

A Logical Theory of Truth-Makers and Falsity-Makers



Neil Tennant

The Oxford Handbook of Truth

Edited by Michael Glanzberg

Print Publication Date: Jul 2018 Subject: Philosophy, Philosophy of Mathematics and Logic

Online Publication Date: Sep 2018 DOI: 10.1093/oxfordhb/9780199557929.013.16

Abstract and Keywords

We explicate the different ways that a first-order sentence can be true (resp., false) in a model M , as formal objects, called (M -relative) truth-makers (resp., falsity-makers). M -relative truth-makers and falsity-makers are co-inductively definable, by appeal to the “atomic facts” in M , and to certain rules of verification and of falsification, collectively called rules of evaluation. Each logical operator has a rule of verification, much like an introduction rule; and a rule of falsification, much like an elimination rule. Applications of the rules (\forall) and (\exists) involve infinite furcation when the domain of M is infinite. But even in the infinite case, truth-makers and falsity-makers are tree-like objects whose branches are at most finitely long. A sentence φ is true (resp., false) in a model M (in the sense of Tarski) if and only if there exists π such that π is an M -relative truth-maker (resp., falsity-maker) for φ . With “ways of being true” explicated as these logical truthmakers, one can re-conceive logical consequence between given premises and a conclusion. It obtains just in case there is a suitable method for transforming M -relative truthmakers for the premises into an M -relative truthmaker for the conclusion, whatever the model M may be.

Keywords: truth-maker, falsity-maker, evaluation rules, atomic facts, domain, model, saturated formula, logical consequence, normalization, bivalence

14.1 Introduction

DUMMETT’S regulative principle C (for “Correspondence,” stated in Dummett 1976: 89) is that

If a statement is true, there must be something in virtue of which it is true.

The thing in question is aptly called a truth-maker. To date, no one really knows what such a thing may be.

In the 1980s three seminal publications appeared, which put truth-makers and truth-making at center stage. These were the 1984 paper “Truth-Makers,” by Kevin Mulligan, Peter Simons, and Barry Smith; the 1987 paper “Truthmaker” by John Fox; and the 1988 discussion of truth-makers by John Bigelow in Part III of his book *The Reality of Numbers*. The field is now well served by the two recent anthologies Beebe and Dodd (2005) and Lowe and Rami (2009), each with a thorough and competent editorial introduction that surveys the issues involved. This happy situation moves the present author to avoid rehashing the material with a survey-style article, and rather to go back to original sources, reflect on their materials, and then attempt to strike out in a new direction that is informed by the author’s own methodological interests. We commend to the reader a new, *logical* theory of truth-makers, a theory that deals with *logically complex* statements as well as logically primitive, or atomic, ones.

(p. 356) 14.2 The Notion of a Truth-Maker

Here is how the aforementioned authors of the three seminal publications interpreted the notion of a truth-maker.

. . . entities *in virtue of which* sentences and/or propositions are true . . . we shall . . . [call] . . . *truthmakers*. (Mulligan et al. 1984: 287–8)

By the truthmaker axiom I mean the axiom that for every truth there is a truth-maker; by a truthmaker for A, I mean something whose very existence entails A. (Fox 1987: 189)

. . . whenever something is true, there must be something in the world which makes it true. I will call this the Truthmaker axiom . . . A truthmaker is simply an object whose existence entails a truth.

(Bigelow 1988: 122 and 128)

These three treatments of truth-makers venture to characterize truth-makers only for atomic sentences or propositions; and the main monograph-length treatment, Armstrong (2004), at best succeeds in characterizing truth-makers only in the atomic case.¹ It would seem that for logically complex sentences (or propositions)²—most notoriously, negations, disjunctions, and universal quantifications—truth-makers are very hard to come by. Such, at least, is the lesson these recent pioneers appear to draw from the tradition:

The glory of logical atomism was that it showed that not every kind of sentence needs its own characteristic kind of truth-maker. Provided we can account for the truth and falsehood of atomic sentences, we can dispense with special truth-makers for, e.g., negative, conjunctive, disjunctive, and identity sentences. (Mulligan et al. 1984: 288)

. . . what could be the truth-makers . . . for universal claims, such as that all ravens are black? The sum of the truthmakers for all the truths of the form “a is black” would not do, for they could exist yet one extra raven exist and be purple. What is required beyond this is that there be no ravens but these; and what would be the truthmaker for *this*?

One strategy for dealing with this is to abandon the generality of the axiom by restricting it to atomic claims.

Such explicit restriction raises the problem of explaining the relation of truth in general to truthmakers . . . [A]rguably it is only atomic truthmaker that can coherently be attributed to the tradition.

(Fox 1987: 204–5)

(p. 357) 14.3 Matters of Background

Some truth-making theorists—especially the Australian school—take themselves to be indulging an unrelenting realism in metaphysical outlook, in hypothesizing that every truth has a truth-maker. But to Dummettian anti-realists there is nothing untoward about this truth-maker axiom, *given the anti-realist’s conception of what truth consists in*. Truth, for the anti-realist, consists in the existence of a truth-maker, albeit a truth-maker of a very special kind—one that is in principle surveyable, or epistemically accessible. For the anti-realist, all truths are knowable. And that is why truth cannot be relied upon to be bivalent.

It is an old point (due to Putnam), but one worth re-emphasizing here, that Tarski’s theory of truth, including its inductive definition of satisfaction and its derivations of all instances of the *T*-schema, are available to the anti-realist even after the anti-realist has restricted herself to intuitionistic logic. (The present author would strengthen this observation: Tarskian truth theory can be carried out within the yet more restricted system of *core logic*, which results from classical logic by pursuing not only constructivizing reforms, but relevantizing ones also.³)

The only “classical” or “realist” feature of Tarski’s theory of truth obtrudes at the point where the theorist tries to prove (in the metalanguage) the Principle of Bivalence: that for every sentence φ of the object language, either φ is true or $\neg\varphi$ is true. The proof of *that* metalinguistic result requires the use, in the metalogic, of a strictly classical rule of negation, such as the law of excluded middle, or the rule of dilemma. But by contrast the derivations of all instances of the *T*-schema proceed in innocence of such classical rules. The derivations are available in core logic.

So the anti-realist—even one who uses only core logic—can hear the truth-making theorist’s asseveration that every truth has a truth-maker, and unequivocally agree.

It follows that if the truth-making theorist wishes to express his metaphysical *realism* by postulating the existence of truth-makers for each and every truth, he needs a more exigent account of what truth-makers are. He needs to characterize them in a way that enables one subsequently to argue, cogently, that for every sentence φ , either there is a truth-maker for φ or there is a truth-maker for $\neg\varphi$.

The present author is about to share with his realist opponents one possible way of doing this. But we shall be careful to point out later (in section 14.11.4) where exactly, in the account about to be given, the anti-realist should demur.

(p. 358) 14.4 The Aims of the Present Study

In this chapter we seek to make good the above-mentioned omission on the part of both vintage and recent truth-maker theorizing, even though it is in realist vein. We offer a new explication of the *kind of things* truth-makers are. The aim is to allow the claim, in full generality, that every truth has a truth-maker.⁴ Ours is a *structural* theory of truth-makers. They are not (to take one extreme) simply individuals, or *Sachverhalte* composed of individuals. Nor, at the other extreme, are they the Great Fact.

Rather, our truth-makers are *proof-like* objects—possibly infinitary, depending on the size of the domain—that clearly *articulate* the grounds for truth of the claim in question.⁵

An immediate note of caution should be entered here. Throughout this discussion, unless otherwise indicated, the phrase “grounds for truth” and its cognates are to be construed *metaphysically* or *ontologically*, rather than epistemically. We are thinking of a species of ontological grounding here, or of constitution—of the actual making-true. It would be a serious error to read the phrase “grounds for truth” as evidential or epistemic in connotation—as having to do with, say, grounds for belief or justification. There might be an interesting relationship, to be explored, between the two kinds of ground; but nothing we say subsequently will be properly understood if we are mistakenly construed, from this point on, as talking about evidential grounds when in fact we are talking about constitutive ones.⁶

Our truth-makers will be able to contain actual individuals. Individuals will be able to be embedded within so-called *saturated formulae* (see section 14.5.1) that occur within truth-makers, such occurrences being like those of ordinary (formal) sentences within deductive proofs (which we shall cast as natural deductions). Moreover, just as a theorem of a mathematical theory can enjoy more than one proof from the axioms of the theory, so too will a given sentence be able to enjoy more than one truth-maker (relative to a given model, or interpretation). One’s immediate intuition is that this is how it ought to be—for in general there are more ways than one for a claim to be true (even within the context of an interpretation, or model, held fixed).

A Logical Theory of Truth-Makers and Falsity-Makers

We stress again that our truth-makers will actually *embody the individuals involved*, embedded within what we are calling *saturated formulae*. So, abstract though they may still be, the truth-makers will “limn the lines of factual composition,” as it were, beginning with the truth (or falsity) of basic predications embedding those individuals. It is difficult to imagine any other way of formulating a theory of truth-makers that can hope (p. 359) to forge any genuine connection with the traditional matters of logical consequence and formal proof. To the metaphysician, to be sure, these might sound like only a logician’s concern—but the present author believes that truth-maker theory can advance only by affording an alternative account of truth, which could be substituted for the usual Tarskian account in the major branches of formal semantics and mathematical logic. Our truth-makers are intended to be the most economical way, logically and mathematically, of *reifying different realizations of truth-conditions* of logically complex sentences.⁷

On our account, it is also the case that any truth-maker Θ for a sentence φ is so constituted that the resulting truthmaking is an *internal* relation, in the sense of Armstrong (2004: section 1.3.3). As Armstrong puts it,

I mean by calling a relation internal that, given just the terms of the relation, the relation between them is necessitated. . . . I suggest it is an attractive ontological hypothesis that such a relation is no addition of being. Given just the terms, we are given the ontology of the situation.

That is what happens here: given just Θ and φ , we are given the ontology of the truth-making situation. The construction Θ makes φ true as a matter of necessity. “[G]iven just the terms of the [truth-making] relation, the relation between them is necessitated.” Justification for this claim (as for other claims made in this section) will have to await the laying out of the logical theory promised, which characterizes exactly what kind of object a truth-maker is.

When they are finite our truth-makers are, as it happens, exactly like proofs, since they possess all the properties that one requires of a proof: that it should admit of a mechanical check for correctness, and that one should be able to “read off” from it *what* it establishes, and *from* what. When, however, our truth-makers are perforce infinite—which can happen when there are infinitely many individuals in the domain of the interpretation in question⁸—what they reveal is that the grounds of truth (of the claim φ in question) are themselves infinite. This is why it would be a mistake to think of “grounds for truth,” as codified by a truth-maker, in an epistemic way. For a given claim φ , it can be the case that all its truth-makers are infinite even if φ can be furnished with a finitary proof within some axiomatic theory of the subject-matter, in which all the axioms (drawn from some decidable set Δ) strike the theorist as self-evident. Many a logically complex claim A , whose grounds of truth are essentially infinite, can qualify as self-evident and be taken as an axiom, i.e. as a member of Δ . In such a case, the proof-based epistemological route to the claim φ in question, by taking A as a deductive starting point, begins with a tremen-

A Logical Theory of Truth-Makers and Falsity-Makers

dously abbreviating short cut.⁹ Proofs are not always (p. 360) truth-makers, in the sense of “truth-maker” to be essayed here. And truth-makers are not always proofs.

An account of truth-makers must mesh with one’s accounts in two other areas of philosophical inquiry—the theory of (perceptual) knowledge, and the logician’s theory of logical consequence. The approaches already cited have something substantial to contribute to the former, but nothing, so far as we can see, to the latter. Our approach should be hospitable to including the best of what is on offer concerning the former. It seeks also to make an innovative contribution to the latter. In the section titled “Conclusion: open questions” of his Introduction to Lowe and Rami (2009), Rami does not list as an open problem that of making any kind of connection between *truth-making*, on the one hand, and, on the other hand, *preservation of truth from premises to conclusions of valid arguments*. From our perspective on the problem of truth-makers, however, this connection is key, as the reader will see in section 14.8.

In developing the beginnings of a theory of truth-makers as structured, logically complex entities, we should remind ourselves of the need for some methodological caution. One should not be over-ambitious. It took an impressive cast of logicians from Frege through to Tarski and Gentzen to master the details of first-order languages involving \neg , \wedge , \vee , \rightarrow , \exists , \forall , and $=$ on both the semantic and the deductive sides. Adequate formal semantics for the modal operators (involving possible worlds) had to await the work of Kripke in the 1960s. And although there are yet more recent formal semantic accounts to be had for such expressions as the counterfactual conditional, the relevant conditional, verbs of propositional attitude, and temporal operators, there is by no means a consensus on how such expressions are to be treated, either in formal semantics or in proof theory. So we restrict ourselves at the outset to a pursuit of an account of truth-makers that will at least work for the logician’s standard extensional, first-order language. The matter to be judged is whether the account succeeds *there*—not on whether, as presented here, it can handle (say) the alethic modal operators, or verbs of propositional attitude, or tense operators.

14.5 Some Basic Notions

All the notions to be defined here can be defined much more rigorously, in a way that would meet the requirements of even the most captious mathematical logician. We are (p. 361) opting in the main part of this study for a slightly more informal presentation, by way of judiciously chosen examples rather than by completely general formal definitions, in order to ease the flow of philosophical discussion. (Section 14.10 provides a much more rigorous formal definition of truth-makers and falsity-makers.)

14.5.1 Saturated terms and formulae

In general, terms and formulae of the object-language may contain free variables. If they do, then they are called *open*. Those that are not open are called *closed*. A closed formula is called a *sentence*. A closed term may be called a (simple or complex) *name*. The seman-

A Logical Theory of Truth-Makers and Falsity-Makers

tic value of a name, when it has one, is an *individual*, which the name is then said to *denote*. The semantic value of a sentence, when it has one, is a *truth-value*, and the sentence is said to be *true* or *false* according as that value is *T* or *F*.

Closing an open term or formula involves substituting closed terms (of the object-language) for free occurrences of variables. Thus one could substitute the object-linguistic name j for the free occurrence of the variable x in the open term $f(x)$, to obtain the closed term $f(j)$ (“the father of John”). Or, to complicate the example slightly, one could substitute the closed object-linguistic term $m(j)$ for that free occurrence, to obtain the closed term $f(m(j))$ (“John’s maternal grandfather”). Likewise, an open formula, say $L(x, y)$, may be closed by substituting closed terms for its free occurrences of variables. One such closing would result in the sentence $L(m(f(j)), f(j))$ (“John’s paternal grandmother loves his father”).

Here we shall introduce an operation on open terms and formulae analogous to the operation of closing, but importantly different from it. The new operation will be called *saturation*. Like the operation of closing, the operation of saturation gets rid of all free occurrences of variables within an object-linguistic term or formula. But the way it does so is importantly different. Instead of substituting closed object-linguistic terms for free occurrences of variables, saturation is effected by substituting *individuals from the domain* for those free occurrences. Thus if α and β are individuals from the domain, one saturation of the open formula $L(x, y)$ would be $L(\alpha, \beta)$. Another one would be $L(m(f(\alpha)), f(\alpha))$, where the saturation is effected by substituting the saturated terms $m(f(\alpha))$ and $f(\alpha)$ for the free occurrences of the variables x and y respectively.

When the domain D supplies all the individuals involved in a saturation operation, the resulting saturated terms are called *saturated D-terms*, and the resulting saturated formulae are called *saturated D-formulae*.

Saturated terms and formulae are object-linguistic and metalinguistic hybrids. But, as mathematical objects, they are well-defined. When one treats, in standard Tarskian semantics, of assignments of individuals to variables, one is assuming such well-defined status for ordered pairs of the form $\langle x, \alpha \rangle$, where x is a free variable of the (p. 362) object-language, and α is an individual from the domain of discourse.¹⁰ Since standard semantics is already committed to the use of such hybrid entities, it may as well take advantage of similar hybrid entities such as saturated terms and formulae.

We shall be taking advantage of them by having them feature in the rules of inference on the right in our description of an illustrative model below. Indeed, such rules will form a constitutive part of the model in question, as will emerge in due course.

14.5.2 Verification and falsification of sentences in models

We shall first describe, in general terms, *interpretations*, or *models*, of a first-order language.

A Logical Theory of Truth-Makers and Falsity-Makers

A *model* consists of

- (i) a domain of individuals;
- (ii) a denotation mapping for names (if there are any names in the object-language);
- (iii) the structure that consists in primitive predicate-extensions; and
- (iv) the structure that consists in the mappings represented by primitive function-signs.

The names, predicates, and function-signs make up the *extra-logical vocabulary* that is being interpreted by the model in question.

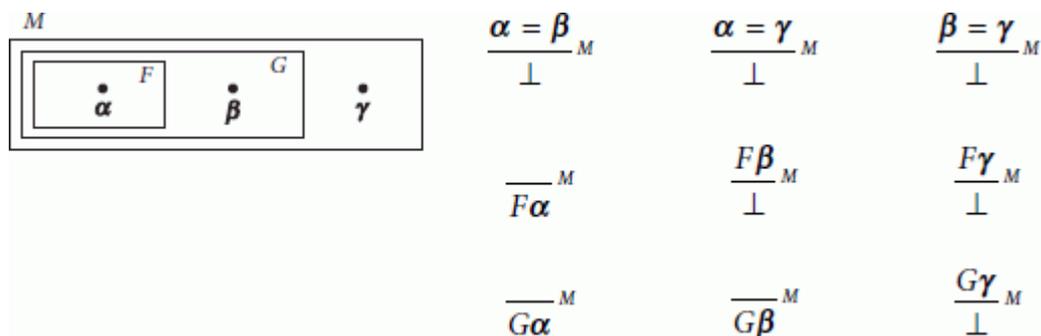
For purposes of illustration, we shall confine ourselves to one-place predicates. So our expressive resources will be quite modest.

We shall now give a simple example of a model (see the figure on the following page). To the left will be a diagram, which can be thought of as the model M itself. The large dots will be the individuals; each one-place predicate extension will be represented by a box, as is usual with Euler-Venn diagrams. (If function-symbols were involved, then each one-place mapping would be represented by arrows—a different style of arrow for each mapping.)

So, for our toy example, we first choose a domain of individuals (here, three). They are labeled α , β , and γ in the *metalanguage*. The extensions of the two monadic predicates F and G are indicated by the boxes in the diagram. That diagram on the left, labeled M , “is” the model.

To the right of the diagram we provide an alternative presentation of the very same model. It takes an “inferentialist” form, using saturated atomic formulae (see section 14.5.1). In the top row to the right of the diagram are three “ M -relative” rules of inference, whose premises are *saturated identity-formulae* (see section 14.5.1 again) and whose conclusions are \perp . These rules ensure that the individuals are pairwise distinct.

(p. 363) In the next two rows to the right of the diagram are some rules that specify the M -extensions of two predicates F and G . These rules can be “read off” from the diagram on the left. Note that this list of rules is exhaustive, dealing with all possible cases, both positive and negative, generated by combining a monadic predicate letter with an individual from the domain.¹¹



A Logical Theory of Truth-Makers and Falsity-Makers

The rules that have saturated formulae as their conclusions are degenerate examples of *truth-makers*; while the rules that have \perp as conclusion are degenerate examples of *falsity-makers*. Just as truth depends on the existence of a truth-maker, so too does falsity depend on the existence of a falsity-maker.¹²

What about true negative claims? Do they fail to have truth-makers? As Rami states the problem (Lowe and Rami 2009: 15),

Most notoriously difficult are so-called negative propositions: basically, negative predications such as the proposition that grass is not black and negative existentials such as the proposition that there are no unicorns. *Intuitively these propositions do not have truth-making entities if they are true.* So it is a counterintuitive consequence if the truth-maker theorist is forced to find such truth-making entities.

(Emphasis added—NT)

We reject the italicized claim, as well as the one following it. We have just seen a falsity-maker for the claim that (in the model M) the individual γ has the property G :

$$\frac{G\gamma_M}{\perp}$$

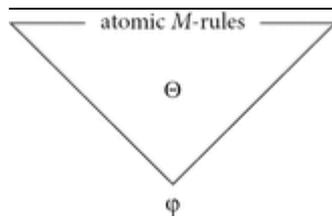
With one step of \neg -Verification (analogous to \neg -Introduction in natural deduction) this can be extended so as to become a *truth-maker* for the *negative* claim $\neg G\gamma$, as follows:

(p. 364)

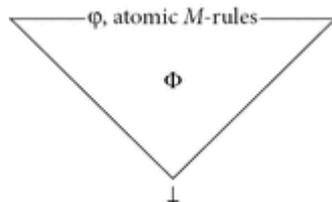
$$\frac{\frac{\text{---}^{(1)}}{G\gamma_M}}{\perp}^{(1)}}{\neg G\gamma}$$

A striking feature of our theory of truth-makers is that it is really a theory, *simultaneously*, *both* of truth-makers *and* of falsity-makers. Ironically, earlier theorists of truth-making have focused their attention on the atomic case in the conviction that Tarskian recursion will take care of the logical operators, without our having to find truth-makers for logically complex sentences. In doing so, however, they have missed a golden opportunity to avail themselves of the *recursive interdependence* of truth-making and falsity-making.¹³ Ours is a unified theory of constructions that are either truth-makers or falsity-makers, as will emerge below.

For the time being, it will be helpful if the reader were to bear in mind that the overall structure of an M -relative truth-maker Θ for φ is that of a rule-governed passage *from* atomic information about (or, better: *from*) the model M to the sentence φ as conclusion:



and that the overall structure of an M -relative falsity-maker Φ for φ is that of a rule-governed passage *from* the sentence φ that it makes false, plus atomic information about (or from) the model M , to the conclusion \perp :



(p. 365) So a truth-maker looks rather like a proof (but from very special premises), while a falsity-maker looks like a *reductio*, or disproof (but again, *modulo* very special premises). But it is to be borne in mind that each kind of construction can contain within it constructions of the other, complementary, kind. Truth-makers can be embedded within falsity-makers, and vice versa. The two kinds of construction are *recursively interdependent*, or *intercalated*.

Moreover, each kind of construction can be infinitary. The reader is reminded, once again, that this means that these constructions are intended to articulate the constitution of truth-making, rather than to serve as epistemic warrants.

In speaking of “rule-governed passages,” the rules to which we advert are what we shall call *evaluation rules*.¹⁴ They are set out in graphic form in section 14.7, and in section 14.10 they are set out again in the form of clauses in an inductive definition of the notions of truth-maker and of falsity-maker. These rules, especially in their graphic form, are similar, but not identical, to the familiar introduction and elimination rules of natural deduction.¹⁵ Corresponding to the introduction rules will be our rules-for-evaluating-as-true; and corresponding to the elimination rules will be our rules-for-evaluating-as-false. And the latter kind of evaluation will consist in “deriving” (in a possibly infinitary way) the conclusion \perp (absurdity) from the sentence that is to be evaluated as false.

14.6 Examples of M -Relative Truth-Makers

We shall immediately show what truth-makers look like (in the finite case), by giving easy examples before stating the general rules that allow for their construction.¹⁶ The model M in section 14.5.2 clearly makes the following sentences true:

$$\exists xFx \quad \forall x(Fx \rightarrow Gx)$$

A Logical Theory of Truth-Makers and Falsity-Makers

In the model M , there is an F ; and all F s are G s. Here is an M -relative truth-maker for the first of these claims (indeed, it is the *only* one available):

$$\frac{\text{---}^M}{F\alpha}$$

$$\frac{}{\exists xFx}$$

(p. 366) And here is a truth-maker for the second:

$$\frac{\frac{\text{---}^M}{G\alpha}}{F\alpha \rightarrow G\alpha} \quad \frac{\frac{\text{---}^M}{G\beta}}{F\beta \rightarrow G\beta} \quad \frac{\frac{\frac{\text{---}^{(1)}}{F\gamma}}{\perp}^M}{F\gamma \rightarrow G\gamma}^{(1)}}{\forall x(Fx \rightarrow Gx)} \quad D = \{\alpha, \beta, \gamma\}$$

The first truth-maker is self-explanatory. The second deserves some comment. It is not the only M -relative truth-maker for $\forall x(Fx \rightarrow Gx)$, since one can provide a different truth-maker for the middle instance $F\beta \rightarrow G\beta$:

$$\frac{\frac{\frac{\text{---}^{(1)}}{F\beta}}{\perp}^M}{F\beta \rightarrow G\beta}^{(1)}$$

The truth of the saturated material conditional $F\beta \rightarrow G\beta$ is, as it were, overdetermined in the model M , since its antecedent is false and its consequent is true. We chose to proceed via the truth of the consequent. The final step of inference in our truth-maker, to the conclusion $\forall x(Fx \rightarrow Gx)$, was taken from *all the instances* of the latter in the domain D of the model M . Those instances are the three saturated formulae

$$F\alpha \rightarrow G\alpha, \quad F\beta \rightarrow G\beta, \quad F\gamma \rightarrow G\gamma$$

That is how we give expression to the fact that α , β , and γ are *all the individuals there are* in M : we allow (the evaluation-analogue of) “universal introduction” from all the instances of the claim to the universal claim in question.¹⁷

14.7 Evaluation Rules

When applying any evaluation rule whose conclusion is \perp , the major premise is to *stand proud*, with no “evaluation work” above it.¹⁸ This ensures that all truth-makers (p. 367) (i.e. evaluations-as-true) and all falsity-makers (i.e. evaluations-as-false) are in *normal form*, in the proof-theorist’s sense.¹⁹

Another requirement on applications of these rules is that there may not be any “vacuous discharges” of assumptions. If a rule indicates that an assumption of a certain form may

A Logical Theory of Truth-Makers and Falsity-Makers

be discharged, then, when the rule is applied, just such an assumption must already be in place (i.e. it must already have been used) and hence be available to be discharged.²⁰

14.7.1 Rules for primitive saturated formulae

$$\frac{A(\alpha_1, \dots, \alpha_n) \quad t_1 = \alpha_1 \quad \dots \quad t_n = \alpha_n}{A(t_1, \dots, t_n)}$$

where A is a primitive n -place predicate, t_1, \dots, t_n are saturated D -terms, and $\alpha_1, \dots, \alpha_n$ are individuals in the domain.

$$\frac{\frac{}{\quad} (i) \quad \frac{A(\alpha_1, \dots, \alpha_n)}{\quad} M \quad \perp \quad t_1 = \alpha_1 \quad \dots \quad t_n = \alpha_n \quad A(t_1, \dots, t_n)}{\quad} (i) \quad \perp$$

where A is a primitive n -place predicate, t_1, \dots, t_n are saturated D -terms, and $\alpha_1, \dots, \alpha_n$ are individuals in the domain.

(p. 368)

$$\frac{f(\alpha_1, \dots, \alpha_n) = \alpha \quad u_1 = \alpha_1 \quad \dots \quad u_n = \alpha_n}{f(u_1, \dots, u_n) = \alpha}$$

where f is a primitive n -place function sign, u_1, \dots, u_n are saturated D -terms and $\alpha, \alpha_1, \dots, \alpha_n$ are individuals in the domain

$$\frac{\frac{}{\quad} (i) \quad \frac{f(\alpha_1, \dots, \alpha_n) = \alpha}{\quad} M \quad \perp \quad u_1 = \alpha_1 \quad \dots \quad u_n = \alpha_n \quad f(u_1, \dots, u_n) = \alpha}{\quad} (i) \quad \perp$$

where f is a primitive n -place function sign, u_1, \dots, u_n are saturated D -terms, and $\alpha, \alpha_1, \dots, \alpha_n$ are individuals in the domain.

$\bar{\alpha} \equiv \bar{\alpha}$ where α is an individual in the domain

14.7.2 Rules for saturated formulae with a connective dominant

The reader will recognize these rules as simply codifying the left-to-right readings of the rows of the respective truth-tables. Remember: all major premises for eliminations stand proud; and vacuous discharges of assumptions are prohibited.

$$\begin{array}{c}
 \frac{}{\varphi}^{(i)} \\
 \vdots \\
 \frac{}{\perp}^{(i)} \\
 \frac{}{\neg\varphi}^{(i)}
 \end{array}
 \quad
 \frac{\neg\varphi \quad \varphi}{\perp}$$

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi}$$

$$\frac{\frac{}{\varphi}^{(i)} \quad \vdots}{\varphi \wedge \psi} \quad \frac{}{\perp}^{(i)} \quad \frac{\frac{}{\psi}^{(i)} \quad \vdots}{\varphi \wedge \psi} \quad \frac{}{\perp}^{(i)}}{\perp}^{(i)}$$

(p. 369)

$$\frac{\varphi}{\varphi \vee \varphi} \quad \frac{\psi}{\varphi \vee \psi}$$

$$\frac{\varphi \vee \psi \quad \frac{}{\perp}^{(i)} \quad \frac{}{\perp}^{(i)}}{\perp}^{(i)}$$

$$\frac{\frac{}{\psi}^{(i)} \quad \frac{}{\perp}^{(i)}}{\varphi \rightarrow \psi} \quad \frac{}{\varphi \rightarrow \psi} \quad \frac{\frac{}{\varphi}^{(i)} \quad \vdots}{\varphi \rightarrow \psi} \quad \frac{}{\perp}^{(i)} \quad \frac{\frac{}{\psi}^{(i)} \quad \vdots}{\varphi \rightarrow \psi} \quad \frac{}{\perp}^{(i)}}{\perp}^{(i)}$$

14.7.3 Rules for saturated formulae with a quantifier dominant

The reader will recognize these rules as capturing the two Wittgensteinian insights often conveyed to students when explaining how the two main quantifiers are to be understood: universal quantifications behave like conjunctions of all their instances; and existential quantifications behave like disjunctions of the same. Once again: all major premises for eliminations stand proud; and vacuous discharges of assumptions are prohibited.

$$\frac{\psi(\alpha_1) \dots \psi(\alpha_n) \dots}{\forall x \psi(x)}$$

where $\alpha_1, \dots, \alpha_n \dots$ are all the individuals in the domain

$$\frac{\frac{}{\psi(\alpha)}^{(i)} \quad \vdots}{\forall x \psi(x)} \quad \frac{}{\perp}^{(i)}}{\perp}^{(i)}$$

$$\frac{\frac{}{\psi(\alpha)}^{(i)} \quad \vdots}{\exists x \psi(x)} \quad \frac{\frac{}{\psi(\alpha_1)}^{(i)} \quad \vdots \quad \dots \quad \frac{}{\psi(\alpha_n)}^{(i)} \quad \vdots \quad \dots}{\perp} \quad \frac{}{\perp}^{(i)}}{\perp}^{(i)}$$

where $\alpha_1, \dots, \alpha_n \dots$ are all the individuals in the domain

(p. 370) 14.7.4 Comments on the rules and the constructions they constitute

One can quickly incur sideways spread in writing down a detailed truth-maker or falsity-maker. This feature militates against the actual construction of these otherwise very illuminating and detailed constructions for sentences (and saturated formulae) relative to a given model M . As soon as one has three or more individuals in the domain of M , along with nested quantifiers (especially when they occasion the use of the two rules that require investigation of all instances of a quantified claim), the blow-up, in the form of sideways spread, is prohibitive. But the resulting construction is only ever as *deep* as the longest branch within the analysis tree of the sentence (or saturated formula) being evaluated.

Moreover, in cases where the domain is *infinite*, some of these truth-makers and falsity-makers will contain steps (for the truth-maker for a universal, or the falsity-maker for an existential) that require *infinitely many* subordinate constructions (in the form of truth-makers or falsity-makers for instances of the quantified claims in question). In such cases the constructions cannot be written down. Instead, they exist only as infinitary mathematical objects: labeled trees (where the labels are at least finite!) that can have infinite branching, albeit only with branches of finite length.²¹ Ultimately, the present “inferentialist” approach to formal semantics via the truth-makers and falsity-makers illustrated above requires no more powerful mathematical machinery than is needed in order to vouchsafe the existence of these (rather modest) kinds of infinitary object.

The combinatorial mathematics that is required in order to vouchsafe our truth-makers and falsity-makers is of very modest consistency strength. In any possible world containing both the individuals involved in the model in question and the symbols that are interpreted by the model, the truth-makers and falsity-makers themselves can be taken to form part of one’s ontology at no further cost. To the extent that the “mathematics of trees” (like the arithmetic of the natural numbers) is part of any reasonably sophisticated conceptual scheme, one can rely on our constructions existing whenever their ultimate constituents exist.

14.7.5 Reflexive stability

Any account of truth-makers faces the potential problem of reflexive instability. If Θ is held, on the account in question, to be a truth-maker for φ , then the question arises: What would be the truth-maker, if any, for the proposition that Θ is indeed a (p. 371) truth-maker for φ ? In facing this question, we are fortunately on firm ground. Our definition of the notions of (M -relative) *truth-maker* and of (M -relative) *falsity-maker* is a co-inductive one. (See section 14.10.) Hence the metalinguistic conclusion “ Θ is an M -relative truth-maker for φ ” (abbreviated as $\mathcal{V}_M(\Theta, \varphi)$) can be displayed as the conclusion of a *metalinguistic truth-maker*, which methodically unravels, step by step (and in the appropriate partial order—even if it is an “infinitely sideways-branching” one), the inductive-definitional grounds for the truth of the theoretical claim $\mathcal{V}_M(\Theta, \varphi)$ itself! This is analogous to

A Logical Theory of Truth-Makers and Falsity-Makers

the way in which one can *prove*, in a weak metatheory, that a (here, finite) tree-like object-linguistic construction is a *proof* (in Peano arithmetic, say) of a particular theorem of first-order arithmetic. It is also analogous to the way in which, in a simple language of color-predication, one can provide a truth-maker for the claim that a particular object α , which happens to be red, is colored. The language contains the analytic, atomic inferential rule

$$\frac{t \text{ is red}}{t \text{ is colored}}$$

If in the model M the individual α is red, then we have recourse to the M -relative atomic rule of inference

$$\frac{}{\alpha \text{ is red}}^M$$

This permits the formation of the M -relative truth-maker

$$\frac{\frac{}{\alpha \text{ is red}}^M}{\alpha \text{ is colored}}$$

for the claim that α is colored. In analogous fashion, within a metatheory containing logically atomic inductive clauses governing the formation of truth-makers and falsity-makers, we can build up truth-makers for metalinguistic claims of the form $\mathcal{V}_M(\Theta, \varphi)$. So reflexive stability is ensured.

14.8 Logical Consequence Defined in Terms of Truth-Makers

The classical conception of logical consequence, due to Tarski (1956 [1936]), requires only *preservation of truth*.

(p. 372) Definition 1 (Tarski consequence)

ψ is a logical consequence of Δ

if and only if

for every interpretation M , if every member of Δ is true in M , then ψ is true in M also.

It is worth noting that the foregoing definiens (the right-hand side of the biconditional) is often written as

for every interpretation M , if M makes every member of Δ true, then M makes ψ true also.

A Logical Theory of Truth-Makers and Falsity-Makers

This invites the reader to (mis)construe the model M itself as a truth-maker. But that would be a serious mistake, resulting from taking the gloss “ M makes φ true” much too literally. The model M is too gross an object to serve as a genuine (even M -relative!) truth-maker. Moreover, on this (mis)construal, M would be serving as *the* truth-maker for each and every sentence φ that is true in M . But the whole point of a more refined theory of truth-makers is to be able to distinguish (with respect to some same model M , whatever it may be) the *different* (but still M -relative) truth-makers that can be had by one and the same sentence φ , and, of course, also to distinguish among the different (but still M -relative) truth-makers that can be had by different sentences. Thinking of M as the sole M -relative truth-maker would be to obliterate all such possible refinements. In the special case where M were the actual world, it would be like taking the Great Fact as the only truth-maker for each and every truth.

The foregoing model-theoretic definition of logical consequence can be stated more formally as follows (where $M \Vdash \varphi$ symbolizes “ φ is true in M ”):

$$\begin{aligned} \Delta \models \psi \\ \Leftrightarrow \\ \forall M([\forall \varphi \in \Delta M \Vdash \varphi] \Rightarrow M \Vdash \psi) \end{aligned}$$

This definition yields no insight, however, into how the grounds of truth-in- M of the conclusion ψ might be related to, or determinable from, the grounds of truth-in- M of the premises in Δ (relative to any of the models M interpreting the extralogical vocabulary involved).²²

There is an alternative conception of logical consequence, fashioned for intuitionistic logic (and for finite sets of premises), which sought to address what was taken to be this deficit of insight, and which is owed to Prawitz (1974).

(p. 373) Definition 2 (Prawitz consequence)

ψ is a logical consequence of $\varphi_1, \dots, \varphi_n$

if and only if

there is an effective method f such that for every interpretation M , and for all M -warrants π_1, \dots, π_n for $\varphi_1, \dots, \varphi_n$ respectively, $f(\pi_1, \dots, \pi_n)$ is an M -warrant for ψ .

More formally:

$$\begin{aligned} \varphi_1, \dots, \varphi_n \models_P \psi \\ \Leftrightarrow \\ \exists \text{ eff. } f \forall M \forall \pi_1 \dots \forall \pi_n [(\mathcal{P}_M(\pi_1, \varphi_1) \wedge \dots \wedge \mathcal{P}_M(\pi_n, \varphi_n)) \Rightarrow \mathcal{P}_M(f(\pi_1, \dots, \pi_n), \psi)] \end{aligned}$$

A Logical Theory of Truth-Makers and Falsity-Makers

Here, $\mathcal{P}_M(\pi, \varphi)$ means that the construction π is an M -warrant for the sentence φ . On Prawitz's account, M is taken to be a so-called *basis* consisting of atomic rules of inference, which together provide some measure of interpretation of the primitive extra-logical expressions involved in them. Prawitz's conception of warrant is anti-realist in inspiration. Warrants are the canonical proofs that Dummett marshalls in arguments that are intended to motivate the adoption of anti-realist semantics. Dummett argues that one should regard as licit only the *recognizable* obtaining of the conditions of truth of a sentence—such as is displayed by a canonical proof. And, given this constraint on manifestation of understanding, he argues further, the Principle of Bivalence is infirmed.²³

So it is important to realize that Prawitz is not in the business of distinguishing metaphysical or ontological grounds for truth (in the constitutive sense essayed here) from justificatory grounds. His warrants are epistemic objects. And for Prawitz, as for Dummett, the truth of any claim *consists in* its possessing such a warrant. As an anti-realist, Prawitz makes no distinction between the constitutive grounds that a realist might believe in, and the contrasting epistemic grounds that are the only grounds the anti-realist can recognize as obtaining. But that should not deter us from trying to appreciate an important feature of Prawitz's characterization of logical consequence, and from trying to apply it profitably in our own characterization of the role that truth-makers can be made to play in a notion of logical consequence more congenial to the realist.

On Prawitz's definition of logical consequence, one is able to interpret the normalization theorem for intuitionistic natural deductions as furnishing a proof of the *soundness* of that proof-system. For the normalization procedure (call it ν) itself qualifies as the (p. 374) effective method f called for on the right-hand side of his definition. To see this, suppose one has an intuitionistic natural deduction Π of the conclusion ψ from (possibly logically complex) premises $\varphi_1, \dots, \varphi_n$. We need to convince ourselves that Π will preserve warranted assertibility from its premises to its conclusion. So suppose further that one furnishes M -warrants π_1, \dots, π_n (which take the form of closed canonical proofs using atomic rules in M) for $\varphi_1, \dots, \varphi_n$ respectively. Append each warrant to the assumption-occurrences of its conclusion in Π :

$$\begin{array}{ccc} \pi_1 & \dots & \pi_n \\ \varphi_1 & & \varphi_n \\ \hline & & \\ & \Pi & \\ & & \psi \end{array}$$

Then by normalizing the result one obtains an M -warrant π for the overall conclusion ψ :

$$\underbrace{\begin{array}{ccc} \pi_1 & \dots & \pi_n \\ \varphi_1 & & \varphi_n \end{array}}_{\Pi} \text{ normalizes to } \psi, \quad \text{where } \pi = \nu(\pi_1, \dots, \pi_n, \Pi)$$

What the foregoing reasoning shows is that

$$\varphi_1, \dots, \varphi_n \vdash_I \psi \Rightarrow \varphi_1, \dots, \varphi_n \models_P \psi.$$

This is the promised soundness result for intuitionistic deduction, with respect to Prawitz’s notion of logical consequence.

The converse of this result:

$$\varphi_1, \dots, \varphi_n \models_P \psi \Rightarrow \varphi_1, \dots, \varphi_n \vdash_I \psi$$

remains as Prawitz’s (as yet unproved) *completeness conjecture* for intuitionistic logic.

A very desirable feature of Prawitz’s definition of consequence is how it *Skolemizes* (as $\nu(\pi_1, \dots, \pi_n, \Pi)$) that which is responsible for the (M -relative) truth of the conclusion ψ . On the classical conception, by contrast, one is given to understand only that if the premises are true, then the conclusion ψ is true also—but without giving one any idea as to how any (M -relative) truth-makers for ψ might depend on (M -relative) truth-makers for the premises.

The desirable feature of Prawitz’s definition can, however, be taken over by the truth-maker theorist even if in the service of a more realist rather than anti-realist conception of truth and consequence. The truth-maker theorist can define logical consequence as follows.

(p. 375) Definition 3 (Truth-maker consequence)

ψ is a logical consequence of $\varphi_1, \dots, \varphi_n$

if and only if

there is a “quasi-effective” method f such that for every interpretation M , and for all M -relative truth-makers π_1, \dots, π_n for $\varphi_1, \dots, \varphi_n$ respectively, $f(\pi_1, \dots, \pi_n)$ is an M -relative truth-maker for ψ .

More formally:

$$\varphi_1, \dots, \varphi_n \models_{\mathbf{T}} \psi$$

\Leftrightarrow

$$\exists q. \text{ - eff. } f \forall M \forall \pi_1 \dots \forall \pi_n [((\mathcal{V}_M(\pi_1, \varphi_1) \wedge \dots \wedge \mathcal{V}_M(\pi_n, \varphi_n)) \Rightarrow \mathcal{V}_M(f(\pi_1, \dots, \pi_n), \psi))]$$

Here, $\mathcal{V}_M(\pi, \psi)$ means that the construction π is an M -relative truth-maker for the sentence ψ .

We have written “quasi-effective method f ” because of problems posed by the infinite case. In the finite case, the method will certainly be effective. But in the infinite case we have to countenance definable operations on infinite sets of constructions, and these by definition cannot be effective (in the sense in which Church’s Thesis states that all effective functions are recursive functions). The functions or methods that we shall employ, however, will be “effective-looking” in the infinite case, because of the obvious, orderly way in which one “follows the recipes” for truth-maker transformation within the infinite context. They are recipes obtained by smooth extrapolation from the finite case. In section 14.9 we give a simple finite example of how this truth-maker transformation is effected, by an appropriate analogue of the normalization procedure for proofs. It is an interesting research problem to characterize as precisely as possible the class of functions that we are here calling, informally, the “quasi-effective” ones.

Note that our statement of “truth-maker transformation” by valid argument is subtler than the so-called *entailment principle* (Lowe and Rami 2009: 26):

For every x, y and z : if x is a truth-maker for y and y entails z , then x is a truth-maker for z .

We do not subscribe to this principle. It is too sweeping. For example, on our account, the truth-maker for a true conjunction $\varphi \wedge \psi$ is a construction that contains as sub-constructions a truth-maker for φ and a truth-maker for ψ . But that does not make the truth-maker for the conjunction a truth-maker for either of its conjuncts. *Containing* a truth-maker for each conjunct does not amount to *being* a truth-maker for either of them.²⁴

(p. 376) 14.9 An Example of Truth-Maker Transformation by Valid Argument

We saw earlier that the sentences

$$\exists x Fx \quad \forall x(Fx \rightarrow Gx)$$

were both true in our chosen example of a very simple model M . Note that we have not yet laid out any M -relative truth-maker for $\exists x Gx$. The reader is asked not to look back at the diagram for M , but to focus instead on the following deductive proof of the conclusion $\exists x Gx$ from the above two sentences as premises. Note that this is a formal proof of the more familiar kind, in the system of natural deduction due to Gentzen (1934–35):

A Logical Theory of Truth-Makers and Falsity-Makers

$$\begin{array}{c}
 \frac{\frac{\forall x(Fx \rightarrow Gx)}{Fa \rightarrow Ga} \quad \text{---}^{(1)}}{Ga} \\
 \frac{\exists xFx \quad \frac{\exists xGx}{\exists xGx}^{(1)}}{\exists xGx}^{(1)}
 \end{array}$$

Consider now what happens when we supply the M -relative truth-makers constructed earlier for the two premises (or undischarged assumptions) of this deductive proof:

$$\begin{array}{c}
 \frac{\frac{\text{---}^M}{G\alpha}}{F\alpha \rightarrow G\alpha} \quad \frac{\frac{\text{---}^M}{G\beta}}{F\beta \rightarrow G\beta} \quad \frac{\frac{\frac{\text{---}^{(1)}}{F\gamma}}{\perp}^M}{F\gamma \rightarrow G\gamma}^{(1)} \quad D = \{\alpha, \beta, \gamma\} \\
 \frac{\forall x(Fx \rightarrow Gx)}{\forall x(Fx \rightarrow Gx)} \\
 \frac{\frac{\frac{\text{---}^M}{F\alpha}}{\exists xFx} \quad \frac{\forall x(Fx \rightarrow Gx)}{Fa \rightarrow Ga} \quad \frac{\text{---}^{(1)}}{Fa}}{\exists xFx \quad \frac{Ga}{\exists xGx}^{(1)}} \\
 \frac{\exists xFx \quad \exists xGx}{\exists xGx}^{(1)}
 \end{array}$$

When the truth-makers above are grafted so that their conclusions are superimposed upon the premise-occurrences of the same sentences in the formal proof below those truth-makers, we obtain a construction of a mixed kind—a combination of the truth-makers “above,” and the formal deductive proof “below”—that invites an analogue of the *normalization process* of the kind familiar to a proof-theorist.

(p. 377) The recipe for normalizing is much the same in the truth-maker setting as it is in the proof setting.²⁵ One treats applications of rules for evaluating-as-true (in M) as though they are applications of introduction rules, and similarly one treats applications of rules for evaluating-as-false (in M) as though they are applications of elimination rules. Only now, one has to countenance the possibility that one will be substituting *individuals* (from the domain of M) for free occurrences of variables in formulae, thereby forcing an erstwhile formal proof to involve *saturated formulae*.

In order to illustrate, we shall normalize the result of grafting the truth-maker for $\exists xFx$ and the truth-maker for $\forall x(Fx \rightarrow Gx)$ onto their respective undischarged assumption-occurrences in the formal proof “below” them. The resulting occurrences of $\exists xFx$ and $\forall x(Fx \rightarrow Gx)$ are then “maximal,” each standing as the conclusion of an “introduction” and as the major premise of the corresponding elimination. Dealing first with the maximal occurrence of $\exists xFx$, we apply the appropriately modified “reduction procedure” to obtain

A Logical Theory of Truth-Makers and Falsity-Makers

$$\begin{array}{c}
 \frac{\frac{\frac{}{G\alpha}}{F\alpha \rightarrow G\alpha}}{\frac{\frac{\frac{}{G\beta}}{F\beta \rightarrow G\beta}}{\frac{\frac{\frac{}{F\gamma}}{F\gamma \rightarrow G\gamma}}{\perp}}{F\gamma \rightarrow G\gamma}}{D = \{\alpha, \beta, \gamma\}}}{\frac{\frac{\frac{\frac{}{\forall x(Fx \rightarrow Gx)}}{F\alpha \rightarrow G\alpha}}{F\alpha}}{G\alpha}}{\exists xGx}}
 \end{array}$$

Note how the individual α has insinuated itself further into the resulting construction. We have left the realm of proofs, and entered a realm of constructions that contain actual individuals (like α) but are not (yet) actual truth-makers. For the construction at this stage is not yet in normal form.

Next we have to deal with the maximal occurrence of $\forall x(Fx \rightarrow Gx)$. We apply the reduction procedure for the universal quantifier in order to replace with an appropriate reduct the construction terminating on the lower occurrence of $F\alpha \rightarrow G\alpha$. The overall result is

$$\frac{\frac{\frac{}{G\alpha}}{F\alpha \rightarrow G\alpha} \quad \frac{}{F\alpha}}{G\alpha}}{\exists xGx}$$

(p. 378) in which the occurrence of $F\alpha \rightarrow G\alpha$ is now (newly) maximal. Applying the reduction procedure for the conditional, we obtain as our final result the (normal!) truth-maker

$$\frac{\frac{}{G\alpha}}{\exists xGx}$$

Now the reader is invited to look back at the model M . This is one of the truth-makers one could have given for the claim $\exists xGx$. The other possibility was

$$\frac{\frac{}{G\beta}}{\exists xGx}$$

But the truth-maker involving β was ignored by the normalization process. This is because the process was directed toward uncovering a truth-maker for $\exists xGx$ that was *implicit* in the combination of (i) the truth-makers actually furnished for $\exists xFx$ and for $\forall x(Fx \rightarrow Gx)$ —the first of which involved α , not β —and (ii) the deductive proof of $\exists xGx$ from those two claims as premises. The last-displayed truth-maker for $\exists xGx$ could not, of course, come to light as a result of this very focused process. It is the Skolemizing nature of the construction of grounds for the truth of the conclusion of a deductive proof, out of grounds for the truth of its premises, which underlies this ability to focus on what is rele-

A Logical Theory of Truth-Makers and Falsity-Makers

vant. We are now in a better position to appreciate how it is that a formal deduction enables one to see how the truth of a conclusion is contained in that of the premises—not, as Frege once wrote,²⁶ as beams are contained in a house, but rather as a plant is contained in the seed from which it grows. And we have helped explicate this insight for the benefit of the Fregean *realist*.

We are now in a position to reflect somewhat critically on the following claim:

Th[e] idea of a perfect parallelism of logical and ontological complexity is the misery of logical atomism . . . [W]e uphold the independence of ontological from logical complexity: ontologically complex objects (those having proper parts) are not for that reason also in some way logically complex, any more than there is reason to suppose that to every logically complex (true) sentence there corresponds an ontologically complex entity which makes it true.

(Mulligan et al. 1984: 298)

With our account of truth-makers and falsity-makers as *appropriately structured* complex entities, we have sought to relieve this particular “misery” of logical atomism. We are not, of course, claiming that ontologically complex objects (those having proper parts) are also in some way logically complex. Rather, we are challenging the claim (p. 379) that there is no “reason to suppose that to every logically complex (true) sentence there corresponds an ontologically complex entity which makes it true.” On the contrary, our theory of truth-makers shows that that is precisely the case.

14.10 A formal Co-inductive Definition of Truth-Makers and Falsity-Makers

Earlier we employed the formal expression $\mathcal{V}_M(\Pi, \varphi)$ to express the claim that Π is an M -relative truth-maker for the sentence (or saturated formula) φ . We shall now elevate the subscript M to be a separate argument of this predication, alongside its domain D of individuals. That way, M can be reconstrued as consisting of *saturated literals*, or the rules that codify them.

A *saturated literal* is a saturation of an atomic formula, or of the negation of an atomic formula. The model M will of course be both *complete* and *coherent*. The *completeness* of M (when it is construed as a set of rules) is a matter of containing, for each n -tuple of individuals drawn from the domain D of M , and for each n -place atomic predicate A , either the “positive” rule

$$\frac{}{A(\vec{\alpha})}^M$$

or the “negative” rule

$$\frac{A(\vec{\alpha})}{\perp} M;$$

whereas the *coherence* of M is a matter of *not* containing, for any such tuple and predicate A , *both* of these rules. Our rule-theoretic conception of a model is therefore thoroughly Fregean and Tarskian, in assuming determinacy of all atomic facts. Indeed, we can consider M as ambiguous between its rule-theoretic characterization, on the one hand, and, on the other hand, the more usual Tarskian characterization, in terms of A -extensions that are subsets of the n -fold Cartesian product of the domain D of M .

Definition 4

The two notions $\mathcal{V}(\Pi, \varphi, M, D)$ and $\mathcal{F}(\Pi, \varphi, M, D)$ (respectively: “the construction Π is a truth-maker for the saturated formula φ modulo the domain D and the set M of saturated literals” and “the construction Π is a falsity-maker for the saturated formula φ modulo the domain D and the set M of saturated literals”) are co-inductively defined by the following metalinguistic axioms and rules of inference:

(p. 380) For saturated literals λ whose individuals are in D :

$$(\lambda\mathcal{V}) \frac{}{\mathcal{V}(\lambda, \lambda, \{\lambda\}, D)}$$

For positive saturated literals A whose individuals are in D :

$$(At\mathcal{F}) \frac{}{\mathcal{F}\left(\frac{\neg A \quad A}{\perp}, A, \{\neg A\}, D\right)} \quad \frac{}{\mathcal{F}\left(\frac{\neg A \quad A}{\perp}, \neg A, \{A\}, D\right)}$$

For non-atomic saturated formulae φ whose individuals are in D :

$$(\neg\mathcal{V}) \frac{\mathcal{F}(\Pi, \varphi, M, D)}{\mathcal{V}\left(\frac{\Pi}{\neg\varphi}, \neg\varphi, M, D\right)}$$

$$(\neg\mathcal{F}) \frac{\mathcal{V}(\Pi, \varphi, M, D)}{\mathcal{F}\left(\frac{\neg\varphi \quad \Pi}{\perp}, \neg\varphi, M, D\right)}$$

For saturated formulae φ_1, φ_2 of any degree of complexity (including positive saturated literals) whose individuals are in D :

$$(\wedge\mathcal{V}) \frac{\mathcal{V}(\Pi_1, \varphi_1, M_1, D) \quad \mathcal{V}(\Pi_2, \varphi_2, M_2, D)}{\mathcal{V}\left(\frac{\Pi_1 \quad \Pi_2}{\varphi_1 \wedge \varphi_2}, \varphi_1 \wedge \varphi_2, M_1 \cup M_2, D\right)}$$

$$(\wedge\mathcal{F}) \frac{\mathcal{F}(\Pi, \varphi_i, M, D)}{\mathcal{F}\left(\frac{\varphi_1 \wedge \varphi_2 \quad \Pi}{\perp}, \varphi_1 \wedge \varphi_2, M, D\right)} \quad (i = 1, 2)$$

$$\begin{aligned}
 (\forall \mathcal{V}) & \frac{\mathcal{V}(\Pi, \varphi_i, M, D)}{\mathcal{V}\left(\frac{\Pi}{\varphi_1 \vee \varphi_2}, \varphi_1 \vee \varphi_2, M, D\right)} \quad (i = 1, 2) \\
 (\forall \mathcal{F}) & \frac{\mathcal{F}(\Pi_1, \varphi_1, M_1, D) \quad \mathcal{F}(\Pi_2, \varphi_2, M_2, D)}{\mathcal{F}\left(\frac{\varphi_1 \vee \varphi_2 \quad \Pi_1 \quad \Pi_2}{\perp}, \varphi_1 \vee \varphi_2, M_1 \cup M_2, D\right)} \\
 (\rightarrow \mathcal{V}) & \frac{\mathcal{V}(\Pi, \varphi_2, M, D)}{\mathcal{V}\left(\frac{\Pi}{\varphi_1 \rightarrow \varphi_2}, \varphi_1 \rightarrow \varphi_2, M, D\right)} \quad \frac{\mathcal{F}(\Pi, \varphi_1, M, D)}{\mathcal{V}\left(\frac{\Pi}{\varphi_1 \rightarrow \varphi_2}, \varphi_1 \rightarrow \varphi_2, M, D\right)}
 \end{aligned}$$

(p. 381)

$$(\rightarrow \mathcal{F}) \frac{\mathcal{V}(\Pi_1, \varphi_1, M_1, D) \quad \mathcal{F}(\Pi_2, \varphi_2, M_2, D)}{\mathcal{F}\left(\frac{\varphi_1 \rightarrow \varphi_2 \quad \Pi_1 \quad \Pi_2}{\perp}, \varphi_1 \rightarrow \varphi_2, M_1 \cup M_2, D\right)}$$

For ψ any unary saturated formula whose individuals are in D :

$$\begin{aligned}
 (\exists \mathcal{V}) & \frac{\mathcal{V}(\Pi, \psi_\alpha^x, M, D)}{\mathcal{V}\left(\frac{\Pi}{\exists x \psi}, \exists x \psi, M, D\right)} \quad \text{where } \alpha \in D \\
 (\exists \mathcal{F}) & \frac{\{\mathcal{F}(\Pi_\alpha, \psi_\alpha^x, M_\alpha, D)\}_{\alpha \in D}}{\mathcal{F}\left(\frac{\exists x \psi \quad \{\Pi_\alpha\}_{\alpha \in D}}{\perp}, \exists x \psi, \bigcup_{\alpha \in D} M_\alpha, D\right)} \\
 (\forall \mathcal{V}) & \frac{\{\mathcal{V}(\Pi_\alpha, \psi_\alpha^x, M_\alpha, D)\}_{\alpha \in D}}{\mathcal{V}\left(\frac{\{\Pi_\alpha\}_{\alpha \in D}}{\forall x \psi}, \forall x \psi, \bigcup_{\alpha \in D} M_\alpha, D\right)} \\
 (\forall \mathcal{F}) & \frac{\mathcal{F}(\Pi, \psi_\alpha^x, M, D)}{\mathcal{F}\left(\frac{\forall x \psi \quad \Pi}{\perp}, \forall x \psi, M, D\right)} \quad \text{where } \alpha \in D
 \end{aligned}$$

The foregoing rules are the basis rules and the inductive rules in the co-inductive definition of the notions $\mathcal{F}(\Pi, \varphi, M, D)$ and $\mathcal{V}(\Pi, \varphi, M, D)$. To complete the definition, we have the *Closure clause*.

If $\mathcal{V}(\Pi, \varphi, M, D)$, then this can be shown by appeal to the foregoing rules. Likewise, if $\mathcal{F}(\Pi, \varphi, M, D)$, then this can be shown by appeal to the foregoing rules.

We shall use the usual notation \overline{D} for the cardinality of D . Note that when D is infinite, the rules $(\forall \mathcal{V})$ and $(\exists \mathcal{F})$ call for the construction of \overline{D} -furcating trees as, respectively, M -relative truth-makers for universals, and M -relative falsity-makers for existentials. So our metatheory needs to furnish such objects as needed.

14.11 Some Results; and Discussion

Metatheorem 1

Modulo a metatheory which contains the mathematics of $\overline{\overline{D}}$ -furchating trees of finite depth, we have, for all models M with domain D ,

$$\exists \Pi \mathcal{V}(\Pi, \varphi, M, D) \Leftrightarrow \varphi \text{ is true in } M$$

where the right-hand side is in the sense of Tarski.²⁷ (p. 382)

Proof. The proof is by the obvious induction on the complexity of the sentence (or saturated formula) φ . Note that the proof is intuitionistic, provided only that the M -relative truth-makers for universals and the M -relative falsity-makers for existentials, in the case where D is infinite, can be assumed to exist, courtesy of the background metamathematics. This observation affects the right-to-left direction in the relevant cases of the inductive step.

Corollary 1

Logical consequence \models in the Tarskian sense of Definition 1 coincides with logical consequence $\models_{\mathbf{T}}$ in the sense of Definition 3 in terms of quasi-effective transformability of truth-makers, but with “quasi-effective” interpreted as imposing no restriction at all on the Skolem function involved.

Proof. Suppose $\varphi_1, \dots, \varphi_n \models \psi$. Let D be an arbitrary domain, and let M be a model with domain D . Let π_1, \dots, π_n be M -relative truth-makers for $\varphi_1, \dots, \varphi_n$ respectively. By Metatheorem 1, $\varphi_1, \dots, \varphi_n$ are true in M in the Tarskian sense. Hence ψ is true in M in the Tarskian sense. By Metatheorem 1 again, there is an M -relative truth-maker for ψ . Choose one such, by whatever method will ensure uniqueness of choice. Take that choice as the value $f(\pi_1, \dots, \pi_n)$. We thereby “construct” (in a possibly non-effective manner) a function f that can serve, in accordance with Definition 3, for the conclusion that $\varphi_1, \dots, \varphi_n \models_{\mathbf{T}} \psi$.

For the converse, suppose $\varphi_1, \dots, \varphi_n \models_{\mathbf{T}} \psi$. By Definition 3, let f be the function that will transform, for any model M , M -relative truth-makers for $\varphi_1, \dots, \varphi_n$ into an M -relative truth-maker for ψ . Let M' be any model in which $\varphi_1, \dots, \varphi_n$ are true in M' in the Tarskian sense. By Metatheorem 1, there are M' -relative truth-makers π_1, \dots, π_n , say, for $\varphi_1, \dots, \varphi_n$. Hence $f(\pi_1, \dots, \pi_n)$ is an M' -relative truth-maker for ψ . By Metatheorem 1 again, ψ is true in M in the Tarskian sense. Hence $\varphi_1, \dots, \varphi_n \models \psi$.

QED

When the domain D is infinite, the furchations within truth-makers and falsity-makers that are called for in connection with the rules $(\exists \mathcal{F})$ and $(\forall \mathcal{V})$ will involve exactly as many branches as there are individuals in the domain. Such infinitary trees are not, of course,

surveyable; but they are well-defined mathematical objects that can serve as representations of truth-makers and of falsity-makers for quantified sentences speaking of the infinitely many individuals in D . They should be congenial to the metaphysical and semantic realist, who entertains no skeptical qualms about the existence of infinitary objects, or about determinate truth-values for claims involving quantification over unsurveyable domains.

Thus far, no formal definition has been given in the truth-making literature, by philosophical logicians, for this quasi-technical notion of analytic metaphysics.²⁸ It is here (p. 383) proposed that the pre-formal notion of truth-maker (resp., falsity-maker) is nicely explicated by the foregoing formal notion of a truth-maker (resp., falsity-maker) of a saturated formula φ , *modulo* a domain D of discourse, from the basic facts represented in a complete and coherent set M of saturated literals.

The notion of a truth-maker for φ enables one also to isolate those possibly “smaller parts of reality” that might suffice to determine the truth of φ . Not all of M need be used, nor all of D surveyed, in order to determine the truth-value of φ . The exact “basic materials” on which that truth-value supervenes, so to speak, are captured by the truth-makers (or falsity-makers) available.

There could well be many different truth-makers for one and the same saturated formula under one and the same interpretation (i.e. *modulo* one and the same domain D , and one and the same set M of saturated literals). These different truth-makers represent different “constitutive routes” to the same truth-value under the interpretation in question. (Similarly for falsity-makers.)

14.11.1 Can we constrain the method of truth-maker transformation without changing the relation of classical logical consequence?

We were originally inspired by an interesting analogy with Prawitz’s definition of intuitionistic logical consequence, when suggesting that we define a more classical notion of consequence in terms of what we called “quasi-effective” transformability of truth-makers for the premises of an argument into a truth-maker for its conclusion. We have seen (Corollary 1) that we obtain exactly the relation of classical logical consequence if we impose no restriction at all on the kind of method used for the transformation of truth-makers. In effect, Corollary 1 established that

$$\varphi_1, \dots, \varphi_n \models \psi$$

$$\Leftrightarrow$$

$$\exists f \forall M \forall \pi_1 \dots \forall \pi_n [((\mathcal{V}_M(\pi_1, \varphi_1) \wedge \dots \wedge \mathcal{V}_M(\pi_n, \varphi_n)) \Rightarrow \mathcal{V}_M(f(\pi_1, \dots, \pi_n), \psi))]$$

where the quantification $\exists f$ is over (set-theoretic) functions *tout court*.

(p. 384) The prima facie risk incurred by seeking to restrict the kind of function f that would be permissible here is that we might end up circumscribing too narrowly the resulting set of supposedly “classically valid” arguments $\varphi_1, \dots, \varphi_n : \psi$. The harder we make it to find such a function f , the more exigent a notion the definiendum \models becomes.

This exigency can be offset, however, by *more narrowly circumscribing* the class of models M with respect to which the f -transformations of truth-makers have to be effected. Noting that the quantification $\forall M$ is really a gloss for

$$(\forall \text{ domains } D)(\forall \text{ models } M \text{ based on domain } D)$$

we can inquire whether we might be able to constrain the kind of domain D involved, so as to offset any restriction of “quasi-effectiveness” that might be imposed on the function f . Obvious candidates are:

1. D is countable;
2. D is effectively enumerable; and/or
3. D is decidable.

Indeed, we might even consider

4. D is the set of natural numbers.

The challenge, as the present author sees it, is to find some formal delimitation of the class of permissible functions f that captures the orderliness of the transformations that are involved when one takes any *classical* natural deduction

$$\underbrace{\varphi_1, \dots, \varphi_n}_{\Pi} \\ \psi$$

and accumulates on its premises respective M -relative truth-makers π_1, \dots, π_n :

$$\begin{array}{ccc} \pi_1 & \dots & \pi_n \\ \varphi_1 & \dots & \varphi_n \\ \hline & \Pi & \\ & \psi & \end{array}$$

and thereupon seeks to “normalize” the resulting construction so that it becomes an M -relative truth-maker for ψ :

$$\begin{array}{ccc}
 \begin{array}{c} \pi_1 \quad \dots \quad \pi_n \\ \varphi_1 \quad \dots \quad \varphi_n \\ \hline \Pi \\ \psi \end{array} & \text{normalizes to} & \begin{array}{c} \pi \\ \psi \end{array}, \quad \text{where } \pi = \nu(\pi_1, \dots, \pi_n, \Pi) \\
 & \mapsto &
 \end{array}$$

(p. 385) We have seen displays like these before, of course, in section 14.8. But now it has to be borne in mind that the “lower” proof Π in this context is *classical*; and that the “upper” constructions π_1, \dots, π_n are (M -relative) *truth-makers*, not finitary M -warrants.

Here is a suggestion, followed by a conjecture. Seek some general but interestingly restrictive characterization \mathcal{E} of the transformation-functions ν that are involved in this kind of normalization process—“general” enough to encapsulate them all, but “restrictive” enough to bring out the Skolemite flavor that was stressed above as providing illumination of the idea that the truth of the conclusion of a valid argument is somehow “contained in” the truth of its premises. The guess is that \mathcal{E} will characterize some class of functions reasonably low in the arithmetical hierarchy.

Now recall Prawitz’s soundness-via-normalization result, and its converse—his completeness conjecture—involving the relations \models_I and \models_P . We submit that there are analogues to be had for each of these.

To see this, recall that we have just talked above of a normalization process involving *classical* natural deductions, and truth-makers for their undischarged assumptions.

First, on the assumption that \mathcal{E} has been chosen so as to capture correctly the kind of method implicit in this normalization process, we shall have an analogue, for the classical case, of Prawitz’s soundness theorem. And this will be the case regardless of any restriction that might be imposed on the domains D —in fact, the stronger any such restriction might be, the easier it will be for functions f with property \mathcal{E} to effect what is required of them.

We propose imposing the restriction that the domain D be *decidable*. Of the four options listed above, this one strikes the present author as the most natural. So the normalization result that constitutes the soundness theorem for classical natural deductions will be stated as follows:

$$\begin{array}{c}
 \varphi_1, \dots, \varphi_n \vdash_C \psi \\
 \Rightarrow \\
 \exists f(\mathcal{E}f \wedge \forall \text{decidable domains } D \forall \text{models } M \text{ based on } D \\
 \forall \pi_1 \dots \forall \pi_n [((\mathcal{V}_M(\pi_1, \varphi_1) \wedge \dots \wedge \mathcal{V}_M(\pi_n, \varphi_n)) \Rightarrow \mathcal{V}_M(f(\pi_1, \dots, \pi_n), \psi))]
 \end{array}$$

That normalizability will hold, in the requisite sense, is made more plausible by Metatheorem 3 below.

Secondly, we venture to assert the converse—now in the form of a completeness conjecture. It is to be hoped that the proposed (and as yet unspecified) restriction \mathcal{E} on our transformation methods f will exactly offset the restriction that the domain D be decidable.²⁹

(p. 386) **Completeness conjecture for classical natural deduction:**

$$\begin{aligned} & \exists f(\mathcal{E}f \wedge \forall \text{decidable domains } D \forall \text{models } M \text{ based on } D \\ & \forall \pi_1 \dots \forall \pi_n [((\mathcal{V}_M(\pi_1, \varphi_1) \wedge \dots \wedge \mathcal{V}_M(\pi_n, \varphi_n)) \Rightarrow \mathcal{V}_M(f(\pi_1, \dots, \pi_n), \psi))] \\ & \Rightarrow \\ & \varphi_1, \dots, \varphi_n \vdash_C \psi \end{aligned}$$

We therefore commend to the reader the research problem of formulating a suitable property \mathcal{E} and establishing (or making highly plausible) the ensuing completeness conjecture in the form just given. One has to confront the problem, of course—which confronts Prawitz’s original completeness conjecture for the intuitionistic case as well—that the use of any *informal* terms such as “effective” and “decidable” render the conjecture incapable of truly *formal* proof. (In the same way, Church’s Thesis to the effect that every effective function on the naturals is recursive is incapable of truly formal proof.) But it is to be hoped that, despite this, some sort of “progress in persuasion” might be made, concerning both Prawitz’s completeness conjecture for the intuitionistic case, and the current completeness conjecture for the classical case that is modeled, by analogy, upon it.

14.11.2 Why falsity-makers?

Armstrong has raised the question (in personal correspondence):

A quick and unconsidered reaction is that a falsitymaker would be a truthmaker for the contradictory of the proposition made false. So why not use this truthmaker as a substitute for the fals[ity]maker?

This suggestion is a very reasonable first reaction. The challenge, however, is to characterize truth-makers for sentences φ *recursively*, unraveling the logical structure of φ . So there has to be a clause for negation. We need a clause along the lines of

T is a truthmaker for $\neg\psi$ if and only if $\dots \psi \dots$

where the blanks have to be filled in.

Compare this with

T is the truth value for $\neg\psi$ if and only if F is the truth value for ψ .

With two-valued truth-tables, we move easily between T and F , never entertaining the possible constraint (analogous to Armstrong’s suggestion above) that we should work on-

A Logical Theory of Truth-Makers and Falsity-Makers

ly with the truth value T . In the truth-tabular context, Armstrong's suggestion would be to the effect that we should regard any sentence ψ as having F for its truth value just in case T is the truth value for its negation $\neg\psi$. But it is the latter condition (p. 387) (T is the truth value for $\neg\psi$) for which one seeks a "recursively unwinding" equivalent that will remove the negation sign; and that is what forces the recourse to the truth value F (for ψ).

Likewise with truth-makers (in place of assigning T) and falsity-makers (in place of assigning F).

Truth consists in the existence of a truth-maker; and falsity consists in the existence of a falsity-maker. The usual Frege-Tarski clause

$\neg\psi$ is true if and only if ψ is false

becomes

$\neg\psi$ has a truthmaker if and only if ψ has a falsitymaker,

or

$\exists \Pi \Pi$ makestrue $\neg\psi$ if and only if $\exists \Sigma \Sigma$ makesfalse ψ .

Now the temptation is irresistible to forge some kind of relation between satisfiers of the left-hand side of this biconditional, and satisfiers of its right-hand side. That is, we want to express the biconditional parametrically, without the existential quantifiers, in a form like

Π makestrue $\neg\psi$ if and only if $f(\Pi)$ makesfalse ψ

—or, with the functional dependency going in the other direction,

Π makesfalse ψ if and only if $g(\Pi)$ makestrue $\neg\psi$.

Here, now, we can see that the function f would be an extraction, from within any truth-maker Π for $\neg\psi$, of a falsity-maker for ψ ; and that the function g would be an elaboration, of any falsity-maker Π for ψ , into a truth-maker for $\neg\psi$.

Both f and g are obviously furnished by the rule ($\neg\mathcal{V}$) above for constructing truth-makers for negations, to wit

if Π makesfalse ψ , then $\frac{\Pi}{\neg\psi}$ makestrue $\neg\psi$.

The converse of this conditional is in effect secured by the closure clause for the co-inductive definition of truth-makers and falsity-makers.

So now, if one is given a falsity-maker for ψ , one knows how to turn it into a truth-maker for $\neg\psi$: just apply a terminal step of the "negation introduction" rule ($\neg\mathcal{V}$). That is the op-

eration g mentioned above. And, if one is given a truth-maker for $\neg\psi$, one knows how to obtain a falsity-maker for ψ : just extract its immediate subordinate construction (the one needed for the terminal application of $(\neg\mathcal{V})$). That is the operation f mentioned above.

(p. 388) 14.11.3 The question of the necessitation of truth by the existence of a truth-maker

We noted in section 14.4 that on our account, any truth-maker Θ for a sentence φ is so constituted that the resulting truthmaking is an *internal* relation, in the sense of Armstrong (2004). Here we clarify how this is so. As Alex Oliver has posed the problem,³⁰ we have to show how it is the case that

. . . the existence of one of [the] truthmakers [as here characterized] entails a truth it makes true. How could that be, given that [these] truthmakers are mathematical objects and can presumably exist even when the sentences they make true are false?

Our answer is as follows. Each truth-maker Θ , if it contains any “domain-fold” branchings (i.e. steps of $(\forall\mathcal{V})$ or $(\exists\mathcal{F})$) must, at each such step, have exactly one subordinate construction for each individual in the domain (D , say). So the existence of this kind of mathematical object Θ necessitates the truth of its conclusion φ when φ is interpreted with its quantifiers ranging over the domain D , and (for any model M with domain D that affords the atomic M -rules that have been used in the truth-maker) as making the relevant claim about M . This, we submit, is about as close as the truth-maker theorist of the current stripe can come to satisfying the Armstrongian demand that the existence of the truth-maker Θ should necessitate the truth of the sentence φ that it purports to make true. There is actually a nice air of generality about the implicit claim: Θ necessitates the truth of φ in *any* model with domain D that affords the atomic rules that have been used in Θ . It is as if the truth-maker theorist can say,

Necessarily, in any world like *this*: . . .

[here pointing to the atomic M -rules used]

and with exactly *these individuals*: . . .

[here indicating the range D of instantiating individuals for each and every step of $(\forall\mathcal{V})$ or $(\exists\mathcal{F})$ within Θ]

this truth-maker: . . .

[here pointing to Θ]

makes-true the claim that φ .

(p. 389) 14.11.4 The threatened slide to realism, and how to resist it

A Logical Theory of Truth-Makers and Falsity-Makers

We foreshadowed earlier that our treatment of truth-makers might provide the realist with some new resources in his debate with the anti-realist. It is time now to reveal how this is so, and also to indicate how this new line of realist argument might be resisted.

The \mathcal{V} - and \mathcal{F} - rules are a straightforward transcription, into a recursive recipe for building constructions of the appropriate kinds, of the familiar inductive clauses in the Tarskian definition of truth or satisfaction. (This observation lies behind the obvious proof of Metatheorem 1.) Since the latter clauses, as already stressed, are acceptable to the anti-realist, it might be thought that the \mathcal{V} - and \mathcal{F} - rules would be acceptable also.

On the assumption (for the time being) that they are indeed acceptable, let us explore some further consequences.

Metatheorem 2 (Principle of Bivalence)

For all φ , either $\exists \Pi \mathcal{V}(\Pi, \varphi, M, D)$ or $\exists \Sigma \mathcal{F}(\Sigma, \varphi, M, D)$.

Proof, and running anti-realist commentary. The proof is reasonably obvious (by induction on the complexity of φ), but it is important to realize that it is strictly classical.

The basis step of the inductive proof requires the completeness of M , in the sense explained above. This is the first point at which a classical conception obtrudes.

Classical metareasoning obtrudes further, at two more places.

First, in the inductive step dealing with saturated formulae of the form $\forall x\psi$ we need, in the metalogic, the assurance that

either every D -instance $\psi\alpha$ has an M -relative truth-maker, or some D -instance $\psi\alpha$ does not have any M -relative truth-maker.

The first disjunct will allow one to construct an M -relative truth-maker for $\forall x\psi$ from the M -relative truth-makers for all its instances. By appeal to the Inductive Hypothesis the second disjunct will imply

some D -instance $\psi\alpha$ has an M -relative falsity-maker,

and this will allow one to construct an M -relative falsity-maker for $\forall x\psi$. Thus we obtain either an M -relative truth-maker or an M -relative falsity-maker for $\forall x\psi$.

Secondly, in the inductive step dealing with saturated formulae of the form $\exists x\psi$ we need, in the metalogic, the assurance that

either every D -instance ψ has an M -relative falsity-maker, or some D -instance ψ does not have any M -relative falsity-maker.

A Logical Theory of Truth-Makers and Falsity-Makers

(p. 390) The first disjunct will allow one to construct an M -relative falsity-maker for $\exists x\psi$ from the M -relative falsity-makers for all its instances. By appeal to the Inductive Hypothesis the second disjunct will imply

some D -instance $\psi\alpha$ has an M -relative truth-maker,

and this will allow one to construct an M -relative truth-maker for $\exists x\psi$. Thus we obtain either an M -relative falsity-maker or an M -relative truth-maker for $\exists x\psi$.

QED

In handling these two quantifier clauses in our co-inductive definition of truth-makers and falsity-makers, the assurance that is needed has the logical form

$$\forall x\Psi x \vee \exists x\neg\Psi x.$$

If this is taken to be available as a theorem of the metalogic, then we are dealing with a *strictly classical* metalogic.³¹ For this principle is *not* acceptable to the intuitionist. So the intuitionist is unable (at least, by *this* method of proof) to deliver Metatheorem 2—which is as it should be.

Metatheorem 3

Any M -relative truth-maker for $\neg\neg\varphi$ contains an M -relative truth-maker for φ . Hence in the object language we have $\neg\neg\varphi \models \varphi$.

Proof. Suppose Π is an M -relative truth-maker for $\neg\neg\varphi$. According to the closure clause of our co-inductive definition of truth-makers and falsity-makers, Π must have been constructed by appeal to the rule ($\neg\mathcal{V}$), and accordingly have the form

$$\frac{\frac{\text{---}^{(i)}}{\neg\varphi}}{\Sigma}}{\perp}^{(i)} \quad \frac{\text{---}^{(i)}}{\neg\neg\varphi}$$

where Σ is an M -relative falsity-maker for $\neg\varphi$. According to the closure clause of our co-inductive definition of truth-makers and falsity-makers, Σ must have been constructed by appeal to the rule ($\neg\mathcal{F}$), and accordingly have the form

$$\frac{\begin{array}{c} \Theta \\ \hline \neg\varphi \quad \varphi \end{array}}{\perp}$$

where Θ is an M -relative truth-maker for φ . QED

(p. 391) The anti-realist's immediate objection to this result will involve challenging the framing of the co-inductive definition of truth-makers and falsity-makers. For the anti-re-

alist envisages situations where it is known that $\neg\varphi$ itself leads to absurdity, without that in itself guaranteeing that there is warrant for the assertion of φ . So the anti-realist's real complaint will be against the closure clause of the inductive definition. For this closure clause, particularly as it applies to falsity-makers for negations, makes the construction rule ($\neg\mathcal{F}$), with its built-in guarantee of a truth-maker for φ , the sole route to the rejection of a negation. While not having space here to develop this criticism on behalf of the anti-realist, we nevertheless wish at least to *identify* the exact issue that has to be joined in any further debate with the realist over this all-too-easy-looking validation of the strictly classical rule of double-negation elimination.

14.11.5 An important application, not hitherto stressed in the truth-making literature

The notion of a truth-maker (or falsity-maker) allows one to analyze what basic facts might be *relevant* to the truth (or falsity) of a saturated formula. To be sure, truth-makers and falsity-makers can be infinitary, but this is the case only when the domain D itself is infinite; and nothing less could be expected of a truth-maker for a universal claim (or of a falsity-maker for an existential one) in such situations.

Because truth-makers and falsity-makers allow us to focus more clearly on the “local” grounds for truth or falsity of a given sentence, they can be employed as important tools when explicating the notion of cognitive significance. A crucial explicandum in this project is the idea of *creative extension* of a given theory by the incorporation of new theoretical vocabulary involved in new hypotheses. In spelling out this idea, the need arises to keep track of how it will always be the case that facts expressible in the observational vocabulary will feature in the “grounds for falsity” of any hypothesis that is able to be refuted by *the observational evidence*. That is how the hypotheses in question acquire cognitive significance, and impart it to the new theoretical terms that they contain. It turns out that this idea is key—and sufficient—for avoiding the well-known Church-style collapses of the resulting formally defined notion of cognitive significance.³² For the purposes of explicating this idea of creative extension and the necessary involvement of observational facts, “whole models” M prove to be too blunt or crude. It turns out that what is needed, instead, are the foregoing notions of M -relative truth-makers and falsity-makers. For a fuller development of these ideas, and an “adequacy metatheorem” concerning the resulting notion of cognitive significance, the reader is referred to Tennant (1997: ch. 11).

(p. 392) 14.11.6 The question of language-independence

A truth-maker theorist who wishes to define truth-makers in such a way as to ensure that they are *language-independent* might take issue with our appeal to predicate-symbols that belong to a particular language.³³ But it is of course open to such a theorist to secure the complete language-independence of truth-makers by stipulating (as for example Prawitz 1968) that any n -place predicate-symbol R is to be interpreted, at all its occurrences within labels of nodes of a truth-maker, as the (language-independent) n -place *re-*

A Logical Theory of Truth-Makers and Falsity-Makers

lation R itself (in the unary case, as REDNESS, say, rather than as “. . . is red” or “. . . est rouge”).

Note that the inductive clauses in the definition of truth-makers and falsity-makers, as labeled trees, can be interpreted in such a way as to make the left-right ordering of immediate subtrees immaterial to the identity of the tree overall. That is to say, the immediate subtrees need not be taken as given in any particular order. This is especially the case with the possibly infinitary quantifier rules ($\exists\mathcal{F}$) and ($\forall\mathcal{V}$), in which the premises for the applications of those rules can be construed as functions mapping individuals from the domain D to falsity-makers, or truth-makers, respectively, for the relevant instances of the quantified matrix.

References

Armstrong, D. M. (2004). *Truth and Truthmakers*. Cambridge: Cambridge University Press.

Ayer, A. J. (1946). *Language, Truth and Logic*, 2nd edn, revised and re-set. London: Victor Gollancz.

Beebe, H. and J. Dodd (eds.) (2005). *Truthmakers: The Contemporary Debate*. Oxford: Oxford University Press.

Bigelow, J. (1988). *The Reality of Numbers: A Physicalist's Philosophy of Numbers*. Oxford: Clarendon Press.

Church, A. (1949). “Review: Alfred Jules Ayer, *Language, Truth and Logic* [review of 2nd edn]. *Journal of Symbolic Logic* 14: 52–3.

Dummett, M. (1976). “What is a Theory of Meaning? II,” in G. Evans and J. McDowell (eds.), *Truth and Meaning*. Oxford: Oxford University Press, 67–137.

Fox, J. F. (1987). “Truthmaker,” *Australasian Journal of Philosophy* 65: 188–207.

Frege, G. (1884; reprinted 1961). *Die Grundlagen der Arithmetik: eine logisch mathematische Untersuchung über den Begriff der Zahl*. Hildesheim: Georg Olms.

Gentzen, G. (1934–35). “Untersuchungen über das logische Schliessen,” *Mathematische Zeitschrift* 1, 2: 176–210, 405–431. Translated as “Investigations into Logical Deduction,” in *The Collected Papers of Gerhard Gentzen*, ed. M. E. Szabo (1969). Amsterdam: North-Holland, 68–131.

Lowe, E. J. and A. Rami (eds.) (2009). *Truth and Truth-Making*. Ithaca: McGill-Queen's University Press.

(p. 393) Mulligan, K., P. Simons, and B. Smith (1984). “Truth-Makers,” *Philosophy and Phenomenological Research* 44: 287–321.

A Logical Theory of Truth-Makers and Falsity-Makers

Prawitz, D. (1968). "Propositions," *Theoria* 34: 134–46.

Prawitz, D. (1974). "On the Idea of a General Proof Theory," *Synthese* 27: 63–77.

Restall, G. "Truth-makers, Entailment and Necessity," *Australasian Journal of Philosophy* 74: 331–40.

Sundholm, G. (1994). "Existence, Proof and Truth-Making: A Perspective on the Intuitionistic Conception of Truth," *Topoi* 13: 117–26.

Tarski, A. (1956) [1936]. "On the Concept of Logical Consequence," in J. H. Woodger (ed.), *Logic, Semantics, Metamathematics*. Oxford: Clarendon Press, 409–20.

Tarski, A. (1956) [in Polish, 1933]. "The Concept of Truth in Formalized Languages," in J. H. Woodger (ed.), *Logic, Semantics, Metamathematics*. Oxford: Clarendon Press, 152–278.

Tennant, N. (1978). *Natural Logic*. Edinburgh: Edinburgh University Press.

Tennant, N. (1979). "Entailment and Proofs," *Proceedings of the Aristotelian Society* 79: 167–89.

Tennant, N. (1987). *Anti-Realism and Logic: Truth as Eternal*. Oxford: Clarendon Library of Logic and Philosophy, Oxford University Press.

Tennant, N. (1992). *Autologic*. Edinburgh: Edinburgh University Press.

Tennant, N. (1997). *The Taming of The True*. Oxford: Oxford University Press.

Tennant, N. (2008). "Natural Logicism via the Logic if Orderly Pairing," in S. Lindström, E. Palmgren, K. Segerberg, and V. Stoltenberg-Hansen (eds.), *Logicism, Intuitionism, Formalism: What has Become of Them?* Synthese Library 341. Dordrecht: Springer, 91–125.

Tennant, N. (2010). "Inferential Semantics," in J. Lear and A. Oliver (eds.), *The Force of Argument: Essays in Honor of Timothy Smiley*. London: Routledge, 223–57.

Tennant, N. (2012). *Changes of Mind: An Essay on Rational Belief Revision*. Oxford: Oxford University Press.

Tennant, N. (2015). "A New Unified Account of Truth and Paradox," *Mind* 124: 571–605

Notes:

(¹) Armstrong has written to the author, ". . . I myself would reject the idea that truth-makers should just be given for "atomic" truths. I'm a truthmaker maximalist—every truth has a truthmaker. I'm not sure, of course, for many truths, what their truthmakers are."

A Logical Theory of Truth-Makers and Falsity-Makers

(²) Henceforth we shall concentrate on just sentences, pending an interesting, theoretically-motivated twist that will emerge in due course.

(³) Core logic is what the present author called *intuitionistic relevant logic* in publications such as Tennant (1987) and Tennant (1997). The new label is preferable in light of an argument in Tennant (unpublished typescript, 2012) for an important revision-theoretic thesis:

Core logic is the minimal inviolable core of logic without any part of which one would not be able to establish the rationality of belief-revision.

(⁴) This claim is intended to survive even in the context of *semantically closed* languages, which are plagued by the logical and semantic paradoxes. See Tennant (2012) for details.

(⁵) Sundholm (1994) points out that *for the intuitionist* truth-makers are finitary proofs. He does not, however, investigate how logical compounding might affect truth-makers (and falsity-makers); nor does he consider infinitary truth-makers as mathematical objects alongside finitary proofs.

(⁶) Discussion with Robert Kraut has been helpful here.

(⁷) Comments from Fraser MacBride have been helpful in prompting this description, for the metaphysician, of the aims of the brand of truth-making on offer here.

(⁸) In some cases, a truth-maker can be finite even when the domain is infinite. Such can be the case with, say, an existential claim. But in an infinite domain any universal claim (or negative existential) can have only infinite truth-makers. Details will emerge below.

(⁹) Two examples: *0 is not the successor of any natural number*; and *every natural number has a successor*. For the vast majority of mathematicians, these are self-evident, and are adopted as axioms. That is to say, they serve as their own *epistemic* grounds. Yet their grounds of truth involve each and every natural number, and hence are infinite.

One can seek to mitigate the gap between epistemic grounds and grounds of truth by seeking even deeper foundations, as the logicist does, for the mathematical axioms that are usually taken as starting points for mathematical proofs. Those axioms can then be derived in a non-trivial fashion as theorems, from yet more “obvious” and logical-looking first principles in the logicist’s theory. (See e.g. the constructive logicist derivation of the Peano postulates provided by Tennant (1987) and Tennant (2008).) Still, however, the grounds of truth of the latter principles are in all likelihood going to be infinite.

(¹⁰) This is true not only of Tarski’s original treatment (Tarski 1956; first published, in Polish, in 1933), which invoked *infinite sequences* of individuals correlated with object-linguistic variables, but also of the treatment (in Tennant 1978) of Tarski’s approach that appeals, more modestly, to *finitary* assignments of individuals to the free variables in a formula.

A Logical Theory of Truth-Makers and Falsity-Makers

⁽¹¹⁾ This means that, if one had a *binary* predicate to interpret, and the same three individuals in the domain, one would have to furnish the right kind of rule for each of the *nine* ($= 3^2$) possible combinations of that predicate with *ordered pairs* of individuals from the domain.

⁽¹²⁾ We prefer the term “falsity-maker” to the term “false-maker” of Armstrong (2004: section 1.4). In contrast with “truth” we need the noun “falsity” rather than the adjective “false.”

⁽¹³⁾ Throughout the recent revival of interest in truth-makers and truth-making, there has been a curious blindness to the possibility that truth-makers might be as variegated and as structurally complex as proofs are. In his Introduction to Lowe and Rami (2009), Rami writes (10),

Among truth-maker theorists there is considerable disagreement concerning the question which kinds of entity should be accepted as truth-makers: some only accept states of affairs as truth-making entities;[fn] some only accept individuals and states of affairs;[fn] and some only accept individuals and particular properties.
[fn]

Notably absent from this list of extant possibilities is any conception of truth-makers along the lines proposed here.

⁽¹⁴⁾ Again, the reader is not to think of evaluation as an epistemic process, or as involving human judgment. It is meant here in a more neutral sense, as when a logician says that a universal quantification $\forall xG(x)$ over all the real numbers is to be “evaluated” as true if its predicate G is true of each real number. If preferred, the reader is welcome to read “evaluation” as “constitution” throughout (and similarly for cognates).

⁽¹⁵⁾ The first detailed and systematic presentation of these evaluation rules, with discussion of their relation to the rules of natural deduction, can be found in Tennant (2010). A reasonably well-cultured germ of the main ideas is to be found in Tennant (1997: section 11.3.2).

⁽¹⁶⁾ The finitude of our example must not lure the reader into thinking that truth-makers in general are like epistemic warrants. We have been at pains to distinguish the two notions. That having been said, however, it is the case that *finitary* truth-makers can serve as epistemic warrants. This point was made earlier.

⁽¹⁷⁾ Likewise, we allow (the evaluation-analogue of) “existential elimination” from the major (existential) premise, plus falsity-makers for all instances thereof, to the conclusion \perp . See section 14.7.3.

⁽¹⁸⁾ Again, “work” is here not to be construed as the result of intellectual effort, or an outcome representing some sort of epistemic achievement. It refers only to possible con-

A Logical Theory of Truth-Makers and Falsity-Makers

structions (truth-makers) that constitute the grounds for truth of the sentence in question, in the ontological or metaphysical sense explained earlier.

(¹⁹) This way of ensuring normal form—but of ordinary *deductive proofs*—was first treated in Tennant (1992). Here we are applying the idea to the kinds of construction that we are calling truth-makers and falsity-makers.

(²⁰) This is a very important precondition for the *relevance* of premises to conclusions in natural deduction. It was stated in Tennant (1979). It is exactly the proof-theoretic insight that is needed in order to put appropriate proof-theoretic resources at the disposal of the truth-maker theorist. For the operation of determining the truth-value of a compound from the truth-values of its immediate constituents (i.e. subsentences, or instances of quantifications) is, crucially, a *relevant* one. The truth-value of the compound *depends on* those of its immediate constituents. No “proof-like” conception of truth-makers (and falsity-makers) could possibly work if one did not pay attention to this fact, and *build that very feature into one’s account*.

None of this, of course, is to be construed as turning truth-makers (or falsity-makers) into epistemic warrants or justifications. They remain *metaphysical* objects. They are constituted by the mathematics of finite and infinite labeled trees, whose labels (which are often saturated formulae) can embed individuals. Proof-theoretic notions happen to be applicable to these trees only because such trees have been studied extensively in proof theory. But, whereas the proofs of traditional proof theory have always been understood as epistemic warrants, the more generalized trees that we are dealing with here, and which we are calling truth-makers and falsity-makers, are being put into the service of metaphysics and ontology. The truth-making and falsity-making involved can be as “epistemically transcendent” as any realist reader might wish. (This anti-realist author is really running with the wolves here.)

(²¹) Note that the branches, all of them finite, are bounded in length by the length of the longest branch in the analysis tree of the sentence being evaluated.

(²²) The word “determinable” here is to be taken in a constitutive, not epistemic, sense. One should certainly not expect the method of determination, in general, to be effective.

(²³) For a critique of this “manifestation argument,” and an attempt to improve upon it, see Tennant (1997: ch. 7).

(²⁴) This puts us in disagreement with Restall (1996: 333), who writes,

“There is also a desirable result connecting truthmaking and conjunction. If something makes both *A* and *B* true, then it also makes their conjunction true, and vice versa.”

A Logical Theory of Truth-Makers and Falsity-Makers

(²⁵) This remark holds even when we consider *infinitary* truth-makers “above,” in combination with (perforce finitary) proofs “below.” In the current illustration, however, we are dealing with finitary truth-makers, because M is finite.

(²⁶) See Frege (1884; reprinted 1961: section 88).

(²⁷) We abbreviated this right-hand side as $M \Vdash \varphi$ in section 14.8.

(²⁸) Perhaps the best example of construction coming quite close to being truth-makers in our sense would be the infinitary proofs proposed by Carnap for arithmetic using the so-called ω -rule

$$\frac{\Psi_0 \quad \Psi_{s0} \quad \Psi_{ss0} \quad \dots \quad \Psi_n \quad \dots}{\forall n \Psi_n}$$

which would be the \mathbb{N} -relative evaluation rule of “universal introduction” in our sense. It would need to be complemented, of course, by a dual rule for “existential elimination”:

$$\frac{\begin{array}{ccccccc} \frac{\text{---}(i)}{\Psi_0} & \frac{\text{---}(i)}{\Psi_{s0}} & \frac{\text{---}(i)}{\Psi_{ss0}} & \dots & \frac{\text{---}(i)}{\Psi_n} & \dots & \\ \vdots & \vdots & \vdots & \dots & \vdots & \dots & \\ \exists n \Psi_n & \perp & \perp & \dots & \perp & \dots & \end{array}}{\perp} (i)$$

(²⁹) We know already by the downward Löwenheim-Skolem Theorem that restricting domains D to be both countable and decidable will not result in any unwanted curtailment of the classical logical consequence relation.

(³⁰) Personal correspondence.

(³¹) This point applies equally well to the case where the classically needed but intuitionistically disputed principle has the logical form $\forall x \neg \Psi x \vee \exists x \Psi x$.

(³²) See Church’s famous review (Church 1949) of Ayer (1946), in which Church showed how to trivialize Ayer’s notion of indirect verifiability (which Ayer had offered as his formal explication of what we call cognitive significance).

(³³) The author is grateful to Kit Fine for raising this problem.

Neil Tennant

Neil Tennant, Arts and Sciences Distinguished Professor in Philosophy, Ohio State University

