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Parts, classes and *Parts of Classes*: an anti-realist reading of Lewisian mereology

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Abstract This study is in two parts. In the first part, various important principles of classical extensional mereology are derived on the basis of a nice axiomatization involving 'part of' and fusion. All results are proved here with full Fregean (and Gentzenian) rigor. They are chosen because they are needed for the second part. In the second part, this natural-deduction framework is used in order to regiment David Lewis's justification of his Division Thesis, which features prominently in his combination of mereology with class theory. The Division Thesis plays a crucial role in Lewis's informal argument for his Second Thesis in his book *Parts of Classes*. In order to present Lewis's argument in rigorous detail, an elegant new principle is offered for the theory that combines class theory and mereology. The new principle is called the Canonical Decomposition Thesis. It secures Lewis's Division Thesis on the strong construal required in order for his argument to go through. The exercise illustrates how careful one has to be when setting up the details of an adequate foundational theory of parts and classes. The main aim behind this investigation is to determine

This paper was accepted for presentation at the SAC conference on David Lewis's contributions to formal philosophy, held in Copenhagen in September 2007, but, in the event, the author was unable to attend. The paper was submitted for the proceedings immediately after the conference was held; this is its first publication. The author offers this study as a tribute to the legacy of David's unfailingly rigorous but catholic intellect, and in grateful memory of particularly helpful and encouraging philosophical conversations with David during his annual visits to Australia in the late eighties and early nineties. Thanks are owed to Gabriel Uzquiano for helpful correspondence that was a stimulus to this project. Ben Caplan provided detailed comments on an earlier version, which helped avert a formal error. Comments by Kevin Scharp and Nicholaos Jones led to improvements in exposition. Salvatore Florio gave two later versions particularly eagle-eyed attention, uncovering errors of composition in some of the formal derivations. Fred Muller kindly provided some LaTeX codes for mereological symbols. The author is solely responsible for any errors that remain.

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whether an anti-realist, inferentialist theorist of meaning has the resources to exhibit Lewis's argument for his Second Thesis—which is central to his marriage of class theory with mereology—as a purely conceptual one. The formal analysis shows that Lewis's argument, despite its striking appearance to the contrary, can be given in the constructive, relevant logic *IR*. This is the logic that the author has argued, elsewhere, to be the correct logic from an anti-realist point of view. The anti-realist is therefore in a position to regard Lewis's argument as purely conceptual.

Keywords Mereology · Parts · Classes · Inferentialism · Canonical Decomposition Thesis · Lewis's Division Thesis · Lewis's Second Thesis

Part I: Natural deduction for mereology

1 Natural deduction and the inferentialist theory of meaning

Seminal work by Dummett (1977,1978,1991) and Prawitz (1974,1977) has led to the development of an anti-realist, inferentialist theory of meaning. This theory stresses the central role of harmoniously balanced introduction and elimination rules. Such rules capture the meanings of the logico-mathematical operators with which they deal. Moreover, they do so constructively. Introduction and elimination rules provide no justification for the 'strictly classical' rules such as double-negation elimination, dilemma, classical reductio or the law of excluded middle.

Both Dummett and Prawitz confined themselves to logical connectives and quantifiers. These are *sentence*-forming operators. The present author has sought to extend the spirit and methods of their approach so as to deal also with *term*-forming operators. It turns out that such operators, too, can be furnished with carefully crafted introduction and elimination rules. And these rules arguably capture the 'constructive content' of the notions involved—such as 'number of' and 'set'. This natural-deduction analysis is called 'constructive logicism' in the case of a general theory of the natural numbers and of their application in counting. (See Tennant 1987, 2003, 2008 for details.) And in set theory, the natural-deduction analysis interestingly yields extensionality as a *derived* result. (See Tennant (2004) for details.)

Those of an analytic or logicist bent might find this general kind of approach attractive, for it furnishes a principled way of distinguishing that part of a theory that can be said to be analytic from the part that should be conceded to be synthetic. (See the discussion in Tennant (1997, Chap. 9) for more details.) Roughly, the analytic part is generated by the introduction and elimination rules. Analytic results involve either no existential commitments at all, or commitments only to necessary existents. Ironically (in this context) the best example of an analytic portion of a theory is what Quine called 'virtual' set theory—the part of set theory that is free of any existential commitments. (It is precisely this part that is captured by the introduction and elimination rules for the set-abstraction operator.) Furthermore, the analytic results can always be obtained constructively, since one has no truck with classical rules when applying only introduction and elimination rules.

The problem to be addressed here is whether such a treatment can be extended to mereology, with its basic operation of fusion. Exactly what part of mereology is 'purely conceptual'? And pushing the question further: exactly what part of David Lewis's sought marriage of mereology with class theory is 'purely conceptual'? (See Lewis 1991 for the nuptials.) It was with this question in mind that the author addressed Lewis's informal argument for his Second Thesis (to the effect that all parts of classes are classes). On the face of it, Lewis's argument is strikingly non-constructive. Yet it is offered as a purely *conceptual* argument. So: is Lewis's argument *ineliminably* classical? Or can it be constructivized? The question merits a detailed answer, and this investigation seeks to provide one. To foreshadow the main finding: Yes, the argument can be constructivized. Hence, on present evidence, mereology seems to provide a positive testing ground for the aforementioned anti-realist, inferentialist theory of meaning.

2 Mereology

Mereology is the theory of the part-whole relation. Originally, in the pioneering work of Lesniewśki (1983), mereology was intended as an alternative to the theory of classes, and was even intended to supplant it. The Russell Paradox could thereby be avoided. Moreover, Lesniewśki's axiomatization of the part-whole relation answered more satisfactorily—or so he argued—to mathematicians' preformal intuitions concerning the composition of more complex objects out of their constituents.

Mereology cannot, however, aspire to provide an adequate foundation for mathematics if left unsupplemented by class (or set) theory. For it has a one-element model. While this fact will allay any worries one might have as to consistency, it also shows how desperately weak mereology is in consistency strength—an all-important measure, in the wake of Gödel's incompleteness theorems for arithmetic. From the early 1930s onwards, mereology could clearly lay no claim to adequacy as a foundation for mathematics.

At around that time, first-order Zermelo–Fraenkel set theory established itself as an adequate successor to Frege's inconsistent class theory. When mereology was subsequently revived, it was with a new, nominalistic, bent. It concerned itself only with concrete *individuals*, and avoided talking about abstract entities, in particular entities of 'higher type', such as properties or sets. The source of many of our intuitions about the part-whole relation, however, is spatial, or spatio-temporal. Basic claims about mereological operations and relations are particularly easy to motivate in the context, of, say, regions of the Euclidean plane. But, insofar as ordinary physical objects are space-occupying, one can also motivate mereological claims by thinking about extended bodies in three dimensions (or even by thinking about four-dimensional worms, their time-slices, and concoctions thereof). Mereology is a *conceptual* enterprise,[1] and should not be hostage to empirical fortune in such matters as whether there are any *mereological atoms*—that is, individuals that have no proper parts.

For clarity, we shall here reserve the term 'individual' for concrete individuals. Individuals and classes are disjoint species of the ultimate genus *thing*, or *entity*, or *existent*. But individuals and classes do not exhaust the things. That is, one cannot

[1] This point is stressed also by Simons (2006).

simply assume that everything is either an individual or a class. For, among the things one might find irreducibly 'hybrid' things, such as fusions of classes with individuals. And we intend to raise the question (not raised by Lewis) whether *fusions of classes* are yet another hybrid kind of thing, rather than—as we shall see Lewis tacitly assumes—classes.

Leonard and Goodman (1940), in their classic paper, wrote as follows:

> The concept of an individual and that of a class may be regarded as different devices for distinguishing one segment of the total universe from all that remains. In both cases, the differentiated segment is potentially divisible, and may even be physically discontinuous. The difference in the concepts lies in this: that to conceive a segment as a whole or individual offers no suggestion as to what these subdivisions, if any, must be, whereas to conceive a segment as a class imposes a definite scheme of subdivision—into subclasses and members.

In other words, any member of the *class* of all Φs is a Φ; but it need not always hold that any part of the *fusion* (i.e., mereological sum) of all Φs is a Φ. For example, every member of the class of all cats is a cat. But the fusion of all cat-whiskers is part of the fusion of all cats, without being a cat. This is the most crucial formal difference between forming classes and forming fusions. It makes one realize that members of classes do not behave like parts of fusions, even though one might have some intuition to the effect that (non-empty) classes can nevertheless be thought of as having parts.

Lewis's aim was to argue that the parts of a (non-empty) class are exactly its (non-empty) subclasses. In this way, mereological thinking could be extended into the universe of classes. Indeed, Lewis even went so far as to reserve the term 'class' to denote *non-empty* classes, so that his Main Thesis could be stated as follows (*ibid*., p. 7):

Main Thesis. The parts of a class are all and only its subclasses.

One main aim of this study is to regiment and constructivize Lewis's argument for his Main Thesis. That will be done in Part II.

2.1 Mereological relations

The most important mereological *relations* (whether taken as primitive or defined) are the following.[2]

$x \sqsubseteq y$	x is part of y
$x \sqsubset y$	x is a proper part of y
$x \circ y$	x overlaps y
$x \wr y$	x is disjoint from y

By means of any one of the relations just listed, the other three relations can be defined. Here, we shall take $\sqsubseteq$ and identity as primitive. Accordingly, the other three relations will be defined as follows.

[2] Historically, there has been a 'notational jungle' (see Simons 1987, at pp. 98–100) with no canonical choice of notation having been settled upon. We do our best here to choose notations that will be reasonably perspicuous and easy to remember.

$$
\begin{array}{lll}
x \sqsubset y & \equiv_{df} & x \sqsubseteq y \wedge \neg x = y \\
x \circ y & \equiv_{df} & \exists z(z \sqsubseteq x \wedge z \sqsubseteq y) \\
x \rceil y & \equiv_{df} & \neg(x \circ y)
\end{array}
$$

2.2 Mereological operations

The most important mereological *operations* are the following.

$x \sqcup y$	the sum (fusion) of x and y
$x \sqcap y$	the product (nucleus; common part) of x and y
$x - y$	the part of x that is not part of y

The taking of a sum or a product is here represented as a binary operation. But of course the operations in question can be applied more generally, to the members of any given family or class of individuals. In order, however, to avoid presupposing the language and theory of sets in our mereological theorizing about individuals and their sums and products, we can resort to the use of first-order schematic predicates:

$\bigsqcup x\Phi(x)$	the sum (fusion) of all individuals x such that $\Phi(x)$
$\bigsqcap x\Phi(x)$	the product (nucleus) of all individuals x such that $\Phi(x)$

One's axioms and rules of inference for mereology would have to specify appropriate existence conditions for such sums and products. Because these conditions might in some cases not be met, mereology should be based on a *free logic*, such as the system in Tennant (1978, Chap. 7).

Note that $\bigsqcap x\Phi(x)$ can be defined as the fusion of all individuals that are part of every Φ:

$$\bigsqcap x\Phi(x) =_{df} \bigsqcup x\forall y(\Phi y \rightarrow x \sqsubseteq y).$$

Definition 1 (*Sums*) When Φx is $(x = t_1 \vee \cdots \vee x = t_n)$, we may write $\bigsqcup x\Phi x$ as $t_1 + \cdots + t_n$.

Definition 2 (*Differences*) $x - y =_{df} \bigsqcup z(z \sqsubseteq x \wedge z \rceil y)$.

3 Notational conventions for inference and proof

Natural deductions will be set out in tree form below. The reader unfamiliar with this format for proofs is advised that with applications of so-called 'discharge rules' the parenthetically enclosed numeral '(i)' has an occurrence labelling the step at which the indicated assumption-occurrences higher up at 'leaf nodes' of the sub-proof(s) are *discharged* by applying the rule in question. A discharged assumption no longer counts among the assumptions on which the conclusion of the newly created proof depends. Also, we say that a is *parametric* within a sub-proof just in case among the undischarged assumptions, and conclusion, of the sub-proof in question, a occurs only within sentences of the indicated form.

Sometimes, in order to prevent proofs from spreading too wide, applications of (∃-E) and (∧-E) thus:

$$\dfrac{\exists x(\varphi x \wedge \psi x) \qquad \begin{matrix} \underbrace{\dfrac{\overline{\varphi a \wedge \psi a}^{\,(i)}}{\varphi a} \quad \dfrac{\overline{\varphi a \wedge \psi a}^{\,(i)}}{\psi a}} \\ \vdots \\ \theta \end{matrix}}{\theta}\,(i)$$

will be shortened so as to take the form

$$\dfrac{\exists x(\varphi x \wedge \psi x) \qquad \begin{matrix} \underbrace{\overline{\varphi a}^{\,(i)} \quad \overline{\psi a}^{\,(i)}} \\ \vdots \\ \theta \end{matrix}}{\theta}\,(i)$$

Likewise, we shall omit some of the obvious steps involved in proving that t is a proper part of u, by employing the rule

$$\dfrac{t \sqsubseteq u \qquad \begin{matrix} \overline{t = u}^{\,(i)} \\ \vdots \\ \bot \end{matrix}}{t \sqsubset u}\,(i)$$

4 Mereological axioms and rules of inference when ‘part of’ is the relational primitive, and fusion is the primitive abstraction operation

There is a great variety of mutually interpretable mereological theories, arising from the possibility, as indicated above, of taking different mereological relations as primitive. In this section we present only a system of rules for a theory based on $\sqsubseteq$ as the sole relational primitive. Since we are working in a free logic, we shall have frequent need of existential presuppositions of the form $\exists!t$ (short for $\exists x\, x = t$). Rules are stated here in accordance with the notational and graphic conventions of Tennant (1978).

4.1 The abstraction rule for fusions

$$(\sqcup\text{-I}) \qquad \dfrac{\begin{matrix} \underbrace{\overline{\Phi a}^{\,(i)}\, , \overline{\exists! a}^{\,(i)}} \\ \vdots \\ a \sqsubseteq t \end{matrix} \qquad \exists z \Phi z \qquad \exists! t \qquad \begin{matrix} \overline{a \sqsubseteq t}^{\,(i)} \\ \vdots \\ \exists y(\Phi y \wedge a \circ y) \end{matrix}}{t = \bigsqcup x \Phi x}\,(i)$$

In words: Suppose that one has shown t to exist; and shown something to be Φ; and shown that every Φ is part of t; and shown that every part of t overlaps some Φ. Then one may infer that t is the fusion of (all) Φs.

Corresponding to this introduction rule is the following four-part elimination rule.

$$(\bigsqcup\text{-E1})\quad \frac{t = \bigsqcup x \Phi x \quad \Phi(u) \quad \exists! u}{u \sqsubseteq t}$$

$$(\bigsqcup\text{-E2})\quad \frac{t = \bigsqcup x \Phi x \qquad \begin{array}{c} \underbrace{\overline{\Phi a}^{(i)}\ ,\ \overline{\exists! a}^{(i)}} \\ \vdots \\ \psi \end{array}}{\psi}{\scriptstyle (i)}$$

$$(\bigsqcup\text{-E3})\quad \frac{t = \bigsqcup x \Phi x}{\exists! t}$$

$$(\bigsqcup\text{-E4})\quad \frac{t = \bigsqcup x \Phi x \quad u \sqsubseteq t}{\exists y(\Phi y \wedge u \circ y)}$$

($\bigsqcup$-E3) is of course a special case of the rule we already have in free logic, which allows one to infer $\exists! t$ from any atomic predication $A(t)$.

The foregoing introduction and elimination rules for $\bigsqcup$ are ontologically non-committal. All they do is pin down the conceptual connections between fusions, on the one hand, and, on the other, the concepts of part, existence and identity. One may think of this as located within the 'analytical' part of the present account. (Note that fusions are hereby guaranteed to be *unique*.)

4.2 Analytical rules governing 'part'

The parthood relation $\sqsubseteq$ is clearly, on reflection, a partial order. That is, it is governed by the following three rules.

$$(\text{REFLEXIVITY})\quad \frac{\exists! t}{t \sqsubseteq t}$$

$$(\text{TRANSITIVITY})\quad \frac{t \sqsubseteq u \quad u \sqsubseteq v}{t \sqsubseteq v}$$

$$(\text{ANTI- SYMMETRY})\quad \frac{t \sqsubseteq u \quad u \sqsubseteq t}{t = u}$$

4.3 A synthetic rule for fusion

Ontological commitment is incurred only when one postulates that *there are* fusions of such-and-such a kind. The most powerful such postulate is

(UNRESTRICTED EXISTENCE OF FUSIONS) $\dfrac{\exists x\,\Phi x}{\exists!\bigsqcup x\,\Phi x}$

One may think of this as the ‘synthetic’ part of the present account.

5 Some formal results

5.1 Results using only definitions

Lemma 1 $\dfrac{v \circ t \qquad v \sqsubseteq u}{t \circ u}$

Proof

$$\dfrac{\begin{array}{c} v \circ t,\ \text{i.e.} \\ \exists x(x \sqsubseteq v \wedge x \sqsubseteq t)\end{array} \qquad \dfrac{\overset{(1)}{\overline{\exists! a}} \qquad \dfrac{\dfrac{\overset{(1)}{\overline{a \sqsubseteq v \wedge a \sqsubseteq t}}}{a \sqsubseteq t} \qquad \dfrac{\dfrac{\overline{a \sqsubseteq v \wedge a \sqsubseteq t}^{(1)}}{a \sqsubseteq v} \qquad v \sqsubseteq u}{a \sqsubseteq u}}{a \sqsubseteq t \wedge a \sqsubseteq u}}{\exists y(y \sqsubseteq t \wedge y \sqsubseteq u)}}{\begin{array}{c}\exists y(y \sqsubseteq t \wedge y \sqsubseteq u) \\ \text{i.e. } t \circ u\end{array}}\,(1)$$

□

Lemma 2 $\dfrac{v \sqsubseteq \bigsqcup x\,\Phi x}{\exists y(\Phi y \wedge v \circ y)}$

Proof

$$\dfrac{\dfrac{v \sqsubseteq \bigsqcup x\,\Phi x}{\bigsqcup x\,\Phi x = \bigsqcup x\,\Phi x} \qquad v \sqsubseteq \bigsqcup x\,\Phi x}{\exists y(\Phi y \wedge v \circ y)}\,(\bigsqcup\text{-E4})$$

□

Corollary 1 $\dfrac{v \sqsubseteq \bigsqcup x(x = t \vee x = u)}{\exists y((y = t \vee y = u) \wedge v \circ y)}$

Proof For Φx in Lemma 2 take $(x = t \vee x = u)$. □

Lemma 3

$$\dfrac{\begin{array}{c}\underbrace{\overset{(i)}{\overline{a \sqsubseteq t}}\ ,\ \overline{a \sqsubseteq u}^{(i)}} \\ \vdots \\ \bot\end{array}}{t \,\rfloor\, u}\,(i)$$

Proof

$$\dfrac{\dfrac{\begin{array}{c}\overset{(2)}{\overline{t \circ u, \textit{i.e.}}} \\ \exists x(x \sqsubseteq t \wedge x \sqsubseteq u)\end{array} \qquad \begin{array}{c}\underbrace{\overset{(1)}{\overline{a \sqsubseteq t}}\ ,\ \overline{a \sqsubseteq u}^{(1)}} \\ \vdots \\ \bot\end{array}}{\bot}\,(1)}{t \,\rfloor\, u}\,(2)$$

□

Lemma 4 $t \rceil \bigsqcup x(\Phi x \wedge x \rceil t)$

Proof.

$$\dfrac{\dfrac{\dfrac{}{a \sqsubseteq \bigsqcup x(\Phi x \wedge x \rceil t)}(2)}{\exists y((\Phi y \wedge y \rceil t) \wedge a \circ y)}(\text{L2}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{(\Phi b \wedge b \rceil t) \wedge a \circ b}(1)}{a \circ b} \quad \dfrac{}{a \sqsubseteq t}(2)}{b \circ t}(\text{L1}) \quad \dfrac{\dfrac{\dfrac{}{(\Phi b \wedge b \rceil t) \wedge a \circ b}(1)}{\Phi b \wedge b \rceil t}}{b \rceil t}}{\bot}}{\dfrac{\bot}{t \rceil \bigsqcup x(\Phi x \wedge x \rceil t)}(2)\ (\text{L3})}(1)$$

Lemma 5 $\dfrac{t \sqsubseteq \bigsqcup x(\Phi x \wedge x \rceil t)}{\bot}$

$$\dfrac{\dfrac{t \sqsubseteq \bigsqcup x(\Phi x \wedge x \rceil t)}{\exists y(\Phi y \wedge y \rceil t) \wedge t \circ y)}(\text{L2}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{(\Phi a \wedge a \rceil t) \wedge t \circ a}(1)}{\Phi a \wedge a \rceil t}}{a \rceil t} \quad \dfrac{\dfrac{}{(\Phi a \wedge a \rceil t) \wedge t \circ a}(1)}{t \circ a}}{\bot}}{\bot}(1)$$

5.2 Results using only definitions and transitivity

Lemma 6

$$\dfrac{v \circ t \qquad \begin{matrix}\dfrac{}{a \sqsubseteq t}(i) \\ \vdots \\ a \circ u\end{matrix}}{v \circ u}(i)$$

(a parametric in the right-hand subproof)

Proof.

$$\dfrac{\begin{matrix} v \circ t, i.e. \\ \exists y(y \sqsubseteq v \wedge y \sqsubseteq t)\end{matrix} \qquad \dfrac{\begin{matrix}\dfrac{\dfrac{}{a \sqsubseteq v \wedge a \sqsubseteq t}(2)}{a \sqsubseteq t} \\ \vdots \\ a \circ u \ i.e. \\ \exists z(z \sqsubseteq a \wedge z \sqsubseteq u)\end{matrix} \qquad \dfrac{(1)\dfrac{}{\exists! b} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{b \sqsubseteq a \wedge b \sqsubseteq u}(1)}{b \sqsubseteq a} \quad \dfrac{\dfrac{}{a \sqsubseteq v \wedge a \sqsubseteq t}(2)}{a \sqsubseteq v}}{b \sqsubseteq v}(\text{T}) \quad \dfrac{\dfrac{}{b \sqsubseteq a \wedge b \sqsubseteq u}(1)}{b \sqsubseteq u}}{b \sqsubseteq v \wedge b \sqsubseteq u}}{\exists z(z \sqsubseteq v \wedge z \sqsubseteq u),\ i.e.\ v \circ u}}{v \circ u}(1)}{v \circ u}(2)$$

Lemma 7

$$\dfrac{v \sqsubseteq t+u \qquad \begin{matrix}\dfrac{}{a \sqsubseteq t}(i) \\ \vdots \\ a \circ u\end{matrix}}{v \circ u}(i)$$

(a parametric in right-hand subproof)

Proof.

$$\dfrac{\dfrac{v\sqsubseteq t+u}{\exists y((y=t\vee y=u)\wedge v\circ y)}\,(C1)\qquad \dfrac{\dfrac{\overline{(b=t\vee b=u)\wedge v\circ b}^{\,(1)}}{b=t\vee b=u}\qquad \dfrac{\dfrac{\overset{\overline{a\sqsubseteq t}^{\,(3)}}{\vdots}}{a\circ u}\quad \dfrac{\dfrac{\overline{\;}^{\,(2)}\quad \overline{(b=t\vee b=u)\wedge v\circ b}^{\,(1)}}{b=t\qquad v\circ b}}{v\circ t}\,(3)(L6)}{v\circ u}\qquad \dfrac{\dfrac{\overline{\;}^{\,(2)}\quad \overline{(b=t\vee b=u)\wedge v\circ b}^{\,(1)}}{b=u\qquad v\circ b}}{v\circ u}}{v\circ u}\,(2)}{v\circ u}\,(1)$$

5.3 Results using only definitions and the introduction and elimination rules for $\bigsqcup$

Lemma 8 $\dfrac{s\sqsubseteq t \quad s\rfloor u}{s\sqsubseteq(t-u)}$

Proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{s\sqsubseteq t}{\exists! s}\quad \dfrac{s\sqsubseteq t\quad s\rfloor u}{s\sqsubseteq t\wedge s\rfloor u}}{\exists y(y\sqsubseteq t\wedge y\rfloor u)}}{\exists!\bigsqcup y(y\sqsubseteq t\wedge y\rfloor u)}\,(UEF)}{\bigsqcup y(y\sqsubseteq t\wedge y\rfloor u)=\bigsqcup y(y\sqsubseteq t\wedge y\rfloor u)}\quad \dfrac{s\sqsubseteq t\quad s\rfloor u}{s\sqsubseteq t\wedge s\rfloor u}\quad \dfrac{s\sqsubseteq t}{\exists! s}}{s\sqsubseteq\bigsqcup y(y\sqsubseteq t\wedge y\rfloor u)}\,(\sqcup\text{-E1})$$

$$\text{i.e., } s\sqsubseteq(t-u)$$ □

Lemma 9 $\dfrac{\exists!\bigsqcup x(\Phi x\wedge x\sqsubseteq t)}{\exists! t}$

Proof

$$\dfrac{\exists!\bigsqcup x(\Phi x\wedge x\sqsubseteq t)\qquad \dfrac{\overline{b=\bigsqcup x(\Phi x\wedge x\sqsubseteq t)}^{\,(2)}\quad \dfrac{\overline{a\sqsubseteq t}^{\,(1)}}{\exists! t}}{\exists! t}\,(1)\ (\sqcup\text{-E2})}{\exists! t}\,(2)$$ □

Lemma 10 $\dfrac{t=\bigsqcup x(\Phi x\wedge x\sqsubseteq u)\quad \Phi v\quad v\sqsubseteq u\quad v\sqsubseteq s\quad s\rfloor t}{\bot}$

Proof

$$\dfrac{\dfrac{v\sqsubseteq s\qquad \dfrac{t=\bigsqcup x(\Phi x\wedge x\sqsubseteq u)\quad \dfrac{\Phi v\quad v\sqsubseteq u}{\Phi v\wedge v\sqsubseteq u}\quad \dfrac{v\sqsubseteq u}{\exists! v}}{v\sqsubseteq t}\,(\sqcup\text{-E1})}{s\circ t}\qquad s\rfloor t}{\bot}$$ □

Corollary 2 $\dfrac{\Phi v\quad s\sqsubset r\quad s\rfloor t\quad r\sqsubseteq u\quad v\sqsubseteq s\quad t=\bigsqcup y(\Phi y\wedge y\sqsubseteq u)}{\bot}$

Proof

$$\dfrac{t=\bigsqcup y(\Phi y\wedge y\sqsubseteq u)\quad \Phi v\quad \dfrac{\dfrac{v\sqsubseteq s\quad \dfrac{s\sqsubset r}{s\sqsubseteq r}}{v\sqsubseteq r}\,(T)\quad r\sqsubseteq u}{v\sqsubseteq u}\,(T)\quad v\sqsubseteq s\quad s\rfloor t}{\bot}\,(L10)$$ □

5.4 Results using definitions and all rules

Lemma 11 $\dfrac{\exists! u}{\exists! t+u}$

Proof

$$\cfrac{\cfrac{\exists! u}{\exists x(x=t \lor x=u)}\,\text{(Logic)}}{\exists! \bigsqcup x(x=t \lor x=u)}\,\text{(UEF)}$$

$$\text{i.e., } \exists! t+u$$

□

Lemma 12 $\dfrac{\exists! u}{u \sqsubseteq t+u}$

Proof.

$$\cfrac{\cfrac{\exists! u}{\exists! t+u,\ i.e.\ \exists x\, x=t+u}\,\text{(L11)} \qquad \cfrac{\cfrac{\cfrac{}{a=t+u,\ i.e.\ a=\bigsqcup x(x=t\lor x=u)}\,(1) \quad \cfrac{\cfrac{\exists! u}{u=u}}{u=t\lor u=u} \quad \exists! u}{u \sqsubseteq a}\,(\sqcup\text{-E1}) \qquad \cfrac{}{a=t+u}\,(1)}{u \sqsubseteq t+u}\,(1)}{u \sqsubseteq t+u}$$

Lemma 13 $\dfrac{\exists! t}{t=\bigsqcup x\, x=t}$

Proof

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{a=t}\,(1)}{\exists! t}\,\text{(R)}}{t \sqsubseteq t} \quad \cfrac{}{a=t}\,(1)}{a \sqsubseteq t} \qquad \cfrac{\exists! t,\ i.e.}{\exists x\, x=t} \qquad \exists! t \qquad \cfrac{\exists! t \qquad \cfrac{\cfrac{\cfrac{\cfrac{}{a \sqsubseteq t}\,(1)}{\exists! t}}{t=t} \qquad \cfrac{\cfrac{\cfrac{\cfrac{}{a\sqsubseteq t}\,(1)}{\exists! a} \quad \cfrac{\cfrac{\cfrac{\cfrac{}{a\sqsubseteq t}\,(1)}{\exists! a}\,\text{(R)}}{a \sqsubseteq a} \quad \cfrac{}{a\sqsubseteq t}\,(1)}{a\sqsubseteq a \land a \sqsubseteq t}}{\exists z(z\sqsubseteq a \land z \sqsubseteq t),\ i.e.}}{a \circ t}}{t=t \land a\circ t}}{\exists y(y=t \land a \circ y)}}{t=\bigsqcup x\, x=t}\,(1)\ (\sqcup\text{-I})$$

□

Lemma 14 $\dfrac{t \sqsubseteq u}{t \circ u}$

Proof

$$\cfrac{\cfrac{t\sqsubseteq u}{\exists! t} \qquad \cfrac{\cfrac{\cfrac{t \sqsubseteq u}{\exists! t}\,\text{(R)}}{t \sqsubseteq t} \qquad t \sqsubseteq u}{t\sqsubseteq t \land t \sqsubseteq u}}{\exists z(z\sqsubseteq t \land z \sqsubseteq u)),\ i.e.\ t\circ u}$$

□

Lemma 15 $$\dfrac{\exists!t \quad \exists!u \quad \begin{array}{c}\overline{a\sqsubseteq t}^{\,(i)}\\ \vdots \\ a\circ u\end{array}}{t\sqsubseteq u}\,(i)$$ (a parametric in right-hand subproof)

Proof.

$$\dfrac{\dfrac{\exists!t}{t\sqsubseteq t+u}(\text{L12}) \quad \dfrac{\dfrac{\dfrac{\dfrac{}{d=u}(2) \quad \dfrac{\exists!u}{u\sqsubseteq t+u}(\text{L12})}{d\sqsubseteq t+u} \quad \exists!u \quad \dfrac{\exists!u}{\exists!t+u}(\text{L11}) \quad \dfrac{\exists!u \quad \dfrac{\dfrac{\exists!u}{u=u} \quad \dfrac{\dfrac{}{d\sqsubseteq t+u}(2) \quad \begin{array}{c}\overline{a\sqsubseteq t}^{\,(1)}\\ \vdots\\ a\circ u\end{array}}{d\circ u}(1)(\text{L7})}{u=u\wedge d\circ u}}{\exists y(y=u\wedge d\circ y)}}{t+u=\bigsqcup x\, x=u}(2)(\sqcup\text{-I}) \quad \dfrac{\exists!u}{u=\bigsqcup x\, x=u}(\text{L13})}{t+u=u}}{t\sqsubseteq u}$$

Lemma 16 *(Strong Supplementation)* $$\dfrac{\exists!t \quad \exists!u \quad \neg t\sqsubseteq u}{\exists x(x\sqsubseteq t\wedge x\rfloor u)}$$

Proof (using classical logic[3])

$$\dfrac{\dfrac{\exists!t \quad \exists!u \quad \dfrac{\dfrac{\dfrac{\dfrac{}{a\sqsubseteq t}(2)}{\exists!a} \quad \dfrac{\dfrac{}{a\sqsubseteq t}(2) \quad \dfrac{}{\neg a\circ u}(1)}{a\sqsubseteq t\wedge\neg a\circ u}}{\exists x(x\sqsubseteq t\wedge\neg x\circ u)} \quad \dfrac{}{\neg\exists x(x\sqsubseteq t\wedge\neg x\circ u)}(3)}{\dfrac{\bot}{a\circ u}(\text{CR})(1)}}{t\sqsubseteq u}(2)\ (\text{L15}) \quad \neg t\sqsubseteq u}{\dfrac{\bot}{\exists x(x\sqsubseteq t\wedge\neg x\circ u)}(\text{CR})(3)}$$

$$i.e.\ \exists x(x\sqsubseteq t\wedge x\rfloor u)$$

□

The use made here of classical reductio renders classical every subsequent result whose proof involves direct or indirect appeal to Lemma 16. Among the latter results is Weak Supplementation (Lemma 18), which will eventually be relied upon in our regimentation of Lewis's informal argument for his Second Thesis. Despite this reliance, however, it will turn out that Lewis's argument can be constructivized.

Lemma 17 $$\dfrac{t\sqsubset u}{u\not\sqsubseteq t}$$

[3] The method of proof of this result is due to Paul Hovda, and was communicated to the present author by Gabriel Uzquiano. Hovda's informal proof work has been regimented here and broken into parts that are essentially the proofs of Lemmas 6, 7 and 15.

Proof

$$\dfrac{\dfrac{\dfrac{\dfrac{t \sqsubset u}{t \sqsubseteq u} \quad \dfrac{}{u \sqsubseteq t}{\scriptstyle(1)}}{t = u}{\scriptstyle(\mathrm{AS})} \qquad t \sqsubset u}{\bot}}{u \not\sqsubseteq t}{\scriptstyle(1)}$$

□

Lemma 18 *(Weak Supplementation)* $\dfrac{u \sqsubset t}{\exists x(x \sqsubseteq t \wedge x \mathrel{\lfloor} u)}$

Proof

$$\dfrac{\dfrac{\dfrac{\begin{array}{c} u \sqsubset t \\ \text{i.e. } u \sqsubseteq t \wedge \neg u = t \end{array}}{u \sqsubseteq t}}{\exists! t} \quad \dfrac{\dfrac{\begin{array}{c} u \sqsubset t \\ \text{i.e. } u \sqsubseteq t \wedge \neg u = t \end{array}}{u \sqsubseteq t}}{\exists! u} \quad \dfrac{u \sqsubset t}{\neg t \sqsubseteq u}{\scriptstyle(\mathrm{L17})}}{\exists x(x \sqsubseteq t \wedge x \mathrel{\lfloor} u)}{\scriptstyle(\mathrm{L16})}$$

□

Note the appeal at the final step to Lemma 16 (Strong Supplementation), which was itself given a classical proof above.

Lemma 19

$$\dfrac{\exists! \bigsqcup x \Phi x \qquad \begin{array}{c} \dfrac{}{\Phi a}{\scriptstyle(i)} \\ \vdots \\ \Psi a \end{array}}{\bigsqcup x \Phi x \sqsubseteq \bigsqcup x \Psi x}{\scriptstyle(i)}$$

Proof.

$$\dfrac{\exists! \bigsqcup x\Phi x \quad \dfrac{\dfrac{\dfrac{\exists! \bigsqcup x\Phi x}{\exists x \Phi x} \quad \dfrac{{\scriptstyle(1)}\dfrac{}{\exists! a} \quad \begin{array}{c} \dfrac{}{\Phi a}{\scriptstyle(1)} \\ \vdots \\ \Psi a \end{array}}{\exists x \Psi x}}{\exists x \Psi x}{\scriptstyle(1)}}{\exists! \bigsqcup x \Psi x}{\scriptstyle(\mathrm{UEF})} \quad \dfrac{\dfrac{\dfrac{\dfrac{{\scriptstyle(3)}\dfrac{}{b \sqsubseteq \bigsqcup x\Phi x}}{\bigsqcup x\Phi x = \bigsqcup x\Phi x} \quad \dfrac{}{b \sqsubseteq \bigsqcup x\Phi x}{\scriptstyle(3)}}{\exists y(\Phi y \wedge b \circ y)}{\scriptstyle(\sqcup\text{-E4})} \quad \dfrac{{\scriptstyle(2)}\dfrac{}{\exists! a} \quad \dfrac{\begin{array}{c} \dfrac{\dfrac{}{\Phi a \wedge b \circ a}{\scriptstyle(2)}}{\Phi a} \\ \vdots \\ \Psi a \end{array} \quad \dfrac{\dfrac{}{\Phi a \wedge b \circ a}{\scriptstyle(2)}}{b \circ a}}{\Psi a \wedge b \circ a}}{\exists x(\Psi x \wedge b \circ x)}}{\exists x(\Psi x \wedge b \circ x)}{\scriptstyle(2)}}{b \circ \bigsqcup x \Psi x}}{\bigsqcup x \Phi x \sqsubseteq \bigsqcup x \Psi x}{\scriptstyle(3)\ (\mathrm{L15})}$$

Lemma 20 $\dfrac{\exists! \bigsqcup x(\Phi x \wedge x \sqsubseteq t)}{\bigsqcup x(\Phi x \wedge x \sqsubseteq t) \sqsubseteq t}$

Proof

$$\dfrac{\dfrac{\exists! \bigsqcup x(\Phi x \wedge x \sqsubseteq t) \qquad \dfrac{\dfrac{}{\Phi a \wedge a \sqsubseteq t}{\scriptstyle(1)}}{a \sqsubseteq t}}{\bigsqcup x(\Phi x \wedge x \sqsubseteq t) \sqsubseteq \bigsqcup x\, x \sqsubseteq t}{\scriptstyle(1)(\mathrm{L19})} \qquad \dfrac{\dfrac{\exists! \bigsqcup x(\Phi x \wedge x \sqsubseteq t)}{\exists! t}{\scriptstyle(\mathrm{L9})}}{t = \bigsqcup x\, x \sqsubseteq t}{\scriptstyle(\mathrm{L13})}}{\bigsqcup x(\Phi x \wedge x \sqsubseteq t) \sqsubseteq t}$$

□

Lemma 21 $$\dfrac{t \sqsubset u}{\bigsqcup y(y \sqsubseteq u \wedge y \rceil t) \sqsubset u}$$

Proof

$$\dfrac{\dfrac{\dfrac{\dfrac{t \sqsubset u}{\exists y(y \sqsubseteq u \wedge y \rceil t)}\,(\text{L}18)}{\exists!\bigsqcup y(y \sqsubseteq u \wedge y \rceil t)}\,(\text{UEF})}{\bigsqcup y(y \sqsubseteq u \wedge y \rceil t) \sqsubseteq u}\,(\text{L}20) \qquad \dfrac{\dfrac{\dfrac{t \sqsubset u}{t \sqsubseteq u} \qquad \overline{\bigsqcup y(y \sqsubseteq u \wedge y \rceil t) = u}^{\,(1)}}{t \sqsubseteq \bigsqcup y(y \sqsubseteq u \wedge y \rceil t)}}{\bot}\,(\text{L}5)}{\bigsqcup y(y \sqsubseteq u \wedge y \rceil t) \sqsubset u}\,(1)$$ □

Lemma 22 $$\dfrac{t = \bigsqcup y(\Phi y \wedge y \sqsubseteq u) \quad \Phi v \quad v \sqsubseteq \bigsqcup y(\Psi y \wedge y \rceil t) \quad \bigsqcup y(\Psi y \wedge y \rceil t) \sqsubseteq u}{\bot}$$

Proof.

$$\dfrac{\dfrac{\dfrac{t = \bigsqcup y(\Phi y \wedge y \sqsubseteq u) \qquad \dfrac{\Phi v \qquad \dfrac{v \sqsubseteq \bigsqcup y(\Psi y \wedge y \rceil t) \quad \bigsqcup y(\Psi y \wedge y \rceil t) \sqsubseteq u}{v \sqsubseteq u}}{\Phi v \wedge v \sqsubseteq u} \qquad \dfrac{v \sqsubseteq \bigsqcup y(\Psi y \wedge y \rceil t)}{\exists! v}}{v \sqsubseteq t}\,(\sqcup\text{-E1})}{v \circ t}\,(\text{L}14) \qquad \dfrac{v \sqsubseteq \bigsqcup y(\Psi y \wedge y \rceil t) \qquad {}^{(\text{L}4)}\overline{t \rceil \bigsqcup y(\Psi y \wedge y \rceil t)}}{v \rceil t}}{\bot}$$

Part II: Combining class theory with mereology

6 Introduction

In his influential monograph Lewis (1991), the late David Lewis sought to combine classical extensional mereology with class (i.e., set) theory in an illuminating and satisfying way. A notable feature of Lewis's monograph is that he used no formal symbols at all. Apart from the use of variables x, y, z etc. in informal quantified reasoning, he eschewed altogether any symbolic abbreviations of mereological relations and functors, preferring instead to write them out in English.

This informal approach was no doubt intended to ensure the widest possible readership. But it allowed for some obscurities of argumentation that this study seeks to eliminate. Lewis's proof of his Second Thesis (*ibid*., pp. 9–10) will be codified here as a rigorous natural deduction—for which, of course, abbreviatory symbols will have to be employed.

7 Lewis's main theses of his combined mereology and class theory

7.1 The clear theses that are easily regimented

We have to bear in mind that classical mereology, since the work of Leonard and Goodman, has been concerned only with individuals, and not at all with classes. When venturing to extend mereology so as to include clasSES both in the fields of mereological relations and in the domains of mereological operations, great care is needed in specifying exactly the ways in which individuals differ from classes (in their 'mereological' behavior) and the ways in which they resemble them. We shall use the sortal specifications Cx (x is a class; i.e., x is a thing that has members) and Ix (x is an individual). Lewis made a start on this project in Lewis (1991). Here we shall elaborate on his theses, presenting them in a different order so as to facilitate both explanation and criticism. We have already noted that for Lewis the following principle holds.

Non-emptiness of classes: Every class has at least one member.

Lewis did not state this as a separate principle, but of course he could have, and should have. So here we state it for him. Expressed as a rule:

(NON-EMPTINESS OF CLASSES)
$$\dfrac{Ct \qquad \begin{array}{c}\overline{a \in t}^{\,(i)} \\ \vdots \\ \psi\end{array}}{\psi}\,(i)$$

We turn now to those theses that Lewis did state explicitly.

First Thesis: One class is part of another iff the first is a subclass of the second.[4]

Expressed as a pair of formal rules:

(FIRST THESIS)
$$\dfrac{Ct \quad Cu \quad t \sqsubseteq u}{t \subseteq u} \qquad \dfrac{Ct \quad Cu \quad t \subseteq u}{t \sqsubseteq u}$$

Comment. Note the restriction to classes. Provided that both t and u *are classes* (which would have to be established independently), we can treat $t \sqsubseteq u$ and $t \subseteq u$ as equivalent.

Priority Thesis: No class is part of any individual.[5]

Expressed as a formal rule:

(PRIORITY)
$$\dfrac{Ct \quad Iu \quad t \sqsubseteq u}{\bot}$$

Comment. The 'priority' here is one of ontological grounding. The motivating idea is that all individuals can be 'given first', and that they are *memberless*. Ontologically, the formation of classes presupposes the formation, 'beforehand', of all individuals.

Fusion Thesis: Any fusion of individuals is itself an individual.[6]

4 *Ibid*., p. 4.

5 *Ibid*., p. 7.

6 *Ibid*., p. 7.

Expressed as a formal rule:

(FUSION)
$$\dfrac{\exists!\bigsqcup x\,\Phi x \qquad \begin{matrix}\overline{\Phi a}^{(i)}\\ \vdots \\ Ia\end{matrix}}{I(\bigsqcup x\,\Phi x)}{\scriptstyle (i)}$$

Comment. This is unproblematic, as a 'pure' thesis of mereology. It involves no mention of classes.

Thesis of Unrestricted Composition: [W]henever there are some things, no matter how many or how unrelated or how disparate in character they may be, they have a mereological fusion.[7]

This is our earlier formal rule

(UNRESTRICTED EXISTENCE OF FUSIONS) $$\frac{\exists x\,\Phi x}{\exists!\bigsqcup x\,\Phi x}\ .$$

Comment. Although controversially strong (for some philosophers) as a thesis of mereology even when restricted to individuals, Lewis makes the composition thesis *even stronger*, by allowing it to apply to *things* in general—among which, of course, will be classes, as well as fusions of classes with individuals, and also fusions of classes with classes. (Note that these are *here* being treated, at this stage, as *prima facie* disparate kinds of entities.)

7.2 The less-than-clear Division Thesis, and how it can be regimented

In his monograph, Lewis states a thesis that calls for some explication.

Division Thesis: Reality divides exhaustively into individuals and classes.[8]

This statement was repeated without revision in his subsequent article Lewis (1993), even though he described that article (p. 3) as 'an abridgement of *Parts of Classes* not as it is, but as it would have been had I known sooner what I know now.' In the later reprint of Lewis (1993) in Lewis (1998), however, he added the following (at p. 208, dated 1996):[9]

> The Division Thesis is badly worded. The meaning that I intended, and that is required by subsequent discussions here and in *Parts of Classes*, is better expressed as follows: everything is either an individual, or a class, or a fusion of an individual and a class. I thank Daniel Nolan for pointing out the problem to me.

Though Lewis thereby mitigates the problem of the obscurity of his original statement of the Division Thesis, the reader is still left with the problem of how that

[7] *Ibid*., p. 7.

[8] *Ibid*., p. 7.

[9] Thanks to Ben Caplan for pointing this out.

thesis—on this clarified construal—is itself to be justified or motivated. Something is offered is this regard below.

We need to remind ourselves that it is *not* the case that everything is either an individual or a class. For, the fusion of the individual Julius Caesar with the class of all natural numbers is neither an individual nor a class. It is what Lewis calls a *mixed fusion*. The *plausible* thought behind the Division Thesis is something like this:

(π) apart from individuals and classes, the only other things that exist will be fusions of them, and fusions of things already thus formed, and fusions of such fusions, … and so on.

(π) is plausible because, in light of the principle of Unrestricted Composition, we know we have to acknowledge the existence of all these fusions anyway; and therefore all we are doing, in stating the Division Thesis on this construal, is saying that what can *thus* be composed is *all* that exists.

A *less* plausible thought, which Lewis subsequently needs, is rather this (which is not an immediate or obvious consequence of the plausible thought (π)):

(λ) apart from individuals and classes, the only other things that exist will be fusions of an individual with a class.

How can this thought (λ) be made plausible? (Recall that it is the thought for which Lewis settled in the footnote referred to above.) Let us consider the kinds of things already countenanced, given the 'formation operations' to which Lewis has already committed himself. One starts with (all) individuals. (These will make true all of classical extensional mereology, in application just to them.) Then one 'adds' classes, using individuals as *Urelemente*. This process of 'addition' (or ontological expansion) takes place in accordance with well-understood and codified principles of class-abstraction, and of outright and conditional class-existence.

But from within mereology, fusion can now expand its reach: one can form fusions of individuals and classes, and indeed do so *unrestrictedly*. Among such fusions will be the following, accreting in a hierarchy of *fusion types*.

(i) Fusions of *individuals* with *individuals*. These, by the Fusion Thesis and by the lights of classical mereology, will be individuals.
(ii) Fusions of *individuals* with *classes*. These are what Lewis calls 'mixed fusions'. They are neither individuals nor classes; but they do not falsify the Division Thesis, suitably explicated.
(iii) Fusions of *classes* with *classes*. (That such fusions might not themselves *be* classes is a possibility that Lewis appears to neglect or overlook.)
(iv) Fusions of *individuals* with *fusions of individuals with classes*.
(v) Fusions of *individuals* with *fusions of classes with classes*.
(vi) Fusions of *fusions of individuals with classes* with *fusions of individuals with classes*.
(vii) Fusions of *classes* with *fusions of individuals with classes*.
(viii) Fusions of *classes* with *fusions of classes with classes*.

It is category (iii) on which we need to focus. Why is it important? First, it is obviously there in logical space, alongside the acknowledged categories (i) and (ii). Secondly, we are trying to excogitate whatever principles govern the co-existence, within an expanded ontology, of individuals, classes, and fusions thereof, as well as the further fusions that appear higher up in the hierarchy of fusion types. In the absence of any reducibility principles, the foregoing list threatens to ramify beyond item (viii), making the Division Thesis difficult to state in formal terms.

It is clear that Lewis was working with the tacit assumption that *the fusion of any class with a class is a class.* This principle should therefore have been made explicit as a second half of the Fusion Thesis. Expressed as a formal rule in two parts, the Fusion Thesis would then read:

(FUSION)

$$\dfrac{\exists!\bigsqcup x\Phi x \qquad \begin{array}{c}\overline{\Phi a}^{\,(i)}\\ \vdots \\ Ia\end{array}}{I(\bigsqcup x\Phi x)}{\scriptstyle (i)} \qquad\qquad \dfrac{\exists!\bigsqcup x\Phi x \qquad \begin{array}{c}\overline{\Phi a}^{\,(i)}\\ \vdots \\ Ca\end{array}}{C(\bigsqcup x\Phi x)}{\scriptstyle (i)}$$

Would this prevent the threatened ramification and unbounded ascent of the hierarchy of fusion types? We would now have the following possibilities.

- (i) Fusions of *individuals* with *individuals*. These, by the Fusion Thesis and by the lights of classical mereology, will be individuals.
- (ii) Fusions of *individuals* with *classes*. These are what Lewis calls ‘mixed fusions’. They are neither individuals nor classes; but they do not falsify the Division Thesis, suitably explicated.
- (iii)′ Fusions of *classes* with *classes*. These, by the amended Fusion Thesis, will be classes.
- (iv) Fusions of *individuals* with *fusions of individuals with classes*.
- (v)′ Fusions of *individuals* with *fusions of classes with classes*. These, by (iii)′, will be cases of (ii), that is, mixed fusions.
- (vi) Fusions of *fusions of individuals with classes* with *fusions of individuals with classes*.
- (vii) Fusions of *classes* with *fusions of individuals with classes*.
- (viii)′ Fusions of *classes* with *fusions of classes with classes*. These, by the amended Fusion Thesis, will be classes.

Together, items (iv), (vi) and (vii) still pose a threat of ramification and unbounded ascent. How is the threat to be averted?

The obvious solution would be to ensure that things of kinds (iv), (vi) and (vii) are actually of kind (ii)—namely, mixed fusions of individuals with classes. We now investigate what deeper principle would secure this—a principle that Lewis omitted to formulate.

Let us abbreviate $\bigsqcup x(x = t \vee x = u)$ as $(t + u)$. Let us use i and c, with numerical subscripts where necessary, as sortal variables ranging over individuals and classes respectively. The obvious way to reduce things of kinds (iv), (vi) and (vii) to things of kind (ii) is to appeal to principles of commutativity and associativity governing fusion, so as to ensure that

(iv) $i_1 + (i_2 + c) = (i_1 + i_2) + c$
(vi) $(i_1 + c_1) + (i_2 + c_2) = (i_1 + i_2) + (c_1 + c_2)$
(vii) $c_1 + (i + c_2) = i + (c_1 + c_2)$

Note that, for any complex fusion-term of the form $t_1 + \cdots + t_n$, no matter how the parentheses are placed within it, the thing denoted by any constituent term t_i is a part of the fusion denoted by the complex fusion-term.

We do not wish, however, to limit ourselves by covertly assuming that all fusion-terms are of the finite form $t_1 + \cdots + t_n$. With the general fusion-term $\bigsqcup x \Phi x$, for example, there could well be infinitely many Φs whose fusion is to be denoted by that term. The general principle that we are seeking would therefore be that everything is the fusion of (i) the fusion of all its class-parts, with (ii) the fusion of all its individual parts (which is clearly what is exemplified in the foregoing identities, given the amended Fusion Thesis). We can call this the *canonical decomposition* of the thing in question.

Canonical Decomposition Thesis. Everything is a fusion of (at most) two things: (i) the fusion of all its class-parts, with (ii) the fusion of all its individual parts.

Expressed formally:

(CANONICAL DECOMPOSITION)

$$\forall x\ x = \bigsqcup z(z = \bigsqcup y(Cy \wedge y \sqsubseteq x) \vee z = \bigsqcup y(Iy \wedge y \sqsubseteq x)),$$

or, using the obvious pattern of abbreviation $x_\Phi =_{df} \bigsqcup y(\Phi y \wedge y \sqsubseteq x)$ for 'the fusion of the Φ-parts of x':

$$\forall x\ x = \bigsqcup z(z = x_C \vee z = x_I).$$

That is, everything is the fusion of the fusion of its C-parts with the fusion of its I-parts. It is not required that both these fusions exist; but at least one of them will.

That leaves open the three mutually exclusive but jointly exhaustive possibilities, concerning any thing x, that

1. x is an individual; or
2. x is a class; or
3. x is a 'mixed fusion' of an individual with a class.

To see this, we reason as follows. If x has both some class-parts and some individual parts, then the Canonical Decomposition Thesis indicates that case (3) is realized: x is a 'mixed fusion' of an individual with a class. If, on the one hand, x has no class-parts, then the canonical decomposition of x, namely

$$\bigsqcup z(z = \bigsqcup y(Cy \wedge y \sqsubseteq x) \vee z = \bigsqcup y(Iy \wedge y \sqsubseteq x))$$

—or, more briefly,

$$\bigsqcup z(z = x_C \vee z = x_I)$$

—boils down to

$$\bigsqcup z\, z = \bigsqcup y(Iy \wedge y \sqsubseteq x), \quad \text{or} \quad \bigsqcup z\, z = x_I,$$

which is provably identical to x_I, hence to x itself and, moreover, by the Fusion Thesis, is itself an *individual*. If, on the other hand, x has no individual parts, then the canonical decomposition of x boils down instead to

$$\bigsqcup z\, z = \bigsqcup y(Cy \wedge y \sqsubseteq x), \text{ or } \bigsqcup z\, z = x_C,$$

which again is provably identical to x_C, hence to x itself and, moreover, by the *amended* Fusion Thesis, is itself a *class*.

We have therefore proved the following

Lemma. *Everything is either an individual, or a class, or the fusion of an individual with a class.*

This Lemma the reader will recognize as the 'less plausible' thought (λ) above—the construal of the informal Division Thesis that Lewis actually needs for his argumentative purposes. It would now appear reasonable to offer the Canonical Decomposition Thesis as a deeper, more elegant ground for Lewis's Division Thesis—provided, of course, that the Fusion Thesis is amended as indicated, so as to deal with classes as well as with individuals.

If one takes the Lemma just stated as a suitably rigorous form of expression for the Division Thesis, a useful formalization would be the trilemmatic rule

(DIVISION)
$$\dfrac{\exists! t \qquad \begin{array}{c} \overline{It}^{(i)} \\ \vdots \\ \psi/\bot \end{array} \qquad \begin{array}{c} \overline{Ct}^{(i)} \\ \vdots \\ \psi/\bot \end{array} \qquad \begin{array}{c} \overline{\exists i \exists c(Ii \wedge Cc \wedge t = i + c)}^{(i)} \\ \vdots \\ \psi/\bot \end{array}}{\psi/\bot}{\scriptstyle (i)}$$

Here the notation '$\psi/\bot$' adverts to the possibility that one or more of the three cases might close off with absurdity. If all three do, then the main conclusion is $\bot$; otherwise, the main conclusion is the sentence ψ that is the conclusion of the non-absurd cases.

The trichotomy expressed by this rule is clearly at work, within Lewis's argument, whenever he invokes his Division Thesis.

8 Further formal results, using Lewis's rules for parts and classes

Lemma 23 $\dfrac{Ct \quad It}{\bot}$

Proof
$$\dfrac{Ct \qquad It \qquad \dfrac{\dfrac{It}{\exists! t}{\scriptstyle (R)}}{t \sqsubseteq t}{\scriptstyle (P1)}}{\bot}$$
□

Lemma 24 $$\frac{\exists!\bigsqcup j(Ij \wedge \Psi j)}{I(\bigsqcup j(Ij \wedge \Psi j))}$$

Proof $$\dfrac{\exists!\bigsqcup x\Phi x \qquad \dfrac{\dfrac{}{Ia \wedge \Psi a}{\scriptstyle(1)}}{Ia}}{I(\bigsqcup x\Phi x)}{\scriptstyle(1)\ (FT)}$$ □

Corollary 3 $$\frac{t = \bigsqcup j(Ij \wedge j \sqsubseteq u)}{It}$$

Proof $$\dfrac{\dfrac{\dfrac{t = \bigsqcup j(Ij \wedge j \sqsubseteq u)}{\exists!\bigsqcup j(Ij \wedge j \sqsubseteq u)}}{I(\bigsqcup j(Ij \wedge j \sqsubseteq u))}{\scriptstyle(L24)} \qquad t = \bigsqcup j(Ij \wedge j \sqsubseteq u)}{It}$$ □

Lemma 25 $$\frac{Cu \qquad \exists!\bigsqcup j(Ij \wedge j \sqsubseteq u)}{\bigsqcup y(y \sqsubseteq u \wedge y \wr \bigsqcup j(Ij \wedge j \sqsubseteq u)) \sqsubset u}$$

Proof $$\dfrac{\dfrac{\dfrac{\exists!\bigsqcup j(Ij \wedge j \sqsubseteq u)}{\bigsqcup j(Ij \wedge j \sqsubseteq u) \sqsubseteq u}{\scriptstyle(L20)} \qquad \dfrac{\dfrac{Cu \quad \dfrac{}{u = \bigsqcup j(Ij \wedge j \sqsubseteq u)}{\scriptstyle(1)}}{C(\bigsqcup j(Ij \wedge j \sqsubseteq u))} \qquad \dfrac{\dfrac{\exists!\bigsqcup j(Ij \wedge j \sqsubseteq u)}{I(\bigsqcup j(Ij \wedge j \sqsubseteq u))}{\scriptstyle(L24)}}{}{\scriptstyle(L23)}}{\bot}}{\bigsqcup j(Ij \wedge j \sqsubseteq u) \sqsubset u}{\scriptstyle(1)}}{\bigsqcup y(y \sqsubseteq u \wedge y \wr \bigsqcup j(Ij \wedge j \sqsubseteq u)) \sqsubset u}{\scriptstyle(L21)}$$

□

Lemma 26 $$\dfrac{\exists!t \qquad \begin{matrix} \underbrace{\dfrac{}{Ia}{\scriptstyle(i)}\ ,\ \dfrac{}{a \sqsubseteq t}{\scriptstyle(i)}} \\ \vdots \\ \bot \end{matrix}}{Ct}{\scriptstyle(i)}$$

Proof.

$$\dfrac{\exists!t \qquad \begin{matrix} \underbrace{\dfrac{}{It}{\scriptstyle(1)}\ ,\ \dfrac{\dfrac{\dfrac{}{It}{\scriptstyle(1)}}{\exists!t}}{t \sqsubseteq t}{\scriptstyle(R)}} \\ \vdots\ {\scriptstyle a/t} \\ \bot \end{matrix} \qquad \dfrac{\dfrac{}{Ct}{\scriptstyle(1)} \qquad \dfrac{\dfrac{}{\exists i\exists c(Ii \wedge Cc \wedge t = i + c)}{\scriptstyle(1)} \qquad \begin{matrix} \underbrace{\dfrac{\dfrac{}{Ib \wedge Cd \wedge t = b + d}{\scriptstyle(2)}}{Ib} \quad \dfrac{\dfrac{\dfrac{}{\exists!b}{\scriptstyle(2)}}{b \sqsubseteq b + d} \quad \dfrac{\dfrac{}{Ib \wedge Cd \wedge t = b + d}{\scriptstyle(2)}}{t = b + d}}{b \sqsubseteq t}} \\ \vdots\ {\scriptstyle a/b} \\ \bot \end{matrix}}{\bot}{\scriptstyle(2)\ 2\times\exists\text{-E}}}{\bot}{\scriptstyle(1)\ (D)}}{Ct}$$

Lemma 27

$$\frac{t = \bigsqcup j(Ij \wedge j \sqsubseteq u) \quad \exists! \bigsqcup y(y \sqsubseteq u \wedge y \rceil t)}{C(\bigsqcup y(y \sqsubseteq u \wedge y \rceil t))}$$

Proof.

$$\dfrac{\exists! \bigsqcup y(y \sqsubseteq u \wedge y \rceil t) \qquad \dfrac{\overline{Ia}^{\,(1)} \quad \overline{a \sqsubseteq \bigsqcup y(y \sqsubseteq u \wedge y \rceil t)}^{\,(1)} \quad t = \bigsqcup j(Ij \wedge j \sqsubseteq u) \quad \dfrac{\exists! \bigsqcup y(y \sqsubseteq u \wedge y \rceil t)}{\bigsqcup y(y \sqsubseteq u \wedge y \rceil t) \sqsubseteq u}\,(\text{L20})}{\bot}\,(\text{L22})}{C(\bigsqcup y(y \sqsubseteq u \wedge y \rceil t))}\,(1)\ (\text{L26})$$

Lemma 28 $\dfrac{t \sqsubseteq u \qquad \neg Ct}{\exists x(Ix \wedge x \sqsubseteq u)}$

Proof.

$$\dfrac{\dfrac{\dfrac{t \sqsubseteq u}{\exists! t} \quad \dfrac{\overline{It}^{\,(2)} \quad t \sqsubseteq u}{It \wedge t \sqsubseteq u}}{\exists x(Ix \wedge x \sqsubseteq u)} \qquad \dfrac{\overline{Ct}^{\,(2)} \quad \neg Ct}{\bot} \qquad \dfrac{\overline{\exists i \exists c(Ii \wedge Cc \wedge t{=}i{+}c)}^{\,(2)} \quad \dfrac{\overline{\exists! i}^{\,(1)} \quad \dfrac{\dfrac{\overline{Ii \wedge Cc \wedge t{=}i{+}c}^{\,(1)}}{Ii} \quad \dfrac{\dfrac{\dfrac{\overline{\exists! i}^{\,(1)}}{i \sqsubseteq i{+}c}\,(\text{L12}) \quad \dfrac{\overline{Ii \wedge Cc \wedge t{=}i{+}c}^{\,(1)}}{t{=}i{+}c}}{i \sqsubseteq t} \quad t \sqsubseteq u}{i \sqsubseteq u}\,(\text{T})}{Ii \wedge i \sqsubseteq u}}{\exists x(Ix \wedge x \sqsubseteq u)}}{\exists x(Ix \wedge x \sqsubseteq u)}\,(1)2{\times}\exists\text{-E}}{\exists x(Ix \wedge x \sqsubseteq u)}\,(2)(\text{D})$$

Lemma 29 $\dfrac{Ct \qquad Iu}{\exists y(y \sqsubseteq t \wedge y \rceil u)}$

Proof

$$\dfrac{\dfrac{\dfrac{Ct \quad Iu \quad \overline{t \sqsubseteq u}^{\,(1)}}{\bot}\,(\text{P})}{t \not\sqsubseteq u}\,(1)}{\exists y(y \sqsubseteq t \wedge y \rceil u)}\,(\text{L16})$$

□

Lemma 30 $\dfrac{Ct \qquad u = \bigsqcup j(Ij \wedge j \sqsubseteq t)}{t - u \sqsubset t}$

Proof

$$\dfrac{\overline{t - u =_{df} \bigsqcup y(y \sqsubseteq t \wedge y \rceil u)} \qquad \dfrac{\dfrac{Ct \quad \dfrac{u = \bigsqcup j(Ij \wedge j \sqsubseteq t)}{\exists! \bigsqcup j(Ij \wedge j \sqsubseteq t)}}{\bigsqcup y(y \sqsubseteq t \wedge y \rceil \bigsqcup j(Ij \wedge j \sqsubseteq t)) \sqsubset t}\,(\text{L25}) \quad u = \bigsqcup j(Ij \wedge j \sqsubseteq t)}{\bigsqcup y(y \sqsubseteq t \wedge y \rceil u) \sqsubset t}}{t - u \sqsubset t}$$

□

Lemma 31 $\dfrac{Ct \qquad u = \bigsqcup j(Ij \wedge j \sqsubseteq t)}{C(t - u)}$

Proof

$$\dfrac{t-u =_{df} \bigsqcup y(y\sqsubseteq t \wedge y\rceil u) \qquad \dfrac{\dfrac{\dfrac{Ct \qquad \dfrac{\dfrac{u=\bigsqcup j(Ij\wedge j\sqsubseteq t)}{\exists!\bigsqcup j(Ij\wedge j\sqsubseteq t)}}{Iu}\,\text{(L24)}}{\exists y(y\sqsubseteq t\wedge y\rceil u)}\,\text{(L29)}}{\exists!\bigsqcup y(y\sqsubseteq t\wedge y\rceil u)} \qquad u=\bigsqcup j(Ij\wedge j\sqsubseteq t)}{C(\bigsqcup y(y\sqsubseteq t\wedge y\rceil u))}\,\text{(L27)}}{C(t-u)}$$

□

Lemma 32 $\dfrac{Ct \quad v\in t}{\{v\}\sqsubseteq t}$

Proof

$$\dfrac{\dfrac{\dfrac{v\in t}{\exists! v}}{C\{v\}} \qquad Ct \qquad \dfrac{v\in t}{\{v\}\subseteq t}}{\{v\}\sqsubseteq t}\,\text{(P)}$$

□

Lemma 33 $\dfrac{\exists! v \quad Ct \quad u=t_I \quad v\notin(t-u)}{\{v\}\not\sqsubseteq(t-u)}$

Proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\exists! v}{C\{v\}} \qquad \dfrac{Ct \quad u=\bigsqcup j(Ij\wedge j\sqsubseteq t)}{C(t-u)}\,\text{(L31)} \qquad \dfrac{}{\{v\}\sqsubseteq(t-u)}\,(1)}{\{v\}\subseteq(t-u)}\,\text{(I)}}{v\in(t-u)} \qquad v\notin(t-u)}{\bot}}{v\not\sqsubseteq(t-u)}\,(1)$$

Corollary 4 $\dfrac{s\sqsubseteq t \quad s\not\sqsubseteq u \quad s\not\sqsubseteq(t-u)}{\exists w(w\sqsubset s\wedge w\rceil u)}$

Proof (classical)[10].

$$\dfrac{\dfrac{s\not\sqsubseteq u}{\exists w(w\sqsubseteq s\wedge w\rceil u)}\,\text{(L16)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{w\sqsubseteq s}\,(3) \qquad \dfrac{\dfrac{\text{(CR)}\,\dfrac{\dfrac{\dfrac{s\sqsubseteq t \quad \dfrac{}{s\rceil u}\,(1)}{s\sqsubseteq(t-u)}\,\text{(L8)} \qquad s\not\sqsubseteq(t-u)}{\bot}}{s\circ u}\,(1) \qquad \dfrac{}{w=s}\,(2)}{w\circ u} \qquad \dfrac{}{w\rceil u}\,(3)}{\bot}}{w\sqsubset s}\,(2) \qquad \dfrac{}{w\rceil u}\,(3)}{w\sqsubset s\wedge w\rceil u} \qquad \dfrac{\dfrac{}{w\sqsubseteq s}\,(3)}{\exists! w}}{\exists w(w\sqsubset s\wedge w\rceil u)}}{\exists w(w\sqsubset s\wedge w\rceil u)}\,(3)$$

10 Thanks to Salvatore Florio for spotting the classical step in this proof.

Lemma 34 $\dfrac{Ct \quad v\in t \quad \exists!v \quad u=t_I \quad v\notin(t-u)}{\exists w(w\sqsubset\{v\}\wedge w\rfloor u)}$

Proof

$$\dfrac{\dfrac{Ct \quad v\in t}{\{v\}\sqsubseteq t}\,(L32) \qquad \dfrac{\dfrac{\exists!v}{C\{v\}} \qquad \dfrac{\dfrac{u=t_I}{It_I}\,(C3) \qquad u=t_I}{Iu}}{\{v\}\not\sqsubseteq u} \qquad \dfrac{\exists!v \quad Ct \quad u=t_I \quad v\notin(t-u)}{\{v\}\not\sqsubseteq(t-u)}\,(L33)}{\exists w(w\sqsubset\{v\}\wedge w\rfloor u)}\,(C4)$$

□

9 Lewis's puzzling preference for fusion over class-formation

The class-theoretic analogue of the mereological Thesis of Unrestricted Composition (our (UEF)) would be something tantamount to Naïve Comprehension. Adapting Lewis's own words, one might try to say

> Whenever there are some things, no matter how many or how unrelated or how disparate in character they may be, they form a class.

But this, of course, would lead to Russell's paradox, for which the reader will need no instruction.

Set theory has survived the paradox by abandoning Naïve Comprehension while yet providing bounteous means of forming sets (or classes), including ones built up out of *Urelemente*, among which can be what are here being calling individuals. There could also, of course, be *abstract Urelemente*, such as the natural numbers (taken to be *sui generis*, and not reductively identified with any canonical progression of pure sets such as the von Neumann ordinals).

From now on we shall use 'set' and 'class' interchangeably. The class-theoretic analogue of the abstraction rule for fusions is the following introduction rule for set- (or class-) abstraction.[11]

({ }-I) $$\dfrac{\begin{array}{c}\overline{\Phi(a)}^{(i)}\ ,\ \overline{\exists!a}^{(i)}\\ \vdots\\ a\in t\end{array} \qquad \exists!t \qquad \begin{array}{c}\overline{a\in t}^{(i)}\\ \vdots\\ \Phi(a)\end{array}}{t=\{x|\Phi(x)\}}\,(i)$$

Corresponding to this introduction rule is the following three-part elimination rule.

({ }-E) $$\dfrac{t=\{x|\Phi(x)\} \quad \Phi(u) \quad \exists!u}{u\in t} \qquad \dfrac{t=\{x|\Phi(x)\}}{\exists!t} \qquad \dfrac{t=\{x|\Phi(x)\} \quad u\in t}{\Phi(u)}$$

These rules for class-formation are ontologically non-committal, since they are formulated with existential presuppositions within a free logic. They serve to pin

[11] See Tennant 2004 for further details.

down the conceptual connections among set-abstraction, membership and predication (or property-possession). Consistently with their non-committal character, however, the set-theorist seeks to provide further axioms or rules specifying which sets exist outright, and which sets exist conditionally upon the existence of others. By way of illustration, it is enough to consider here only the Axiom of Empty Set (outright existence), and the Pair Set Axiom (conditional existence), both expressed here as rules.

Existence of Empty Set. $\overline{\exists!\{x \mid x \neq x\}}$

Existence of Pair Sets. $$\frac{\exists!t \qquad\qquad \exists!u}{\exists!\{x \mid x = t \vee x = u\}}$$

The ‘conditional existence’ rule of Pair Sets employs the existential premises $\exists!t$ and $\exists!u$. It is important to appreciate that this means that we can form the pair set of any two *things* whatsoever—provided only that they are not proper classes. They can be individuals, or ‘set-sized’ classes, or *any other kind of thing* ‘formed’ by steps of fusion and/or class-abstraction. Class-abstraction, like fusion, is a *universally* applicable operation, in the sense that one is not restricted to any particular categories of things in order to find potential members for a class that is to be formed. Provided only that we avoid paradox by not trying to form classes that are too large, there should be no restriction at all on what the *members* of licit classes might be. If we can supply their identity and existence conditions, they should enjoy the same privileges of potential class-membership as do individuals and classes.

It is therefore puzzling that Lewis himself seeks to restrict the currently canonical means of conditional class-formation[12] so as to exclude those hybrid things that he allows in, such as fusions of individuals and classes. He argues as follows:[13]

> If we accept the mixed fusions of individuals and classes, must we also posit some previously ignored classes that have these mixed fusions as members? No; we can hold the mixed fusions to be ineligible for membership. Mixed fusions are forced upon us by the principle of Unrestricted Composition. Classes containing them are not likewise forced upon us by a corresponding principle of unrestricted class-formation. That principle is doomed in any case: we dare not say that whenever there are some things, there is a class of them, because there can be no class of all non-self-members. Nor are classes containing mixed fusions forced upon us in any other way. Let us indulge our offhand reluctance to believe in them.

One can concede the point that ‘[c]lasses containing [mixed fusions] are not … forced upon us by a … principle of unrestricted class-formation’ corresponding to the principle of Unrestricted Composition. But that does *not* guarantee that ‘classes containing mixed fusions [are not] forced upon us *in any other way*.’ [Emphasis added.] For, as we have just pointed out, pair sets (hence also: singletons) of mixed fusions should be

[12] These are: pair sets; unions; power sets; and replacement.

[13] *Ibid*., p. 8.

just as licit as fusions of the same. Whence does Lewis arrive at any *principled* (not: 'offhand') reluctance to follow through with both the letter and the spirit of careful (non-naïve) class-abstraction?

10 Regimenting Lewis's argument for his Second Thesis

We turn now to our final task, which is to regiment Lewis's argument for his Second Thesis.

Second Thesis: No class has any part that is not a class.[14] (Equivalently: All parts of classes are classes.)

Expressed as a formal rule:

$$(\text{SECOND THESIS}) \qquad \frac{t \sqsubseteq u \quad Cu}{Ct}$$

Of the Second Thesis, Lewis wrote that it seemed to him

> far less evident than the First Thesis; it needs an argument. And an argument needs premises.

He sought to construct an argument using as premises only the First Thesis, the Division Thesis, the Priority Thesis, and the (unamended) Fusion Thesis. He also made implicit use of his principle of Unrestricted Composition—though, to be sure, only in application to individuals. Another principle implicitly presupposed is that guaranteeing the existence of remainders (or so-called differences).

Given his premises *as he had stated them*, Lewis had less than fully rigorous justification for the following step in his informal proof:

> Suppose the Second Thesis false: some class x has a part y that is not a class. If y is an individual, x has an individual as part; if y is a mixed fusion of an individual and a class, then again x has an individual as part; *and by the Division Thesis those are the only possibilities*. [Emphasis added.]

The emphasized claim can reasonably be said to follow by dint of our Lemma above; but Lewis himself had not stated the Division Thesis formally or precisely enough to warrant the three-fold disjunction of that Lemma. The closest he came to clarifying exactly what he meant by saying that 'Reality divides exhaustively into individuals and classes' was on p. 8, where he wrote

> All I can say to defend the Division Thesis, and it is weak, is that as yet we have no idea of any third sort of thing that is neither individual nor class *nor mixture of the two*. [Emphasis added.]

By 'mixture' here Lewis clearly meant something more general than 'a fusion of an individual with a class' (else why did he not simply write the latter, or the simple abbreviation 'mixed fusion' that he had used in the previous paragraph, which was

14 *Ibid*., p. 6.

quoted above?). His use of the term ‘mixture’ invites one to think of such things, in addition, as fusions of classes with classes; fusions of mixed fusions with mixed fusions; fusions of classes with mixed fusions … and so on, through all the higher fusion-types. But until one can ensure that all *these* types are reducible to one of the three types mentioned in our Lemma, the quoted conjunct emphasized above (‘and by the Division Thesis those are the only possibilities’) is a non-sequitur.

Once equipped with the Canonical Decomposition Thesis, however, Lewis's argument can be made good. The next section sets it out as a fully formalized natural deduction, employing the principles listed above.

11 A rigorous version of Lewis's argumentation for his Second Thesis

The aim of this section is to regiment Lewis's informal argumentation for his Second Thesis, so that it becomes a natural deduction within a suitable combination of class (set) theory with mereology, based on explicit formal principles. It is necessary to quote at length from Lewis's monograph Lewis (1991) (pp. 9–10). After the quote is given, we shall begin its *Umgestaltung* into a rigorous natural deduction. We shall incorporate symbolic abbreviations, impose some changes of notation, and effect a re-arrangement of the various parts of the informal reasoning into the tree-structure of a formal proof.

> *Lewis's original version.* Suppose the Second Thesis false: some class x has a part y that is not a class. If y is an individual, x has an individual as part; if y is a mixed fusion of an individual and a class, then again x has an individual as part; and by the Division Thesis those are the only possibilities. Let z be the fusion of all individuals that are part of x. Then z is an individual, by the Fusion Thesis. Now consider the difference $x - z$, the residue that remains of x after z is removed. (It is the fusion of all parts of x that do not overlap z.) Then $x - z$ has no individuals as parts, so it is not an individual or a mixed fusion. By the Division Thesis, it must be a class. We now have that x is the fusion of class $x - z$ with an individual z. Since $x - z$ is part of x, and not the whole of x (else there wouldn't have been any z to remove), we have that the class $x - z$ is a proper part of the class x. So, by the First Thesis, $x - z$ must be a proper subclass of x. Then we have v, a member of x but not of $x - z$. According to standard set theory, we then have u, the class with v as its only member. By the First Thesis, u is part of x but not of $x - z$; by the Priority Thesis, u is not part of z; so u has some proper part w that does not overlap z. No individual is part of w; so by the Division Thesis, w is a class. By the First Thesis, w is a proper subclass of u. But u, being one-membered, has no proper subclass. This completes a *reductio*. □

We proceed now with the promised regimentation of Lewis's informal argument. First we spruce it up as follows.

Revised Version 1. Suppose x is a class, y is part of x and y is not a class. By the Division Thesis either (i) y is an individual or (ii) y is a mixed fusion of an individual (say i) and a class. In case (i), x has an individual (namely, y) as part. In case (ii), by transitivity x has an individual (namely, i) as part. So x has an individual as part.

Let z be the fusion of all individuals that are part of x. By the Fusion Thesis z is an individual. Let $x - z$ be the fusion of all parts of x that do not overlap z. Then $x - z$ has no individuals as parts. Hence $x - z$ is not an individual or a mixed fusion. By the Division Thesis, $x - z$ is a class.

So x is the fusion of class $x - z$ with an individual z. Since $x - z$ is part of x, but not identical to x, the class $x - z$ is a *proper* part of the class x. By the First Thesis, $x - z$ is a proper subclass of x. Let v be a member of x that is not in $x - z$. Let u be the singleton $\{v\}$. By the First Thesis, u is part of x but not of $x - z$. By the Priority Thesis, u is not part of z. Hence u has some proper part w that does not overlap z. No individual is part of w. Thus by the Division Thesis, w is a class. By the First Thesis, w is a proper subclass of u. But u, being a singleton, has no proper subclass. This completes a *reductio*. □

Version 1 is at best a clearer version of what is still a very informal argument. More steps are needed in order to turn it into a formal proof. Next we introduce now-familiar symbols for the mereological relations and operations, and use $\{v\}$ in place of u, in order to avoid unnecessary clutter. We write in boldface certain added justifications, to help the reader see where certain background principles of mereology are at work in Lewis's reasoning.

Revised Version 2. Suppose

$$Cx \quad \text{and} \quad y \sqsubseteq x \quad \text{and} \quad \neg Cy.$$

By the Division Thesis either

(i) Iy or
(ii) $\exists i \exists c(Ii \wedge Cc \wedge y = i + c)$.

In case (i), x has an individual (namely, y) as part.
In case (ii), we have $i \sqsubseteq y$, $y \sqsubseteq x$, whence **by transitivity** x has an individual (namely, i) as part.
So in either case

$$\exists j(Ij \wedge j \sqsubseteq x).$$

By Existence of Fusions, there exists such a thing as $\bigsqcup j(Ij \wedge j \sqsubseteq x)$. Call it z. By the Fusion Thesis, Iz. Let

$$x - z =_{df} \bigsqcup k(k \sqsubseteq x \wedge k \urcorner z).$$

Then $x - z$ has no individuals as parts. Hence **by Reflexivity and Transitivity** respectively,

$$\neg I(x - z) \quad \text{and} \quad \neg \exists i \exists c(Ii \wedge Cc \wedge (x - z) = i + c).$$

By the Division Thesis it follows that $C(x - z)$.

So we have:

$$x = (x - z) + z \quad \text{and} \quad C(x - z) \quad \text{and} \quad Iz.$$

Since $(x - z) \sqsubseteq z$ but $(x - z) \neq x$, it follows that $(x - z) \sqsubset x$.
By the First Thesis, $(x - z) \subset x$. Let v be such that

$$v \in x \quad \text{and} \quad v \notin (x - z).$$

By the First Thesis,

$$\{v\} \sqsubseteq x \quad \text{and} \quad \{v\} \not\sqsubseteq (x - z).$$

By the Priority Thesis, $\{v\} \not\sqsubseteq z$. **Hence by Corollary 4,** let w be such that:

$$w \sqsubset \{v\} \quad \text{and} \quad w \rceil z.$$

For no individual i do we have $i \sqsubseteq w$. Thus by the Division Thesis, Cw. (Remember that this means that w is a *non-empty* class.) By the First Thesis, $w \subset \{v\}$, contradicting w non-empty. This completes a *reductio*. □

So far we have changed only the forms of expression and their arrangement on the page. We have not supplied any missing details in the reasoning, except for the odd invocation of a rule or thesis to make it clearer why a particular step may be taken. What appear as primitive steps in this passage of reasoning are, however, far from being so within the combined theory of classes, individuals and parts. The missing details for what *appear* to be primitive inferences will be given in due course. At this juncture, we shall interpolate various foreshadowings of the extra work that remains to be done, by re-writing the last version with comments pointing out which passages of reasoning need to be 'textured' further. We shall use the Greek symbols Γ, Ξ, Σ, Ω and Θ to denote subproofs that will need to be supplied. (This will be done only after Version 3 has been laid out.)

Version 3. Suppose

$$Cx \quad \text{and} \quad y \sqsubseteq x \quad \text{and} \quad \neg Cy.$$

By the Division Thesis either

(i) Iy or
(ii) $\exists i \exists c (Ii \wedge Cc \wedge y = i + c)$.

In case (i), x has an individual (namely, y) as part.
In case (ii), we have $i \sqsubseteq y$, $y \sqsubseteq x$, whence by transitivity x has an individual (namely, i) as part.

So in either case

$$\exists j (Ij \wedge j \sqsubseteq x).$$

[The reasoning from assumptions $y \sqsubseteq x$ and $\neg Cy$ to conclusion $\exists j(Ij \wedge j \sqsubseteq x)$ will be called Ω below.]

By Existence of Fusions, there exists such a thing as $\bigsqcup j(Ij \wedge j \sqsubseteq x)$.
Call it z. By the Fusion Thesis, Iz.

—[We now add:

By Lemma 20, $z \sqsubseteq x$. By the Priority Thesis, $z \neq x$. Hence $z \sqsubset x$. By Weak Supplementation, $\exists k(k \sqsubseteq x \wedge k \rceil z)$. Hence by Existence of Fusions there is such a thing as

$$\bigsqcup k(k \sqsubseteq x \wedge k \rceil z).\textbf{]—}$$

Let

$$x - z =_{df} \bigsqcup k(k \sqsubseteq x \wedge k \rceil z).$$

Then $x - z$ has no individuals as parts. Hence

$$\neg I(x-z) \quad \text{and} \quad \neg \exists i \exists c(Ii \wedge Cc \wedge (x-z) = i + c).$$

By the Division Thesis it follows that $C(x-z)$.

[The reasoning from assumption $z = \bigsqcup j(Ij \wedge j \sqsubseteq x)$ to conclusion $C(x-z)$ will be called Σ below.]

So we have:

$$x = (x-z)+z \quad \text{and} \quad C(x-z) \quad \text{and} \quad Iz.$$

Since $(x-z) \sqsubseteq z$ but $z \neq x$, it follows that $(x-z) \sqsubset x$.

[The reasoning from assumption $z = \bigsqcup j(Ij \wedge j \sqsubseteq x)$ to conclusion $(x-z) \sqsubset x$ will be called Ξ below.]

Recall that $C(x-z)$ and Cx. By the First Thesis, $(x-z) \subset x$. Let v be such that

$$v \in x \quad \text{and} \quad v \notin (x-z).$$

By the First Thesis,

$$\{v\} \sqsubseteq x \text{ but } \{v\} \not\sqsubseteq (x-z).$$

By the Priority Thesis, $\{v\} \not\sqsubseteq z$. Hence by Corollary 4, let w be such that:

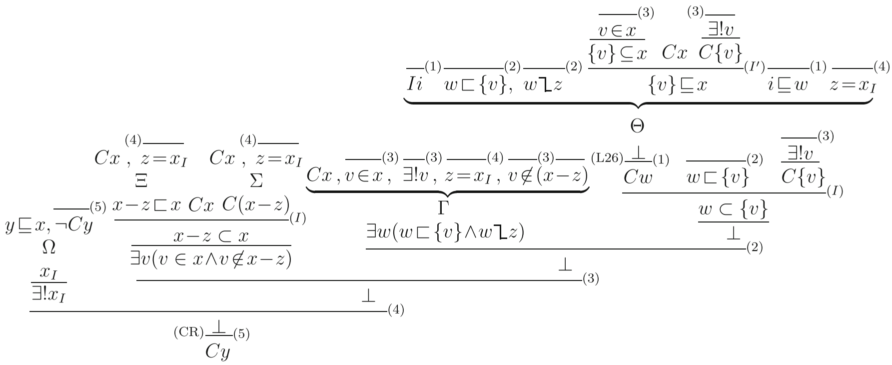
$$w \sqsubset \{v\} \text{ and } w \rceil z.$$

For no individual i do we have $i \sqsubseteq w$.

[The joint reductio of assumptions Ii and $i \sqsubseteq w$, *modulo* the assumptions $z = \bigsqcup j(Ij \wedge j \sqsubseteq x)$, $w \sqsubset \{v\}$, $w \urcorner z$ and $\{v\} \sqsubseteq x$ will be called Θ below.]

Thus by the Division Thesis, Cw. (Remember that this means that w is a *non-empty* class.) By the First Thesis, $w \subset \{v\}$, contradicting w non-empty. This completes a *reductio*. □

At this level of analysis, we can at least foreshadow the tree-structure of the eventual natural deduction that will regiment Lewis's entire argument. Pending the details of the subproofs Ω, Ξ, Σ, Γ and Θ, we can regiment the reasoning at the current level of manifest structural detail as follows, where we recall that $\bigsqcup j(Ij \wedge j \sqsubseteq x)$ may be abbreviated as x_I.

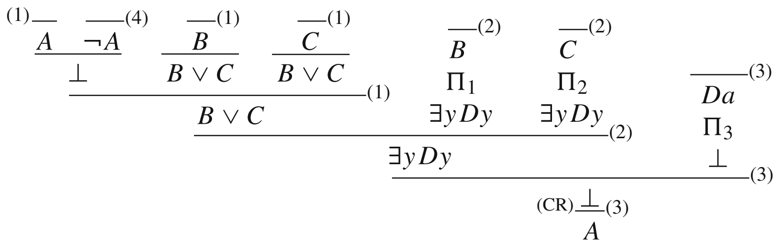


The proof Ω is a suitable substitution instance of the proof of Lemma 8.
The proof Θ is a suitable substitution instance of the proof of Corollary 2.
The proof Γ is a suitable substitution instance of the proof of Lemma 34 (within which one finds the appeal to Corollary 4).
The proof Ξ is a suitable substitution instance of the proof of Lemma 30.
The proof Σ is a suitable substitution instance of the proof of Lemma 31.

12 Constructivizing Lewis's argument

As it stands, Lewis's argument is strictly classical, for it assumes $\neg Cy$, reduces that assumption to absurdity ($\bot$), and then applies classical *reductio* in order to infer Cy. Moreover, the argument contains an application of Weak Supplementation (Lemma 18) which, as seen above, depends for its proof on yet another application of classical reductio; and the argument also makes use of Corollary 4, whose proof contains an application of classical reductio. Yet insofar as Lewis's argument is intended to be purely conceptual, it ought also to be constructive. In this section is is shown that the argument can indeed be constructivized.

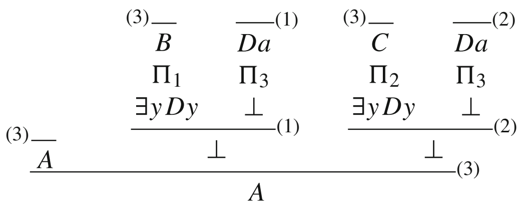
What we have here is a proof of the form



That this is the form of the proof above can readily be grasped by means of the correspondence

$$
\begin{array}{lcl}
A & : & Cy \\
B & : & Iy \\
C & : & \exists i \exists c(Ii \wedge Cc \wedge y{=}i{+}c) \\
\exists y Dy & : & \exists ! \bigsqcup j(Ij \wedge j \sqsubseteq x)
\end{array}
$$

With this form of proof, the appearance of non-constructivity is the price paid for making do with only one copy of the sub-proof Π_3. The trilemma of DIVISION marked (1) is of course hidden within the sub-proof Ω. We may assume that DIVISION is constructive in nature. Notice that its application here can be 'shuffled down' so as to become the terminal step of a completely constructive proof-schema of the form



Here, the tradeoff for constructivity is the repetition of the sub-proof Π_3. That, however, is a small price to pay in order to perfect the philosophical argument as a constructive one—an argument that draws on the inherent conceptual content of its main terms, so as to negotiate the passage from premises to conclusion without any applications of classical *reductio*. Moreover, as is usually done in informal reasoning, the wholesale repetition of Π_3 can be obviated in an informal argument by simply saying something like 'the reasoning in this case is similar to that in the earlier case'. Alternatively, the *reductio* of *Da* could be stated as a separate lemma, with abbreviating appeals to it at both points within the overall proof where it is needed.

But wait! (the reader will say). The sub-proofs Γ, Ξ and Σ all depend (as pointed out above) on applications of Strong Supplementation (Lemma 16 above). Moreover, Γ involves Corollary 4. And there were appeals to classical *reductio* in the proofs of Lemma 16 and of Corollary 4. So, will not those classical steps compromise the allegedly constructive character of the overall proof?

The answer is negative. For we are dealing with a *reductio ad absurdum* in each of the two main sub-proofs within the 'constructivized' proof-schema just given for (the overall form of) the proof of Lewis's argument. This means that any applications therein of classical *reductio* will disappear upon normalizing the overall proof. The Gödel–Gentzen–Glivenko theorem guarantees that any *reductio*, in classical first-order logic, of a set of assumptions Δ can be transformed into a *reductio*, in intuitionistic first-order logic, of $\neg\neg[\forall\neg\neg]\Delta$. The operation $\neg\neg[\forall\neg\neg]$ on a sentence φ places a double negation immediately after any universal quantifier occurrence within φ, and then prefixes the result with a double negation (see Tennant 1978, pp. 129–130).

For our purposes, it suffices that there will be an intuitionistic *reductio* of $[\forall\neg\neg]\Delta$ (without the initial double negations). Inspection reveals that, since the proof of Lemma 16 involved classical *reductio* only on existentials, we can also drop the '$[\forall\neg\neg]$' part of the operation as well. Our regimentation of Lewis's argument can be both relevantized and constructivized.

The lessons learned from this exercise in formalization and transformation can be applied in a reformulation of the informal argument itself. Here is how Lewis could have set out his reasoning, at his chosen level of informal rigor.

Constructive and relevant version of Lewis's informal argument.
Suppose x is a (non-empty) class and y is part of x.
Suppose now for *reductio ad absurdum* that y has an individual as a part. Then so does x, by Transitivity. So the fusion of all individual parts of x exists. Call it z. By the Fusion Thesis, z is an individual. By the Division Thesis, $x-z$ is a class; and is clearly also a proper part of x. By the First Thesis, $x-z$ is a proper subclass of x.
Let v be a member of x but not of $x-z$.
Since v is a member of x, it follows that $\{v\}$ is a class—indeed, a subclass of the class x. Hence by the First Thesis, $\{v\}$ is a part of x.
Since v is not a member of $x-z$, it follows that $\{v\}$ is not a subclass of $x-z$. Hence by the First Thesis, $\{v\}$ is not a part of $x-z$.
By the Priority Thesis, $\{v\}$ is not part of $x-z$.
We now have:

$\{v\}$ is a part of x;
$\{v\}$ is not a part of z; and
$\{v\}$ is not a part of $x-z$.

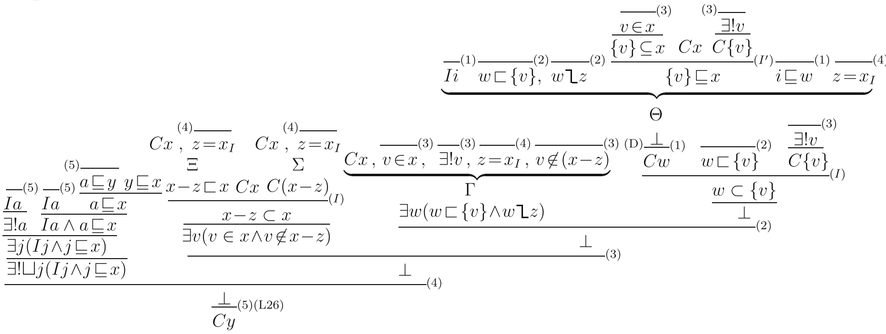
Hence (by Corollary 4) there is a proper part of $\{v\}$ disjoint from z. Let w be such. Since $\{v\} \subseteq x$ and both these are classes, we have by the First Thesis that $\{v\} \sqsubseteq x$.
Suppose for *reductio ad absurdum* that w has an individual i as a part. Then:

i is an individual part of w, which is a proper part of $\{v\}$, which is part of x; w is disjoint from z; z is the fusion of all individual parts of x.

By Transitivity it follows that i is an individual part of x. Hence i is part of z. Since i is also part of w, w overlaps z. But this contradicts the claim that w is disjoint from z. So w has no individual as a part, after all.
By the Division Thesis (here in the form of Lemma 26), it follows that w is a class. Since w is also a proper part of the class $\{v\}$, it follows by the First Thesis that w is a proper subclass of $\{v\}$. But this is impossible.

So y has no individual as a part, after all. By the Division Thesis (again in the form of Lemma 26), y is a class. □

This constructive train of reasoning is formalizable as follows, where Ξ, Σ, Γ and Θ are as before. Note the absence of any explicit applications of classical *reductio* in this proof-schema. As explained above, the application of classical *reductio* that was made in the proof of Lemma 16 (and which now lurks within the sub-proofs Ξ, Σ and Γ) is non-essential, since it is located within the overall reductio whose terminal step is labelled (3).



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