deducibility statements. They require of proofs that witness their deducibility claims only that their premises be members of the antecedent Δ, not that they collectively exhaust Δ. But there is absolutely no reason not to allow for the same on the *right*, with the succedent—*insofar as deducibility* is concerned. And let us further stress: we are venturing to use the reddened turnstile, temporarily, *only on Kleene's behalf*, so as to make his reasoning accessible to the core logician. We do not propose that the reading of the reddened turnstile should henceforth be replacing the established reading of the black turnstile, according to which $\Delta \vdash_{\mathcal{S}} \varphi$ if and only if there is an $\mathcal{S}$-proof whose conclusion is φ and whose premises are drawn from Δ.

If we do allow the proposed venture into the use of the reddened turnstile, then we can revisit the reasoning in which Kleene engaged in his attempt to show that ω-consistency implies simple consistency. That reasoning goes through with only minor adjustment for the worker in the framework of type $\left[\frac{\mathbb{C}}{\mathbb{C}+}\right]$. The formulation of ω-consistency becomes

$$\forall\psi(\forall n\, T \vdash_{\mathbb{C}+} \psi\underline{n} \;\rightarrow\; T \nvdash_{\mathbb{C}+} \neg\forall x\,\psi x),$$

and then the reasoning goes through crisply as follows:

> Suppose for (constructive) *reductio* that $T \vdash_{\mathbb{C}+} \bot$. Suppose also that T is ω-consistent, that is, that
>
> $$\forall\psi(\forall n\, T \vdash_{\mathbb{C}+} \psi\underline{n} \;\rightarrow\; T \nvdash_{\mathbb{C}+} \neg\forall x\,\psi x).$$
>
> Let χ be any unary formula. It follows that
>
> $$\forall n\, T \vdash_{\mathbb{C}+} \chi\underline{n} \;\rightarrow\; T \nvdash_{\mathbb{C}+} \neg\forall x\,\chi x.$$
>
> Let m be any number. Since $T \vdash_{\mathbb{C}+} \bot$, we have
>
> $$T \vdash_{\mathbb{C}+} \chi\underline{m}\,.$$
>
> But m is arbitrary. It follows that
>
> $$\forall n\, T \vdash_{\mathbb{C}+} \chi\underline{n}\,.$$
>
> Detaching, we infer that
>
> $$T \nvdash_{\mathbb{C}+} \neg\forall x\,\chi x\,.$$
>
> This contradicts $T \vdash_{\mathbb{C}+} \bot$.

Note: *there has been no appeal to EFQ as a rule of inference in the logic of the object language*. We *have*, to be sure, employed the metalogical form of inference

$$\frac{\Delta \vdash_{\mathbb{C}+} \bot}{\Delta \vdash_{\mathbb{C}+} \varphi}.$$

And this form of inference is an immediate consequence of our proposed definition of $\vdash_{\mathbb{C}+}$. It *looks* like THINNING, but it is not. Its metalogical validity follows immediately from the definition of the reddened turnstile $\vdash_{\mathbb{C}+}$. This inference amounts (roughly: see below) to the metalinguistic claim that THINNING is (trivially) *admissible* for the system $\mathbb{C}^+$. It is *not* the claim (nor does it justify the claim) that THINNING is *derivable in*, or is a primitive proof-forming *rule of*, the system $\mathbb{C}^+$, as a means of forming (sequent) proofs. The latter rule would be stated as

$$\frac{\Delta : \perp}{\Delta : \varphi}.$$

There is a world of difference between having this rule as a rule *of* or *in* one's system, and merely having it be *admissible for* one's system.

What we have said here about THINNING holds equally for CUT. The metalinguistic inference

$$\frac{\Delta \vdash_{\mathbb{C}+} \varphi \qquad \Gamma, \varphi \vdash_{\mathbb{C}+} \psi}{\Delta, \Gamma \vdash_{\mathbb{C}+} \psi}$$

is valid on our proposed definition of $\vdash_{\mathbb{C}+}$, but all it states is the *admissibility* of CUT for the system $\mathbb{C}^+$. This is a far cry from having an applicable *rule* of CUT

$$\frac{\Delta : \varphi \qquad \Gamma, \varphi : \psi}{\Delta, \Gamma : \psi}$$

in the system, as a means of forming (sequent) proofs.

The validity of the metalinguistic inference

$$\frac{\Delta \vdash_{\mathbb{C}+} \varphi \qquad \Gamma, \varphi \vdash_{\mathbb{C}+} \psi}{\Delta, \Gamma \vdash_{\mathbb{C}+} \psi}$$

is an easy corollary of the main metatheorem of Tennant [18], which states

> There is an effective method [,] that transforms any two proofs
>
> $$\begin{array}{ccc} \Delta & A, \Gamma & \\ \Pi & \Sigma & \text{(where } A \notin \Gamma \text{ and } \Gamma \text{ may be empty)} \\ A & \theta & \end{array}$$
>
> in Classical Core Logic into a proof $[\Pi, \Sigma]$ in Classical Core Logic *of* θ *or of* $\perp$ from (some subset of) $\Delta \cup \Gamma$.

Here, again, one could say in light of this metatheorem that CUT is *admissible* for classical core logic, even though it is not derivable (and, also, is not a rule *in* the system). This talk of admissibility in the case of THINNING and in the case of CUT appeals, to be sure, to a slightly revised conception of a rule's admissibility. As an anonymous referee pointed out:

> Typically, a one-premise rule is said to be admissible in some sequent calculus $\mathcal{S}$ if, and only if, whenever there is a derivation of the premise-sequent in $\mathcal{S}$, there is also a derivation of the conclusion-sequent in $\mathcal{S}$.

The conception of admissibility that we have deployed in the foregoing is captured by the following slight revision (which we give for the general case n rather than 1, since CUT involves *two* premise-sequents):

> An n-premise rule is said to be admissible in some sequent calculus $\mathcal{S}$ if, and only if, whenever there are derivations in $\mathcal{S}$ of the n premise-sequents, there is also a derivation in $\mathcal{S}$ of *some subsequent of* the conclusion-sequent.

10 Representability of Recursive Functions

Lemma 13 *In* Q*, the formula* $x + y = z$ *represents the basic function of addition.*

Proof By Definition 19, we need to show the following:

if $m+n=k$, then $\left\{\begin{array}{l}\mathsf{Q}, \underline{m}+\underline{n}=a \vdash a=\underline{k} \\ \mathsf{Q}, a=\underline{k} \vdash \underline{m}+\underline{n}=a\end{array}\right\}$,
where a is parametric.

So suppose $m+n=k$. By Lemma 11, we have[10]

$$\mathsf{Q} \vdash \underline{m}+\underline{n}=\underline{k}.$$

This means there is a Q-proof Π of the form

$$\begin{array}{c}\mathsf{Q}\\ \Pi\\ \underline{m}+\underline{n}=\underline{k}\end{array}$$

Its extension

$$\dfrac{\begin{array}{c}\mathsf{Q}\\ \Pi\\ \underline{m}+\underline{n}=\underline{k}\end{array} \quad \underline{m}+\underline{n}=a}{a=\underline{k}}$$

shows that

$$\mathsf{Q}, \underline{m}+\underline{n}=a \vdash a=\underline{k}.$$

Likewise, its extension

$$\dfrac{\begin{array}{c}\mathsf{Q}\\ \Pi\\ \underline{m}+\underline{n}=\underline{k}\end{array} \quad a=\underline{k}}{\underline{m}+\underline{n}=a}$$

shows that

$$\mathsf{Q}, a=\underline{k} \vdash \underline{m}+\underline{n}=a. \qquad \square$$

Lemma 14 *The formula $x \times y=z$ represents the basic function of multiplication.*

Proof This is like the proof of Lemma 13, but appealing to Lemma 12. □

Lemma 15 *For $1 \leq k \leq m$, the formula $x_1=x_1 \wedge \ldots \wedge x_m=x_m \wedge y=x_k$ represents the projection function id_k^m, which is basic.*

Proof The action of the projection function, for given $1 \leq k \leq m$, is

$$id_k^m(n_1, \ldots, n_m) = n_k.$$

By Definition 19, we need to show the following:

if $id_k^m(n_1, \ldots, n_m) = n_k$,

then $\left\{\begin{array}{l}\mathsf{Q}, \underline{n_1}=\underline{n_1} \wedge \ldots \wedge \underline{n_m}=\underline{n_m} \wedge a=\underline{n_k} \vdash a=\underline{n_k} \\ \mathsf{Q}, a=\underline{n_k} \vdash \underline{n_1}=\underline{n_1} \wedge \ldots \wedge \underline{n_m}=\underline{n_m} \wedge a=\underline{n_k}\end{array}\right\}$,

where a is parametric.

These deducibilities are trivial, given reflexivity of identity and the rules of introduction and elimination for conjunction. □

Lemma 16 *The formula $(x_1=x_2 \wedge y=1) \vee (\neg x_1=x_2 \wedge y=0)$ represents the characteristic function $c_=$ of identity, which is basic.*

Proof The characteristic function $c_{=}(m,n)$ of identity is given by

$$c_{=}(m,n) = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

By Definition 19, we need to show the following:

$$\text{if } c_{=}(k_1,k_2)=k, \text{ then } \left\{ \begin{array}{l} \mathsf{Q}, (\underline{k_1}=\underline{k_2}\wedge a=1) \vee (\neg\underline{k_1}=\underline{k_2}\wedge a=0) \vdash a=\underline{k} \\ \mathsf{Q}, a=\underline{k} \vdash (\underline{k_1}=\underline{k_2}\wedge a=1) \vee (\neg\underline{k_1}=\underline{k_2}\wedge a=0) \end{array} \right\},$$

where a is parametric.

There are two mutually exclusive and collectively exhaustive cases to consider:

1. $k_1 = k_2$ and $k = 1$;
2. $k_1 \neq k_2$ and $k = 0$.

In Case 1, we have $k_1 = k_2$, so we have, as an instance of reflexivity of identity,

$$\overline{\underline{k_1}=\underline{k_2}}$$

In Case 1, we also have $k = 1$, so we likewise have

$$\overline{\underline{k}=1}$$

The two required deducibilities in Case 1 are accordingly established by the following proofs. (Note that they do not make use of any axioms of Q.)

$$\begin{prooftree}
\AxiomC{$(\underline{k_1}=\underline{k_2}\wedge a=1)\vee(\neg\underline{k_1}=\underline{k_2}\wedge a=0)$}
\AxiomC{}
\RightLabel{(1)}
\UnaryInfC{$\underline{k_1}=\underline{k_2}\wedge a=1$}
\UnaryInfC{$a=1$}
\AxiomC{}
\RightLabel{(1)}
\UnaryInfC{$\neg\underline{k_1}=\underline{k_2}\wedge a=0$}
\UnaryInfC{$\neg\underline{k_1}=\underline{k_2}$}
\AxiomC{}
\UnaryInfC{$\underline{k_1}=\underline{k_2}$}
\BinaryInfC{$\bot$}
\RightLabel{(1)}
\TrinaryInfC{$a=1$}
\AxiomC{}
\UnaryInfC{$\underline{k}=1$}
\BinaryInfC{$a=\underline{k}$}
\end{prooftree}$$

$$\begin{prooftree}
\AxiomC{}
\UnaryInfC{$\underline{k_1}=\underline{k_2}$}
\AxiomC{$a=\underline{k}$}
\AxiomC{}
\UnaryInfC{$\underline{k}=1$}
\BinaryInfC{$a=1$}
\BinaryInfC{$\underline{k_1}=\underline{k_2}\wedge a=1$}
\UnaryInfC{$(\underline{k_1}=\underline{k_2}\wedge a=1)\vee(\neg\underline{k_1}=\underline{k_2}\wedge a=0)$}
\end{prooftree}$$

In Case 2, we have $k_1 \neq k_2$, hence (by Lemma 1) a proof within Q of the form

$$\begin{array}{c} \underbrace{\mathsf{Q}, \underline{k_1}=\underline{k_2}} \\ \vdots \\ \bot \end{array}$$

In Case 2, we also have $k = 0$. Therefore we have, as an instance of reflexivity of identity,

$$\overline{\underline{k}=0}$$

The two required deducibilities modulo Q are accordingly established in Case 2 by the following proofs:

$$\dfrac{\dfrac{(\underline{k_1}=\underline{k_2}\wedge a=1)\vee(\neg\underline{k_1}=\underline{k_2}\wedge a=0) \qquad \begin{matrix}\underbrace{\mathsf{Q}\ ,\ \dfrac{\dfrac{}{\underline{k_1}=\underline{k_2}\wedge a=1}(1)}{\underline{k_1}=\underline{k_2}}} \\ \vdots \\ \bot\end{matrix} \qquad \dfrac{\dfrac{}{\neg\underline{k_1}=\underline{k_2}\wedge a=0}(1)}{a=0}}{a=0}(1) \qquad \dfrac{}{\underline{k}=0}}{a=\underline{k}}$$

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\underbrace{\mathsf{Q}\ ,\ \dfrac{}{\underline{k_1}=\underline{k_2}}(1)} \\ \vdots \\ \bot\end{matrix}}{\neg\underline{k_1}=\underline{k_2}}(1) \qquad \dfrac{a=\underline{k} \qquad \dfrac{}{\underline{k}=0}}{a=0}}{\neg\underline{k_1}=\underline{k_2}\wedge a=0}}{(\underline{k_1}=\underline{k_2}\wedge a=1)\vee(\neg\underline{k_1}=\underline{k_2}\wedge a=0)}$$

□

Lemma 17 *Suppose*

the formula $\psi_1(x_1,\ldots,x_{k+1})$ *represents the* k*-place function* $g_1(n_1,\ldots,n_k)$;

$\vdots$

the formula $\psi_m(x_1,\ldots,x_{k+1})$ *represents the* k*-place function* $g_m(n_1,\ldots,n_k)$;
the formula $\varphi(x_1,\ldots,x_{m+1})$ *represents the* m*-place function* $f(n_1,\ldots,n_m)$.

Then the formula

$$\exists y_1\ldots\exists y_m\big(\wedge_{i=1}^{m}\psi_i(\overrightarrow{x},y_i)\wedge\varphi(y_1,\ldots,y_m,y)\big)$$

represents the composed function

$$f\big(g_1(x_1,\ldots,x_k),\ldots,g_m(x_1,\ldots,x_k)\big).$$

Proof

$$\dfrac{\exists y_1\ldots\exists y_m(\wedge_{i=1}^{m}\psi_i(\underline{\overrightarrow{n}},y_i)\wedge\varphi(y_1,\ldots,y_m,a)) \qquad \begin{matrix}\dfrac{\dfrac{\dfrac{}{\wedge_{i=1}^{m}\psi_i(\underline{\overrightarrow{n}},b_i)\wedge\varphi(b_1,\ldots,b_m,a)}(1)}{\varphi(b_1,\ldots,b_m,a)} \qquad \left\{\begin{matrix}\dfrac{\dfrac{}{\wedge_{i=1}^{m}\psi_i(\underline{\overrightarrow{n}},b_i)\wedge\varphi(b_1,\ldots,b_m,a)}(1)}{\psi(\underline{\overrightarrow{n}},b_j)} \\ \vdots \\ \text{(since } \psi_j \text{ represents } g_j) \\ \vdots \\ b_i=\underline{g_j(\overrightarrow{n})}\end{matrix}\right\}_{j=1,\ldots,m}}{\varphi(\underline{g_1(\overrightarrow{n})},\ldots,\underline{g_m(\overrightarrow{n})},a)} \\ \vdots \\ \text{(since } \varphi \text{ represents } f) \\ \vdots \\ a=\underline{f(g_1(\overrightarrow{n}),\ldots,g_m(\overrightarrow{n}))}\end{matrix}}{a=\underline{f(g_1(\overrightarrow{n}),\ldots,g_m(\overrightarrow{n}))}}(1)$$

The step marked (1) is an m-fold existential elimination, using the parameters $b_1,\ldots,b_m$.

$$
\dfrac{\dfrac{\begin{array}{ccc} \begin{array}{c}\text{(since } \psi_1 \\ \text{represents } g_1)\\ \vdots \\ \psi_1(\underline{\vec{n}}, \underline{g_1(\vec{n})})\end{array} & \begin{array}{c}\text{(since } \psi_m \\ \text{represents } g_m)\\ \vdots \\ \psi_m(\underline{\vec{n}}, \underline{g_m(\vec{n})})\end{array} & \begin{array}{c} a=\underline{f(g_1(\vec{n}),\ldots,g_m(\vec{n}))} \\ \vdots \\ \text{(since } \varphi \\ \text{represents } f)\\ \vdots \\ \varphi(\underline{g_1(\vec{n})},\ldots,\underline{g_m(\vec{n})},a)\end{array}\end{array}}{\wedge_{i=1}^{m}\psi_i\big(\underline{\vec{n}}, \underline{g_i(\vec{n})}\big) \wedge \varphi\big(\underline{g_1(\vec{n})},\ldots,\underline{g_m(\vec{n})},a\big)}}{\exists y_1 \ldots \exists y_m\big(\wedge_{i=1}^{m}\psi_i(\underline{\vec{n}}, y_i) \wedge \varphi(y_1,\ldots,y_m,a)\big)} \qquad \square
$$

Lemma 18 *Suppose*

the $(n+1)$*-place function* f *is regular ;*
the formula $\varphi(\vec{x}, y, z)$ *represents* f *in* Q;
the n*-place function* g *is obtained by minimization of* f *—*
that is, for all $m_1, \ldots, m_n$, $g(\vec{m}) = \lambda p[f(\vec{m}, p)=0]$.

Then g *is represented in* Q *by the formula*

$$\varphi(\vec{x}, y, 0) \wedge \forall z \forall w\big(sw+z=y \rightarrow \neg\varphi(\vec{x}, z, 0)\big).$$

Note that this last formula can be read as

$$\varphi(\vec{x}, y, 0) \wedge \forall z\big(z<y \rightarrow \neg\varphi(\vec{x}, z, 0)\big).$$

Proof Since $f(\underline{\vec{m}}, \underline{g(\vec{m})})=0$, and $\varphi(\vec{x}, y, z)$ represents f in Q, there exists (recall Observation 7) a Q-proof

$$\begin{array}{c}\Sigma \\ \varphi(\underline{\vec{m}}, \underline{g(\vec{m})}, 0)\end{array}$$

Recall Lemma 3, with a change of schematic letter:

$$\forall n>0 \;\; \mathsf{Q}, \big\{\psi(\underline{k}) \mid k < n\big\}, sb+a=\underline{n} \vdash_{\mathbb{C}} \psi(a).$$

This tells us that there is a (core) Q-proof of the form

$$\begin{array}{c}\underbrace{\psi(0), \psi(1), \ldots, \psi(\underline{n-1}), sb+a=\underline{n}} \\ \vdots \\ \psi(a)\end{array}$$

For $\psi(x)$, we substitute $\neg\varphi(\underline{\vec{m}}, x, 0)$. We now have a (core) Q-proof of the form

$$\begin{array}{c}\underbrace{\neg\varphi(\underline{\vec{m}}, 0, 0), \neg\varphi(\underline{\vec{m}}, 1, 0), \ldots, \neg\varphi(\underline{\vec{m}}, \underline{n-1}, 0), sb+a=\underline{n}} \\ \vdots \\ \neg\varphi(\underline{\vec{m}}, a, 0)\end{array}$$

Bear in mind that $g(\vec{m})$ is the *least* number p such that $f(\vec{m}, p) = 0$. So for all $i < g(\vec{m})$, we have $f(\vec{m}, i) \neq 0$; whence again by representability (recall Lemma 1 and Observation 8) there is, for each $i < g(\vec{m})$, a Q-proof

$$\Xi^i:\quad \dfrac{\begin{array}{c}\overline{\varphi(\underline{\vec{m}},\underline{i},0)}\,{}^{(1)}\\ \vdots\\ \bot\end{array}}{\neg\varphi(\underline{\vec{m}},\underline{i},0)}\,{\scriptstyle (1)}$$

That means we can now assemble a (core) Q-proof

$$\begin{array}{c}\underbrace{\begin{array}{c}\Xi^0\\ \neg\varphi(\underline{\vec{m}},0,0)\end{array}\ ,\ \begin{array}{c}\Xi^1\\ \neg\varphi(\underline{\vec{m}},1,0)\end{array}\ ,\ \ldots,\ \begin{array}{c}\Xi^{n-1}\\ \neg\varphi(\underline{\vec{m}},\underline{n-1},0)\end{array}\ ,\ sb+a=\underline{n}}\\ \vdots\\ \neg\varphi(\underline{\vec{m}},a,0)\end{array}$$

which we shall abbreviate as

$$\begin{array}{c}sb+a=\underline{n}\\ \Xi\\ \neg\varphi(\underline{\vec{m}},a,0)\end{array}$$

In order to establish that g is represented in Q by the formula

$$\varphi(\vec{x},y,0)\wedge\forall z\forall w\big(sw+z=y\to\neg\varphi(\vec{x},z,0)\big),$$

we need two Q-proofs, one for each direction involved. We embed Σ and Ξ as subproofs in order to obtain them:

$$\dfrac{\dfrac{\begin{array}{c}\Sigma\\ \varphi(\underline{\vec{m}},\underline{g(\vec{m})},0)\end{array}\quad \dfrac{\dfrac{\dfrac{\begin{array}{c}\overline{sb+a=\underline{g(\vec{m})}}\,{}^{(1)}\\ \Xi\\ \neg\varphi(\underline{\vec{m}},a,0)\end{array}}{sb+a=\underline{g(\vec{m})}\to\neg\varphi(\underline{\vec{m}},a,0)}\,{\scriptstyle (1)}}{\forall w\big(sw+a=\underline{g(\vec{m})}\to\neg\varphi(\underline{\vec{m}},a,0)\big)}}{\forall z\forall w\big(sw+z=\underline{g(\vec{m})}\to\neg\varphi(\underline{\vec{m}},z,0)\big)}}{\varphi(\underline{\vec{m}},\underline{g(\vec{m})},0)\wedge\forall z\forall w\big(sw+z=\underline{g(\vec{m})}\to\neg\varphi(\underline{\vec{m}},z,0)\big)}\qquad a=\underline{g(\vec{m})}}{\varphi(\underline{\vec{m}},a,0)\wedge\forall z\forall w\big(sw+z=a\to\neg\varphi(\underline{\vec{m}},z,0)\big)}$$

$$\dfrac{\dfrac{\begin{array}{c}\Sigma\\ \varphi(\underline{\vec{m}},\underline{g(\vec{m})},0)\end{array}\quad \dfrac{{}^{(1)}\overline{sb+\underline{g(\vec{m})}=a}\quad \dfrac{\dfrac{\dfrac{\varphi(\underline{\vec{m}},a,0)\wedge\forall z\forall w(sw+z=a\to\neg\varphi(\underline{\vec{m}},z,0))}{\forall z\forall w(sw+z=a\to\neg\varphi(\underline{\vec{m}},z,0))}}{\forall w(sw+\underline{g(\vec{m})}=a\to\neg\varphi(\underline{\vec{m}},\underline{g(\vec{m})},0))}}{sb+\underline{g(\vec{m})}=a\to\neg\varphi(\underline{\vec{m}},\underline{g(\vec{m})},0)}}{\neg\varphi(\underline{\vec{m}},\underline{g(\vec{m})},0)}}{\bot}\quad \dfrac{\dfrac{\varphi(\underline{\vec{m}},a,0)\wedge\forall z\forall w(sw+z=a\to\neg\varphi(\underline{\vec{m}},z,0))}{\varphi(\underline{\vec{m}},a,0)}\quad \begin{array}{c}{}^{(1)}\overline{sb+a=\underline{g(\vec{m})}}\\ \Xi\\ \neg\varphi(\underline{\vec{m}},a,0)\end{array}}{\bot}}{a=\underline{g(\vec{m})}}\,{\scriptstyle (1)}\ \ \mathrm{L10}$$

□

Theorem 1 (Representability of recursive functions in Q) *For every recursive function $f(x_1,\ldots,x_n)$—call it $f(\vec{x})$ for short—there is a formula $\varphi(\vec{x},y)$ in $\mathcal{L}$ such that for all $k_1,\ldots,k_n$ the following two inferences hold modulo* Q*:*

$$\dfrac{\varphi(\underline{\vec{k}},a)}{a=\underline{f(\vec{k})}},\ \textit{and its converse}\ \dfrac{a=\underline{f(\vec{k})}}{\varphi(\underline{\vec{k}},a)},$$

where the parameter a is chosen so as to occur only in the two positions displayed.

Proof The result follows by induction on the pedigree of recursive functions. The basis cases concerning the basic functions are covered by Lemmas 13, 14, and 15, and the operations for forming new recursive functions from old ones—namely composition and minimization of regular functions—are covered by Lemmas 17 and 18. □

11 Representing Theories

11.1 Some general results about effective methods In this section, we shall continue to deploy the *informal* or *preformal* notions of effective procedure and of computable or effectively calculable function. We are *not* (yet) availing ourselves of the formally precise notion of a recursive function.

Observation 10 *For any consistent set Δ, we can employ the inference rule*

$$\frac{\Delta \vdash \bot}{\bot}.$$

Note that the upper occurrence of $\bot$ is the absurdity sign in the *object-language*, that is, the first-order language $\mathcal{L}$ of arithmetic, while the lower occurrence of $\bot$ is the absurdity sign of the *metalanguage*, in which we are reasoning *about* the object-language. So the inference rule in question is a metalinguistic one.

Definition 27 Suppose P is a (monadic) property of natural numbers. Then the *characteristic function of* P, called c_P, is defined by the following inference rules:

$$\frac{P(n)}{c_P(n) = 1}, \qquad \frac{\neg P(n)}{c_P(n) = 0}.$$

Definition 28 Suppose that μ is an effective function defined on all (metalinguistic) sentences of the form $P(n)$ (where n is a natural number), and that the range of μ is $\{0, 1\}$. Then:

> μ is an effective method for deciding, of any given natural number n, whether
> $P(n)$
> if and only if
> for every n, the following inferences hold:
>
> $$\frac{P(n)}{\mu(P(n)) = 1}, \qquad \frac{\neg P(n)}{\mu(P(n)) = 0}.$$

Definition 29 A property P of natural numbers is *decidable* if and only if there is an effective method for deciding, of any given natural number n, whether $P(n)$.

Definition 30 Suppose f is a function with domain D and range $\{0, 1\}$. Then $\tilde{f}$ is the function with domain D and range $\{0, 1\}$ defined by the following inference rules (for each $d \in D$):

$$\frac{f(d) = 1}{\tilde{f}(d) = 0}, \qquad \frac{f(d) = 0}{\tilde{f}(d) = 1}.$$

Lemma 19 *If P is a decidable property of natural numbers, then so is* $\neg P$.

Proof Let μ be an effective method for deciding, of any given natural number n, whether $P(n)$. Then $\tilde{\mu}$ is an effective method for deciding, of any given natural number n, whether $\neg P(n)$. □

Definition 31 Δ is *representing* if and only if Δ is consistent and for every decidable property P of natural numbers there is some unary formula $\psi(x)$ such that for every natural number n, the following inferences hold:

$$\frac{P(n)}{\Delta \vdash \psi(\underline{n})}, \text{ and its converse } \frac{\Delta \vdash \psi(\underline{n})}{P(n)}.$$

Definition 32 Δ is *strongly representing* if and only if for every decidable property P of natural numbers there is some unary formula $\psi(x)$ such that for all natural numbers n, the following inferences hold:

$$\frac{P(n)}{\Delta \vdash \psi(\underline{n})}, \qquad \frac{\neg P(n)}{\Delta, \psi(\underline{n}) \vdash \bot}.$$

Observation 11 *Suppose Δ is strongly representing and $\Delta \subseteq \Delta'$. Then Δ' is strongly representing.*

Lemma 20 *Suppose Δ is consistent. Suppose Δ is strongly representing. Then Δ is representing.*

Proof Since Δ is strongly representing, we already have the inference

$$\frac{P(n)}{\Delta \vdash \psi(\underline{n})}.$$

The following proof establishes the converse inference, using only logical rules of inference and rules of inference justified by the main suppositions:

$$\dfrac{\Delta \vdash \psi(\underline{n}) \qquad \dfrac{\dfrac{}{\neg P(n)}{\scriptstyle (1)}}{\Delta, \psi(\underline{n}) \vdash \bot}}{\dfrac{\dfrac{\Delta \vdash \bot}{\bot}}{P(n)}{\scriptstyle (1)}}\;{\scriptstyle \textsc{cut for } \bot}$$

Note that the final step, which is formally one of classical *reductio ad absurdum* (in the metalanguage), is constructively acceptable, since the sentence $P(n)$ is, by hypothesis, decidable. Note also that ‘CUT for $\bot$’ is an admissible rule for core logic. □

Definition 33 Δ is *axiomatizable* if and only if there is an effective enumeration of all and only the sentences that follow from Δ.

Definition 34 Δ is *decidable* if and only if there is an effective method for deciding, of any sentence φ in the language of arithmetic, whether $\Delta \vdash \varphi$.

Theorem 2 *Every decidable theory is axiomatizable.*

Proof Suppose Θ is a decidable theory. One can effectively enumerate its theorems by listing them as follows. First, effectively enumerate all sentences $\varphi_0, \varphi_1, \ldots$ of the language. Then generate the sought (sub)list consisting of exactly the theorems of Θ as follows. Take each sentence φ_i in turn. Effectively decide whether φ_i is in Θ. If so, append φ_i to the list of theorems, and advance to φ_{i+1}; if not, simply advance to φ_{i+1}. □

Theorem 3 (Janiczak) *Suppose Δ is consistent. Suppose Δ is complete. Suppose Δ is axiomatizable. Then Δ is decidable.*

(This result was first proved in Janiczak [4].)

Proof We seek to reveal an effective method for answering the question 'Does φ follow from Δ?' concerning any given sentence φ of the language of arithmetic.

The axiomatizability of Δ guarantees that the sentences that follow from Δ can be effectively enumerated as $\varphi_0, \varphi_1, \ldots$. Given any sentence φ in the language of arithmetic, the completeness of Δ guarantees that either φ occurs as some φ_n, or $\neg\varphi$ does. If φ so occurs, then we are done: we answer 'yes' to the question 'Does φ follow from Δ?' But if $\neg\varphi$ so occurs, then the consistency of Δ guarantees that φ does *not* so occur. So once again we are done: we answer 'no.' Thus we can effectively determine, of any given φ, whether φ follows from Δ. □

Theorem 4 *No set Δ is both representing and decidable.*

Proof Suppose Δ is representing and decidable. Let $\psi_0, \psi_1, \ldots$ be an effective enumeration of all unary formulae. (There is such an enumeration.) Consider the following property of natural numbers n:

$$\Delta \nvdash \psi_n(\underline{n}).$$

This property is decidable. For, given any natural number n, we can decide whether $\Delta \nvdash \psi_n(\underline{n})$ as follows.

> Effectively find the unary formula ψ_n. Effectively substitute the numeral $\underline{n}$ for every free occurrence of the free variable in ψ_n. The result is $\psi_n(\underline{n})$. Since Δ is decidable, there is an effective method (say μ) for deciding, of any given natural number m, whether $\Delta \vdash \psi_m(\underline{m})$. It follows that $\tilde{\mu}$ is an effective method for deciding, of any given natural number m, whether $\Delta \nvdash \psi_m(\underline{m})$. Now apply $\tilde{\mu}$ to decide whether $\Delta \nvdash \psi_n(\underline{n})$.

Since Δ is representing, there is, for this effectively decidable property $\Delta \nvdash \psi_n(\underline{n})$ of natural numbers, some unary formula ψ such that for every natural number n we can employ the rules of inference

$$\frac{\Delta \nvdash \psi_n(\underline{n})}{\Delta \vdash \psi(\underline{n})} \quad \text{and its converse} \quad \frac{\Delta \vdash \psi(\underline{n})}{\Delta \nvdash \psi_n(\underline{n})}.$$

But the unary formula ψ occurs in the effective enumeration of all unary formulae as ψ_k, for some k. Thus the rules of inference that we may employ, for any natural number n, become (for this fixed choice of k, and for any n)

$$\frac{\Delta \nvdash \psi_n(\underline{n})}{\Delta \vdash \psi_k(\underline{n})} \quad \text{and} \quad \frac{\Delta \vdash \psi_k(\underline{n})}{\Delta \nvdash \psi_n(\underline{n})}.$$

Now take as an instance of n the number k. Then we may employ the rules

$$\frac{\Delta \nvdash \psi_k(\underline{k})}{\Delta \vdash \psi_k(\underline{k})} \quad \text{and} \quad \frac{\Delta \vdash \psi_k(\underline{k})}{\Delta \nvdash \psi_k(\underline{k})}.$$

But these rules establish the equivalence of the statement $\Delta \vdash \psi_k(\underline{k})$ with its own negation. Contradiction. □

Theorem 5 *No set Δ is consistent, complete, strongly representing, and axiomatizable.*

Proof

$$\underbrace{\underbrace{\Delta \text{ consistent}\ ,\ \Delta \text{ strongly representing}}_{\substack{\vdots\ (\text{Lemma } 20)\\ \Delta \text{ representing}}}\ ,\ \underbrace{\Delta \text{ consistent}\ ,\ \Delta \text{ complete}\ ,\ \Delta \text{ axiomatizable}}_{\substack{\vdots\ (\text{Theorem } 3)\\ \Delta \text{ decidable}}}}_{\substack{\vdots\ (\text{Theorem } 4)\\ \bot}}$$

□

11.2 Proving the first incompleteness theorem by appealing to Church's thesis but without constructing a 'Gödel-sentence'

Theorem 6 Q *is strongly representing.*

Proof Suppose P is a decidable property of natural numbers. Then its characteristic function c_P is computable. By Church's thesis, c_P is recursive. (This is the first time we have appealed to Church's thesis.) By Theorem 1, there is a formula $\varphi(x, y)$ such that for all n the following two inferences hold modulo Q, with a parametric:

$$\text{(I)}\quad \frac{\varphi(\underline{n}, a)}{a = \underline{c_P(n)}}, \qquad\qquad \text{(II)}\quad \frac{a = \underline{c_P(n)}}{\varphi(\underline{n}, a)}.$$

We now show that the formula $\varphi(x, \underline{1})$ strongly represents P, by establishing the two inferences

$$\text{(i)}\qquad \frac{P(n)}{\mathsf{Q} \vdash \varphi(\underline{n}, \underline{1})}$$

and

$$\text{(ii)}\qquad \frac{\neg P(n)}{\mathsf{Q}, \varphi(\underline{n}, \underline{1}) \vdash \bot}.$$

Ad (i): Suppose $P(n)$. Then $c_P(n) = 1$. Substituting 1 for $c_P(n)$ in (II) above, we have that the following inference holds modulo Q, with a parametric:

$$\frac{a = \underline{1}}{\varphi(\underline{n}, a)}.$$

Putting now $\underline{1}$ in place of the parameter a, we obtain, modulo Q, the inference

$$\frac{\underline{1} = \underline{1}}{\varphi(\underline{n}, \underline{1})}.$$

But the premise is a logical theorem. Thus $\mathsf{Q} \vdash \varphi(\underline{n}, \underline{1})$.

Ad (ii): Suppose $\neg P(n)$. Then $c_P(n) = 0$. Substituting 0 for $c_P(n)$ in (I) above, we have that the following inference holds modulo Q, with a parametric:

$$\frac{\varphi(\underline{n}, a)}{a = 0}.$$

Putting now $\underline{1}$ in place of the parameter a, we obtain, modulo Q,

$$\frac{\varphi(\underline{n}, \underline{1})}{\underline{1} = 0}.$$

But $\mathsf{Q}, \underline{1} = 0 \vdash \bot$. Hence $\mathsf{Q}, \varphi(\underline{n}, \underline{1}) \vdash \bot$. □

Corollary 1 $\mathrm{Th}(\mathbb{N})$ *is strongly representing.*

Proof By Theorem 6, Q is strongly representing. Recall Observation 4: Q $\subseteq$Th($\mathbb{N}$). By Observation 11, Th($\mathbb{N}$) is strongly representing. □

Theorem 7 Th($\mathbb{N}$) *is not axiomatizable.*

Proof By definition, Th($\mathbb{N}$) is consistent and complete. By Corollary 1, Th($\mathbb{N}$) is strongly representing. Hence, by Theorem 5, Th($\mathbb{N}$) is not axiomatizable. □

11.3 Proving the first incompleteness theorem by appealing to Church's thesis to construct a 'Gödel-sentence' by diagonalizing We shall suppose that we are given a mapping

$$\varphi \mapsto \overline{\varphi}$$

of sentences to closed terms of $\mathscr{L}$.

Definition 35 Let θ be a unary formula. We shall say that Δ is *self-representing via* θ just in case

$$\text{for every sentence } \varphi, \quad \theta(\overline{\varphi}) \in \Delta \ \leftrightarrow \ \varphi \in \Delta .$$

Equivalently, we can use the following two (metalinguistic) rules of inference:

$$\frac{\Delta \vdash \varphi}{\Delta \vdash \theta(\overline{\varphi})}, \qquad \frac{\Delta \vdash \theta(\overline{\varphi})}{\Delta \vdash \varphi}.$$

We shall suppose also that we are given a mapping

$$\psi \mapsto \overline{\overline{\psi}}$$

of formulae with one free variable to closed terms of $\mathscr{L}$.

Definition 36 Given a binary formula $\delta(x,y)$, we shall say that δ is a *diagonal* for Δ if and only if for every unary formula ψ we have

$$\delta(\overline{\overline{\psi}}, a) \dashv^{\Delta}\vdash a = \overline{\psi(\overline{\overline{\psi}})},$$

with a parametric. This means that there are (core) proofs Π^a_ψ and Σ^a_ψ in the object-language as follows:

$$\begin{array}{c} \underbrace{\Delta\, ,\, \delta(\overline{\overline{\psi}}, a)} \\ \Pi^a_\psi \\ \hline a = \overline{\psi(\overline{\overline{\psi}})} \end{array} \qquad \begin{array}{c} \underbrace{\Delta\, ,\, a = \overline{\psi(\overline{\overline{\psi}})}} \\ \Sigma^a_\psi \\ \hline \delta(\overline{\overline{\psi}}, a) \end{array}$$

Thus a diagonal represents a mapping that assigns to each unary formula ψ the sentence $\psi(\overline{\overline{\psi}})$, which results from substituting the term $\overline{\overline{\psi}}$ for the free variable in ψ. In applications below, $\overline{\varphi}$ will be $\underline{\sharp\varphi}$, and $\overline{\overline{\psi}}$ will likewise be $\underline{\sharp\psi}$.

Observation 12 *A diagonal for any theory is a diagonal for any of its extensions.*

Lemma 21 *Suppose*

Δ *is self-representing via* $\theta(x)$,
Δ *has a diagonal* $\delta(x,y)$.

Set

$$\varepsilon(y) \ =_{df} \ \forall x\big(\delta(y,x) \rightarrow \neg\theta(x)\big)$$

and

$$\gamma \ =_{df} \ \varepsilon(\overline{\overline{\varepsilon}}).$$

Then

$$\Delta, \gamma, \theta(\overline{\gamma}) \vdash \bot$$

and

$$\Delta, \neg\theta(\overline{\gamma}) \vdash \gamma.$$

Proof There are the following proofs in the object-language. In the first one, the parameter a is replaced by the term $\overline{\gamma}$.

$$\dfrac{\dfrac{\begin{matrix}\underbrace{\Delta\,,\overline{\gamma} = \overset{\text{By defn. of }\gamma:}{\overline{\varepsilon(\overline{\overline{\varepsilon}})}}}_{\Sigma_\varepsilon^{\overline{\gamma}}} \\ \delta(\overline{\overline{\varepsilon}},\overline{\gamma})\end{matrix} \qquad \dfrac{\begin{matrix}\gamma,\ \text{i.e.} \\ \forall x(\delta(\overline{\overline{\varepsilon}},x) \to \neg\theta(x))\end{matrix}}{\delta(\overline{\overline{\varepsilon}},\overline{\gamma}) \to \neg\theta(\overline{\gamma})}}{\neg\theta(\overline{\gamma})} \qquad \theta(\overline{\gamma})}{\bot}$$

$$\begin{matrix}\dfrac{\dfrac{\dfrac{\neg\theta(\overline{\gamma}) \qquad \dfrac{\dfrac{\begin{matrix}\underbrace{\Delta\,,\ \overline{\delta(\overline{\overline{\varepsilon}},a)}^{(2)}}_{\Pi_\varepsilon^a} \\ a = \varepsilon(\overline{\overline{\varepsilon}})\end{matrix} \qquad \overline{\gamma} = \overset{\text{By defn. of }\gamma:}{\overline{\varepsilon(\overline{\overline{\varepsilon}})}}}{a = \overline{\gamma}} \qquad \overline{\theta(a)}^{(1)}}{\theta(\overline{\gamma})}}{\dfrac{\bot}{\neg\theta(a)}(1)}}{\delta(\overline{\overline{\varepsilon}},a) \to \neg\theta(a)}(2)}{\forall x\big(\delta(\overline{\overline{\varepsilon}},x) \to \neg\theta(x)\big)} \\ \text{i.e., } \gamma\end{matrix}$$

These two proofs respectively justify the following two deducibility claims:

$$\Delta, \gamma, \theta(\overline{\gamma}) \vdash \bot \quad ; \quad \Delta, \neg\theta(\overline{\varphi}) \vdash \gamma.$$ □

Theorem 8
The following set of conditions on Δ are jointly inconsistent:

Δ *is a theory*,
Δ *is consistent*,
Δ *is complete*,
Δ *is self-representing (via $\theta(x)$, say)*,
Δ *has a diagonal ($\delta(x, y)$, say).*

Proof Let γ be constructed from θ and δ as before. The hypotheses of Lemma 21 are satisfied. So we are entitled to the deducibility claims below.

$$\dfrac{\dfrac{\Delta \vdash \gamma}{\Delta \vdash \theta(\overline{\gamma})} \qquad \dfrac{\Delta \vdash \gamma \qquad \Delta, \gamma, \theta(\overline{\gamma}) \vdash \bot}{\Delta, \theta(\overline{\gamma}) \vdash \bot}}{\Delta \vdash \bot}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\Delta, \neg\theta(\overline{\varphi}) \vdash \gamma \qquad \Delta, \gamma \vdash \bot}{\Delta, \neg\theta(\overline{\varphi}) \vdash \bot}\,\text{CR}}{\Delta \vdash \theta(\overline{\varphi})}}{\Delta \vdash \gamma} \qquad \Delta, \gamma \vdash \bot}{\Delta \vdash \bot}$$

Note the appeal to classical reductio *in the object-language* in the second proof. The second proof, however, is still a core proof at the metalevel.

These two metalinguistic core proofs show that no matter whether a self-representing theory Δ with a diagonal proves or refutes γ, Δ is inconsistent. So, contraposing (within core metalogic): if a *self-representing theory* Δ *with a diagonal* is consistent, then Δ neither proves nor refutes 'its' γ; whence Δ is incomplete. □

The question to confront now is this: What consistent theories might be self-representing and be possessed of a diagonal? *Consistency* can be ensured by considering only subtheories of $Th(\mathbb{N})$. But what about being *self-representing* and *having a diagonal*?

This is where the indicated appeal to Church's thesis (in the heading of this section) comes in. We proceed now to show that Q has a diagonal, and for any Δ such that $\mathsf{Q} \subseteq \Delta \subseteq Th(\mathbb{N})$, if Δ is axiomatizable (i.e., effectively enumerable), then Δ is self-representing.

The choice of Q as Δ (at the bottom end of the just-mentioned inclusion range) merely coheres with what one already knows about the weakness of Q: its incompleteness is obvious from any of the aforementioned examples of sentences that are self-evidently true in $\mathbb{N}$ but false in certain pathological models of Q.

Let us now put Church's thesis to work, hand-in-hand with the representability theorem (i.e., Theorem 8). We already know by Theorem 6 that Q is strongly representing, and we have already used Church's thesis in proving that result. The proofs of our next two theorems likewise appeal to Church's thesis.

Theorem 9 Q *has a diagonal.*

Proof Suppose $\sharp$ is a numerical coding of formulae with one free variable and $\natural$ is a numerical coding of sentences. Codings and their inverses are effective functions. Therefore the mapping

$$n \mapsto \natural\big(\sharp^{-1}(n)(\underline{n})\big)$$

is itself effective. By Church's thesis it is recursive. Hence by the representability theorem there is a formula $\delta(x, y)$ such that for all n

$$\delta(\underline{n}, a) \dashv^{\mathsf{Q}}\vdash a = \underline{\natural\big(\sharp^{-1}(n)(\underline{n})\big)}.$$

If θ is a formula with one free variable, let

$$\overline{\overline{\theta}} = \underline{\sharp\theta},$$

and if θ is a sentence, let

$$\overline{\theta} = \underline{\natural\theta}.$$

Then for every unary formula ψ, we have

$$\delta(\underline{\sharp\psi}, a) \dashv^{\mathsf{Q}}\vdash a = \underline{\natural\big(\sharp^{-1}(\sharp\psi)(\underline{\sharp\psi})\big)},$$

that is,

$$\delta(\overline{\overline{\psi}}, a) \dashv^{\mathsf{Q}}\vdash a = \overline{\psi(\overline{\overline{\psi}})}.$$ □

By Observation 12, the diagonal for Q serves as a diagonal for any extension of Q.

Theorem 10 *Any axiomatizable subtheory of Th*($\mathbb{N}$) *containing* Q *is self-representing.*

Proof Suppose Δ is an axiomatizable subtheory of Th($\mathbb{N}$) containing Q. Let Δ be given by the effective enumeration

$$\theta_0, \theta_1, \ldots$$

of all its theorems, and let

$$\varphi_0, \varphi_1, \ldots$$

be an effective enumeration of all sentences. Consider the decidable relation R of natural numbers defined by

$$\varphi_m = \theta_n,$$

that is, the mth sentence of the language is the nth theorem of Δ. By Church's thesis, the characteristic function c_R is recursive. By the representability theorem, there is a ternary formula $\psi(x, y, z)$ such that for all m, n we have

$$\psi(\underline{m}, \underline{n}, a) \dashv^{\mathsf{Q}}\vdash a = \underline{c_R(m, n)}.$$

Hence $\varphi_m = \theta_n$ if and only if $Q \vdash \psi(\underline{m}, \underline{n}, 1)$. We now show that

$$\varphi_m \in \Delta \text{ if and only if } \exists x \psi(\underline{m}, x, 1) \in \Delta.$$

(i) Suppose $\varphi_m \in \Delta$. Then for some n, we have $\varphi_m = \theta_n$; whence $c_R(m, n) = 1$. Thus $\mathsf{Q} \vdash \psi(\underline{m}, \underline{n}, 1)$. By $\exists$I, we have

$$\mathsf{Q} \vdash \exists x \psi(\underline{m}, x, 1),$$

and so $\exists x \psi(\underline{m}, x, 1) \in \Delta$.

(ii) Conversely, suppose $\exists x \psi(\underline{m}, x, 1) \in \Delta$. Since $\Delta \subseteq Th(\mathbb{N})$, it follows that $\mathbb{N} \models \exists x \psi(\underline{m}, x, 1)$. So for some n, $\mathbb{N} \models \psi(\underline{m}, \underline{n}, 1)$. Now if $c_R(m, n) = 0$, then $\mathsf{Q} \vdash \neg\psi(\underline{m}, \underline{n}, 1)$; whence $\mathbb{N} \models \neg\psi(\underline{m}, \underline{n}, 1)$. Thus $c_R(m, n) = 1$ (that is, $\varphi_m = \theta_n$). Thus $\varphi_m \in \Delta$.

Since $\underline{m} = \overline{\varphi_m}$, we have for all φ

$$\varphi \in \Delta \text{ if and only if } \exists x \psi(\overline{\varphi}, x, 1) \in \Delta.$$

So Δ is self-representing via the predicate $\exists x \psi(y, x, 1)$. $\square$

11.4 Proving the first incompleteness theorem without appealing to Church's thesis and by directly constructing a 'Gödel-sentence' This is what Gödel did.

Notes

1. Please note that some notation in this article appears in color online at https://10.1215/00294527-2022-0033.

2. This extraction theorem was first proved in Tennant [14]. A similar extraction theorem holds for classical core logic $\mathbb{C}^+$, with 'intuitionistic proof' replaced by 'classical proof', and 'core proof' replaced by 'classical core proof'. For proofs of all the other metatheorems about the core systems stated in this study, see Tennant [20].

3. This was first proved in Tennant [19].

4. See Friedman [3]. A useful exposition of this conservative extension result, using Friedman's A-translation, can be found in Selinger [10].

5. If one were using a free logic, then the rule of induction would be framed as follows:

$$\dfrac{\begin{array}{c}\\ \\ \vdots\\ \Phi(0)\end{array}\qquad \begin{array}{c}\underbrace{\overset{\diamond\,\overline{}\,(i)}{\Phi(a)}\;,\;\;\overset{\diamond\,\overline{}\,(i)}{\exists! a}}\\ \vdots\\ \Phi(sa)\end{array}}{\Phi(t)}\,(i)$$

where the parameter a occurs only where indicated

and t may be any closed term.

6. Note that the symbol '$<$' in some of these examples is not primitive, but defined. See Definition 10. Foreshadowing: $x < y$ is defined as short for $\exists z\, sz + x = y$. We have to be very careful to stick to this definition when reasoning within Q, precisely because Q's parlous stock of theorems makes the usual cavalier mathematical manipulations (such as thinking it will not matter if we write $\exists z\, x + sz = y$ instead) out of order.

7. This is the approach of Boolos and Jeffrey [1]. However, our derivations will have certain advantages over theirs.

8. Note that one would not wish to have the comment 'n is not the code number of any syntactic item in Σ' itself be a syntactic item in Σ ! This is a seldom-considered complication speaking in favor of avoiding any self-referential language, even in the very weak sense of having a sentence about syntactic items itself count as a syntactic item. This weak sense does not even amount to the usual kind of *semantic* closure that gives rise to paradoxes such as the Liar.

9. For the various subtleties involved, see Tennant [15], especially Chapters 6 and 8.

10. One could appeal here to Lemma 4 to the same effect.

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