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Does Choice Really Imply Excluded Middle? Part I: Regimentation of the Goodman–Myhill Result, and Its Immediate Reception†

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ABSTRACT

The one-page 1978 informal proof of Goodman and Myhill is regimented in a weak constructive set theory in free logic. The decidability of identities in general ($a = b \vee \neg a = b$) is derived; then, of sentences in general ($\psi \vee \neg\psi$). Martin-Löf's and Bell's receptions of the latter result are discussed. Regimentation reveals the form of Choice used in deriving Excluded Middle. It also reveals an abstraction principle that the proof employs. It will be argued that the Goodman–Myhill result does not provide the constructive set theorist with a dispositive reason for not adopting (full) Choice.

1. INTRODUCTION

The aim of this study is to provide a logically neutral framework that will enable us to gain clarity about the inferential details, hence also the overall significance, of the result of Goodman and Myhill [1978]. This result purports to commit the constructive set theorist, upon adopting the Axiom of Choice,

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to the absolutely general Law of Excluded Middle —

$$\psi \vee \neg\psi$$

— which of course the constructivist eschews. Such a result — provided that it passes close critical muster — provokes both foundational and philosophical reflection.

The immediate aim is to *regiment* the Goodman–Myhill proof as a natural deduction in (free) constructive logic. Such regimentation is a useful exercise with any result that appears to have foundational or philosophical consequences. For it helps us to be absolutely certain as to what, exactly, the *premises* are on which the conclusion of the proof actually rests; and, also, to be certain that absolutely no logical fallacies are hidden along the way in the *informal* reasoning that meets the expository standards of 'informal rigor'. In this particular case, the regimentation serves a further purpose that is central to the project — that of confirming that the reasoning is, indeed, fully *constructive*.

The formal proof is constructed by filling in all the necessary steps that the informal expositor often omits to take or mention explicitly. This, of course, makes the formal proof more prolix. *But* — and this is important — the formal proof is a *faithful homologue* of the informal one that it regiments. It embodies the same overall line of argument, which can be abstracted from it by suppressing the extra explicit detail that has been supplied, in regimentation, in order to ensure absolute formal rigor.

The reader should be aware at the outset that the undertaking to furnish a completely formal regimentation of the Goodman–Myhill reasoning is in earnest. For some, this will mean that they should feel free to skip those passages of formalization that are of bits of informal reasoning so obvious to them that they may think their formalization is unnecessary.

2. THE LOGIC OF SETS

The work about to be undertaken will be in free first-order constructive logic, using natural-deduction rules for (constructive) set theory. These rules remain unchanged when one classicizes. All that changes when classicizing is that one is entitled to use (as the constructivist is not) any of the strictly classical rules of negation — Classical Reductio, Double Negation Elimination, Law of Excluded Middle, and Dilemma.

2.1. Introduction and Elimination Rules for Set-Abstraction

$\exists! t$ is the familiar abbreviation for $\exists x\, x = t$.

The rule of introduction for the abstraction operator that forms set terms from predicates is

$$\{\,\}\mathrm{I}:\qquad \cfrac{\begin{array}{c}\underbrace{{}^{(i)}\overline{\exists! a}\ ,\ \overline{\Phi^x_a}^{(i)}}\\ \vdots \\ a\in t\end{array}\qquad \exists! t \qquad \begin{array}{c}\overline{a\in t}^{(i)}\\ \vdots\\ \Phi^x_a\end{array}}{t=\{x|\Phi\}}{}^{(i)}\ ,\qquad \text{where } a \text{ is parametric.}$$

The corresponding elimination rules for $\{\,\}$ are

$$\begin{array}{ll}
\{\,\}\mathrm{E}_1: & \dfrac{t=\{x|\Phi\}\quad \exists! v\quad \Phi^x_v}{v\in t}\\[2ex]
\{\,\}\mathrm{E}_2: & \dfrac{t=\{x|\Phi\}}{\exists! t}\\[2ex]
\{\,\}\mathrm{E}_3: & \dfrac{t=\{x|\Phi\}\quad v\in t}{\Phi^x_v}\ .
\end{array}$$

These introduction and elimination rules are *ontologically neutral* — they characterize only the *logic* of one's talk about sets, not one's theory about what sets actually exist. To say *what sets there are* one has to employ various outright and conditional set-existence principles. At present, none is offered. Nor, for that matter, will any definite set of axioms or axiom schemata be offered (for the would-be constructive set theorist) by the end of this study.

A brief digression. The aim here is only to explore certain problems attending any final selection of outright and conditional set-existence principles — on behalf of the constructivist, in pursuit of a faithful foundational formalization of constructive analysis — in light of the various considerations provoked by a study of the Goodman–Myhill result.

One possible extreme outcome of such an investigation is that the constructivist might conclude that we have to *abandon* the supposedly foundational project of formulating a constructive version of (classical) *set theory* in order to found constructive *analysis*. This would be on the grounds that the untoward intrusions of classicism (courtesy of Goodman and Myhill, once their formal reasoning is more deeply understood) turn out to be unavoidable in the context, quite generally, of an *extensional* set theory with any prospect of serving as a foundational basis for constructive real analysis. It might be that the constructive real analyst will have to resort to entities, concepts, and methods that are more *sui generis* for real number theory (much in the way that Bishop himself did), rather than seeking to 'get' the real numbers by means of certain set-theoretic surrogates — that is, to *construct* them, via set-theoretic definitions that can be deployed in subsequent theorizing without any need to have recourse to strictly classical logical inferences. In this regard, we note with some bemusement the following two quotes:

> ... *a real number should not be defined as an equivalence class of Cauchy sequences of rational numbers*; there is no need to drag in the equivalence classes. [Bishop, 1967, p. 9, emphasis added]

> The system **C** is adequate for the usual set theoretic foundation of ordinary analysis ... The system **B** is a subsystem of [**C**] obtained from analyzing which parts of it are actually used in constructive analysis. ... We now come to the actual set theoretic development of constructive analysis within **B**. ... *Real numbers are equivalence classes of Cauchy sequences of rationals.* [Friedman, 1977, pp. 4–7, emphasis added]

So it would appear that Friedman's system **B**, which he said was to provide 'a set theoretic foundation for constructive analysis which is strikingly analogous to the usual set theoretic foundations for ordinary [*i.e.*, classical] analysis' was guided more by the admitted analogy than it was by the aim of 'faithful formalization' of Bishop's constructive analysis.
End of digression.

Because our introduction and elimination rules for the set-abstraction operator are part of a *free* logic,[1] Russell's Paradox does not pose itself as an inherent problem, as it did for Frege's class theory based on an *unfree* logic. Instead, in the foregoing system of rules against the background of a *free* logic, the reasoning behind Russell's Paradox yields a straightforward proof of the inconsistency of the claim that the Russell set exists. Thus the rules produce the theorem

$$\neg\exists!\{x|\neg x \in x\}.$$

An important rule in free logic is the Rule of Atomic Denotation (RAD). This rule says that atomic truths entail the existence of the individuals purportedly involved:

$$\frac{A(\ldots t \ldots)}{\exists! t}\ , \qquad \text{where } A \text{ is an atomic predicate.}$$

In the context of set theory, where both $\in$ and $=$ are primitive predicates, RAD will find applications of the forms

$$\frac{u \in t}{\exists! t} \qquad \frac{u \in t}{\exists! u} \qquad \frac{u = t}{\exists! t} \qquad \frac{u = t}{\exists! u}\ .$$

In free logic the Reflexivity of Identity is expressed by the rule

$$\text{Ref} = \qquad \frac{\exists! t}{t = t}\ .$$

[1] See [Tennant, 1978, Ch. 7].

The reader who is unfamiliar with the natural deduction rules here provided for the set-abstraction operator needs to be made aware of the important fact that they afford an outright (constructive) proof of the Axiom of Extensionality.[2] This very important axiom of Zermelo's set theory thereby turns out to be part of the *logic* of sets, and is derivable without appealing to any principles that incur commitment to the existence of any particular sets. The rules accomplish, rather, the *conceptual* task of clarifying the connections among set-abstraction, predication, and membership.[3]

Here is a very simple, but important, such connection. Let us employ the usual abbreviation $\{u\}$ ('singleton u') for the abstractive term $\{x|x=u\}$.

Lemma 1. $\dfrac{t \in \{u\}}{t=u}$

Proof.

$$\Lambda_1: \quad \dfrac{\dfrac{\dfrac{\begin{array}{c} t \in \{u\},\ \text{i.e.,} \\ t \in \{x|x=u\} \end{array}}{\exists!\{x|x=u\}}}{\{x|x=u\}=\{x|x=u\}} \qquad \begin{array}{c} t \in \{u\},\ \text{i.e.,} \\ t \in \{x|x=u\} \end{array}}{t=u}\{\ \}\mathrm{E}_3 \qquad . \qquad \square$$

In due course the reader will see that the converse inference

$$\frac{t=u}{t \in \{u\}}$$

will also be provable, but only after adopting an axiom (or rule) that vouchsafes the existence of the singleton of any individual. So this converse inference requires at least some ontological commitment within one's set theory; whereas the inference of Lemma 1 is part of the ontologically non-committal, free *logic* of sets.

2.2. Singletons

Here we explore the logic of singletons a little further. The rule $\{\ \}$I has as a special instance (taking $x=u$ for Φ)

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[2] See [Tennant, 1978, p. 173] and [Tennant, 2004, p. 119].

[3] The reader will see in due course, however, that Goodman and Myhill themselves do employ a certain set-existence abstraction principle as a premise for their proof; and will be invited to consider whether that principle is at all warranted from the point of view of the constructive set theorist.

$$\dfrac{\begin{array}{c}\underbrace{\overline{\exists! a}^{\,(i)} \;,\; \overline{a=u}^{\,(i)}}\\ \vdots \\ a\in t\end{array} \qquad \exists! t \qquad \begin{array}{c}\overline{a\in t}^{\,(i)}\\ \vdots\\ a=u\end{array}}{t=\{x|x=u\}}{\scriptstyle (i)}\ .$$

In the foregoing instance of { }I, one does not need to have used the dischargeable assumption $\exists! a$ at top left. This is because any such apparent assumption could be 'covered' by the other assumption $a=u$ of the leftmost subordinate proof, by application of free logic's Rule of Atomic Denotation.

As we have seen, the identity predicate is primitive, and not defined; so it can be taken as an instance of (binary) A in the Rule of Atomic Denotation. So too can the binary membership predicate $\in$. The following inference rule (call it Singleton-Introduction) therefore holds:

$$\mathrm{SI}: \qquad \dfrac{\begin{array}{c}\overline{a=u}^{\,(i)}\\ \vdots \\ a\in t\end{array} \qquad \exists! t \qquad \begin{array}{c}\overline{a\in t}^{\,(i)}\\ \vdots\\ a=u\end{array}}{t=\{u\}}{\scriptstyle (i)}\ .$$

The corresponding instance of $\{\ \}\mathrm{E}_1$ (taking once again $x=u$ for Φ) is

$$\dfrac{t=\{x|x=u\} \quad \exists! v \quad v=u}{v\in t}\ .$$

Thus one has the result

$$\begin{array}{c}\underbrace{t=\{u\}\,,\ \exists! u}\\ \Sigma_0\\ u\in t\end{array} \quad : \qquad \dfrac{t=\{u\} \qquad \exists! u \qquad \dfrac{\exists! u}{u=u}{\scriptstyle \mathrm{Ref}=}}{u\in t}{\scriptstyle \{\ \}\mathrm{E}_1}\ .$$

Note that one cannot blithely help oneself to the would-be rule

$$\dfrac{t=\{u\}}{u\in t}\ .$$

This is because the conclusion $u\in t$ is atomic, and therefore commits one (by the Rule of Atomic Denotation) to the existence of u. Yet for non-denoting terms u, singleton u *can be taken to exist*, courtesy of the axiomatic principle Existence of the Empty Set. Hence the need for the extra premise $\exists! u$ when inferring from 't is singleton u' to 'u is in t'. The little proof Σ_0 just given furnishes the more careful inference employing the extra premise, using only ontologically non-committal rules.

2.3. Pairs

The Axiom of Pairs, in familiar notation, is

$$\forall x \forall y \exists! \{x, y\}.$$

In the official language of set theory, this would be written

$$\forall x \forall y \exists! \{z | z = x \vee z = y\}.$$

Note that the Axiom of Pairs yields the singleton $\{t\}$ of any object t as well — one simply instantiates the two universal quantifiers in the prefix $\forall x \forall y$ with respect to t, and notes that any disjunction of the form $\varphi \vee \varphi$ is logically equivalent to φ.

The following proof Ξ, using Pairs, shows that if t and u both exist then t is a member of $\{t, u\}$. Note how the quantifier rules of free logic involve existential presuppositions. The step marked (1) is an application of Existential Elimination, with parameter a.

$$\begin{array}{c}\underbrace{\exists!t,\ \exists!u,\ \text{Pairs}}\\ \Xi\\ t \in \{t, u\}\end{array} \quad :$$

$$\cfrac{\cfrac{\exists!u \qquad \cfrac{\exists!t \qquad \begin{array}{c}\text{Pairs}:\\ \forall x \forall y \exists! \{z | z = x \vee z = y\}\end{array}}{\forall y \exists! \{z | z = t \vee z = y\}}}{\exists! \{z | z = t \vee z = u\}} \qquad \cfrac{\cfrac{\cfrac{}{a = \{z | z = t \vee z = u\}}{\scriptstyle (1)} \qquad \cfrac{\cfrac{\exists!t}{t = t}}{t = t \vee t = u} \qquad \exists!t}{t \in a}{\scriptstyle \{\ \}\mathrm{E}_1} \qquad \cfrac{}{a = \{z | z = t \vee z = u\}}{\scriptstyle (1)}}{t \in \{z | z = t \vee z = u\}}}{t \in \{z | z = t \vee z = u\}}{\scriptstyle (1)}$$

A similar proof Ξ', from the same premises, yields the conclusion $u \in \{t, u\}$.

Returning to our brief preoccupation with singletons, we see that an easy adaptation (call it Ω) of the foregoing proof, obtained by substituting u for t, vouchsafes that if u exists then u is a member of $\{u\}$:

$$\Omega : \qquad \begin{array}{c}\underbrace{\exists!u\,,\ \text{Pairs}}\\ \Omega\\ u \in \{u\}\end{array} \quad .$$

This enables us to prove the converse of Lemma 1.

Lemma 2. $\dfrac{t = u}{t \in \{u\}}$ (*modulo* Pairs).

Proof. Λ_2:

$$\dfrac{\begin{array}{c}\underbrace{\dfrac{t=u}{\exists!u}\,,\ \text{Pairs}}\\ \Omega\\ u\in\{u\}\end{array}\qquad t=u}{t\in\{u\}}$$

□

Lemma 3. $\dfrac{\exists!u}{\exists!\{u\}}$ (*modulo* Pairs).

Proof. Λ_3:

$$\dfrac{\begin{array}{c}\underbrace{\exists!u\ ,\ \text{Pairs}}\\ \Omega\\ u\in\{u\}\end{array}}{\exists!\{u\}}\ .$$

□

Lemmas 1 and 2 (*modulo* Pairs) justify any inference from one sentence to another that results from it by substituting, in at least one place, a formula of the form $t\in\{u\}$ for one of the form $t=u$, or *vice versa*. An important example of such an inference is this one:

$$\dfrac{\exists y\forall x(x\in y\leftrightarrow(x\in\{0\}\wedge\psi))}{\exists y\forall x(x\in y\leftrightarrow(x=0\wedge\psi))}\ ,$$

which will be employed in the final regimentation of the Goodman–Myhill proof. The detailed formal proof (call it Υ) of this inference in free logic appeals to both Lemma 2 and Lemma 1 and is strategically simple but formally tedious.

Υ :

$$\dfrac{\exists y\forall x(x\in y\leftrightarrow(x\in\{0\}\wedge\psi))\qquad\dfrac{\dfrac{\dfrac{\begin{array}{c}\dfrac{{\scriptstyle(L1)}\dfrac{\dfrac{\dfrac{{\scriptstyle(1)}\dfrac{}{c\in a}\quad\dfrac{{\scriptstyle(3)}\dfrac{}{\forall x(x\in a\leftrightarrow(x\in\{0\}\wedge\psi))}\quad\dfrac{}{\exists!c}{\scriptstyle(2)}}{c\in a\leftrightarrow(c\in\{0\}\wedge\psi)}}{c\in\{0\}\wedge\psi}}{c\in\{0\}}}{c=0}\qquad\dfrac{\dfrac{{\scriptstyle(1)}\dfrac{}{c\in a}\quad\dfrac{{\scriptstyle(3)}\dfrac{}{\forall x(x\in a\leftrightarrow(x\in\{0\}\wedge\psi))}\quad\dfrac{}{\exists!c}{\scriptstyle(2)}}{c\in a\leftrightarrow(c\in\{0\}\wedge\psi)}}{c\in\{0\}\wedge\psi}}{\psi}}{c=0\wedge\psi}\\ \vdots\\ c=0\wedge\psi\end{array}\qquad\dfrac{\dfrac{{\scriptstyle(2)}\dfrac{}{\exists!c}\quad{\scriptstyle(3)}\dfrac{}{\forall x(x\in a\leftrightarrow(x\in\{0\}\wedge\psi))}}{c\in a\leftrightarrow(c\in\{0\}\wedge\psi)}\qquad\dfrac{{\scriptstyle(L2)}\dfrac{\dfrac{\dfrac{}{c=0\wedge\psi}{\scriptstyle(1)}}{c=0}}{c\in\{0\}}\quad\dfrac{\dfrac{}{c=0\wedge\psi}{\scriptstyle(1)}}{\psi}}{c\in\{0\}\wedge\psi}}{c\in a}}{c\in a\leftrightarrow(c=0\wedge\psi)}{\scriptstyle(1)}}{\forall x(x\in a\leftrightarrow(x=0\wedge\psi))}{\scriptstyle(2)}\quad\dfrac{}{\exists!a}{\scriptstyle(3)}}{\exists y\forall x(x\in y\leftrightarrow(x=0\wedge\psi))}}{\exists y\forall x(x\in y\leftrightarrow(x=0\wedge\psi))}{\scriptstyle(3)}$$

Let us recognize the existence of two distinct objects, 0 and 1:

$$\dfrac{}{\exists!0}\qquad\dfrac{}{\exists!1}\qquad\dfrac{0=1}{\perp}$$

It does not matter what these two objects 0 and 1 actually *are*. It is not to be presumed that the former is the empty set, nor that the latter is its singleton.

All one needs is their existence and distinctness. Of course, as soon as one has just a smidgeon of *ontologically committal* set *theory*, one will be able to take the empty set for 0, and its singleton for 1.[4]

For the time being, use will be made only of the rule stating that 0 is distinct from 1. By the end of the regimentation of the Goodman–Myhill proof (see Π_8 in §3.2.1) use will also have been made, on their behalf, of the premises $\exists!0$, $\exists!1$, the Axiom of Pairs, a certain form of Choice, and an abstraction principle that can be construed as an instance of Separation.

An important reason for regimenting the proof is that we need to determine exactly 'how much' of Choice is needed for the proof to secure its conclusion, so that we can then inquire whether the amount of Choice thus revealed is constructively acceptable. In advance of that, however, it is worth reminding the reader of what Bishop himself wrote [1967, p. 9] about the axiom of choice:

> This axiom is unique in its ability to trouble the conscience of the classical mathematician, but in fact it is not a real source of the unconstructivities of classical mathematics. A choice function exists in constructive mathematics, because *a choice is implied by the very meaning of existence.*

This author, however, is not proposing that any version of 'full' Choice should find its way into any constructively acceptable set theory. An important point to bear in mind is that the weaker (hence: more acceptable, to the constructivist) the form of Choice appealed to by Goodman and Myhill turns out to be, the more prepared the constructivist should be to question any of the *other* premises on which the unfortunate conclusion of Excluded Middle has been shown to rest. And among these, as we shall see, is an instance of the Axiom Scheme of Separation (in its usual, unrestricted form).

3. REGIMENTATION OF THE GOODMAN–MYHILL PROOF

The width of the page limits the extent to which one can construct chunks of completely formal proof. Let us therefore proceed in a perforce modular fashion, presenting completely formal 'prooflets' that can obviously be assembled in the tree-like way needed in order to construct, finally, a single overall proof tree establishing the sought result.

The formal reconstruction of the Goodman–Myhill proof will take place in the two main stages discernible in their informal reasoning. The first stage will establish the decidability of identity $(a=b\vee\neg a=b)$,[5] using only Choice and the distinctness of 0 and 1. The second stage will build on this result. It will use as further premises a particular 'abstraction principle' (which is really an instance

[4] The 'smidgeon' in question could be just $\exists!\{x|\neg x=x\}$ and $\forall x\exists!\{y|y=x\}$ — that is, the existence of the empty set, plus the (conditional) existence of singletons.

[5] Here, both a and b are proof-theoretic *parameters*, indicating absolute generality of reasoning. One would be able to substitute any closed singular term for either of them, and the reasoning would still go through.

of Separation) and the innocuous premise $\exists!0$. And it will end by deriving, for arbitrary ψ, the conclusion $\psi\vee\neg\psi$, *i.e.*, Excluded Middle in full generality.

3.1. The First Stage

3.1.1. A Special Predicate Embedding a Simple Function

Let f be a one-place function sign and let $\xi^f(x)$ abbreviate the formula

$$(x=a \wedge fx=0) \vee (x=b \wedge fx=1),$$

in which a and b are parameters that will admit of uniform substitutions by any terms one cares to choose as one's reasoning proceeds. Note that both of the instances $\xi^f(a)$ and $\xi^f(b)$ of $\xi^f(x)$ contain both a and b.

Consider the following constructive proofs Π_1–Π_3. They employ color only to aid the eye.

$$\underbrace{\xi^f(b),\, fb=0}_{\textstyle\Pi_1}\atop a=b\vee\neg a=b \quad : \quad \dfrac{\dfrac{(b=a\wedge fb=0)\vee(b=b\wedge fb=1) \qquad \dfrac{\dfrac{{\scriptstyle(1)}\dfrac{}{b=a\wedge fb=0}}{b=a}}{a=b} \qquad \dfrac{\dfrac{\dfrac{{\scriptstyle(1)}\dfrac{}{b=b\wedge fb=1}}{fb=1}\qquad fb=0}{0=1}}{\bot}}{a=b}{\scriptstyle(1)}}{a=b\vee\neg a=b}$$

Note that the step labeled (1) is easily derivable in Intuitionistic Logic, but appears here more succinctly as an application of a primitive $\vee$-Elimination rule of Core Logic that allows one, when either one of the case proofs ends with $\bot$, to bring down as the main conclusion the conclusion of the other case proof.

$$\underbrace{fa=0,\, fb=1}_{\textstyle\Pi_2}\atop a=b\vee\neg a=b \quad : \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{{\scriptstyle(1)}\dfrac{}{a=b}\qquad fa=0}{fb=0}\qquad fb=1}{0=1}}{\bot}{\scriptstyle(1)}}{\neg a=b}}{a=b\vee\neg a=b}$$

$$\underbrace{\xi^f(a),\, fa=1}_{\textstyle\Pi_3}\atop a=b\vee\neg a=b \quad : \quad \dfrac{\dfrac{(a=a\wedge fa=0)\vee(a=b\wedge fa=1) \qquad \dfrac{\dfrac{\dfrac{{\scriptstyle(1)}\dfrac{}{a=a\wedge fa=0}}{fa=0}\qquad fa=1}{0=1}}{\bot} \qquad \dfrac{{\scriptstyle(1)}\dfrac{}{a=b\wedge fa=1}}{a=b}}{a=b}{\scriptstyle(1)}}{a=b\vee\neg a=b}$$

We have now the three proofs

$$\underbrace{\xi^f(b),\, fb=0}_{\textstyle\Pi_1}\atop a=b\vee\neg a=b \qquad \underbrace{fa=0,\, fb=1}_{\textstyle\Pi_2}\atop a=b\vee\neg a=b \qquad \underbrace{\xi^f(a),\, fa=1}_{\textstyle\Pi_3}\atop a=b\vee\neg a=b$$

Remember that the only non-logical rule that these three proofs employ is the one saying that 0 is distinct from 1. '$\neg\, 0 = 1$' has not been listed as an explicit underbraced premise in the complex names for these proofs. That premise will always be in the background, and available for use at any time. Only those premises are listed whose subsequent role in the overall proof work will be important. The reader is advised, however, that all proofs labeled Π_i (i a numerical subscript) *do* have '$\neg\, 0=1$' as an implicit premise. (The proofs Π_1, Π_2, and Π_3 already serve as examples of this.) But those proofs that are being labeled Σ_i do *not* have '$\neg\, 0=1$' as an implicit premise. (The proof Σ_0 already serves as an example of this.)

Two more proofs will now be constructed in the Π-series, each embedding one or two of the earlier ones.

$$\begin{array}{c}\underbrace{\xi^f(b), fa=0}\\ \Pi_4\\ a=b\vee\neg a=b\end{array} :$$

$$\dfrac{\begin{array}{c}\xi^f(b),\ \text{i.e.,}\\ (b=a\wedge fb=0)\vee(b=b\wedge fb=1)\end{array}\qquad \begin{array}{c}\underbrace{\xi^f(b)\ ,\ \dfrac{\overline{b=a\wedge fb=0}^{\,(1)}}{fb=0}}\\ \Pi_1\\ a=b\vee\neg a=b\end{array}\qquad \begin{array}{c}\underbrace{fa=0\ ,\ \dfrac{\overline{b=b\wedge fb=1}^{\,(1)}}{fb=1}}\\ \Pi_2\\ a=b\vee\neg a=b\end{array}}{a=b\vee\neg a=b}{\scriptstyle (1)}$$

and then

$$\begin{array}{c}\underbrace{\xi^f(a), \xi^f(b)}\\ \Pi_5\\ a=b\vee\neg a=b\end{array} :$$

$$\dfrac{\begin{array}{c}\xi^f(a),\ \text{i.e.,}\\ (a=a\wedge fa=0)\vee(a=b\wedge fa=1)\end{array}\qquad \begin{array}{c}\underbrace{\xi^f(b)\ ,\ \dfrac{\overline{a=a\wedge fa=0}^{\,(1)}}{fa=0}}\\ \Pi_4\\ a=b\vee\neg a=b\end{array}\qquad \begin{array}{c}\underbrace{\xi^f(a)\ ,\ \dfrac{\overline{a=b\wedge fa=1}^{\,(1)}}{fa=1}}\\ \Pi_3\\ a=b\vee\neg a=b\end{array}}{a=b\vee\neg a=b}{\scriptstyle (1)}$$

Let us pause to take stock of what has been proved with Π_5. It establishes constructively (and relevantly) the argument

$$\dfrac{\begin{array}{c}\xi^f(a)\\ \xi^f(b)\end{array}}{a=b\vee\neg a=b}$$

— that is, since $\xi^f(x)$ is the formula

$$(x=a\wedge fx=0)\vee(x=b\wedge fx=1)$$

— the argument

$$\dfrac{\begin{array}{c}(a=a\wedge fa=0)\vee(a=b\wedge fa=1)\\ (b=a\wedge fb=0)\vee(b=b\wedge fb=1)\end{array}}{a=b\vee\neg a=b}\ .$$

(The reader should not forget that Π_5 also uses the implicit premise $\neg\, 0=1$.) Noteworthy is that the conclusion of this constructively proved argument is of excluded-middle form.

3.1.2. *A Special Predicate Embedding an Arbitrary Sentence*

Let ψ be any sentence, and let $\chi(x)$ be the biconditional formula

$$x \in a \leftrightarrow (x=u \wedge \psi),$$

in which u is a term and a is still the parameter in use thus far.

The rule (SI) is now employed as follows, to construct the proof

$$\underbrace{\psi\ ,\ \exists!a\ ,\ \forall x\chi(x)}_{\substack{\Sigma_1\\ a=\{u\}}} \quad : \quad \dfrac{\dfrac{\dfrac{\overset{(1)}{\overline{c=u}}\quad \psi}{c=u\wedge\psi}\qquad \dfrac{\forall x(x\in a\leftrightarrow(x=u\wedge\psi))\quad \dfrac{\overset{(1)}{\overline{c=u}}}{\exists!c}}{c\in a\leftrightarrow(c=u\wedge\psi)}}{c\in a}\qquad \exists!a \qquad \dfrac{\dfrac{\dfrac{\forall x(x\in a\leftrightarrow(x=u\wedge\psi))\quad \dfrac{\overset{(1)}{\overline{c\in a}}}{\exists!c}}{c\in a\leftrightarrow(c=u\wedge\psi)}\qquad \overset{(1)}{\overline{c\in a}}}{c=u\wedge\psi}}{c=u}}{a=\{u\}}\,{\scriptstyle (1)}\,\mathrm{SI}$$

Next construct the proof

$$\underbrace{a=\{u\}\ ,\ \forall x\chi(x)\ ,\ \exists!u}_{\substack{\Sigma_2\\ \psi}} \quad : \quad \dfrac{\dfrac{\underbrace{a=\{u\},\exists!u}_{\substack{\Sigma_0\\ u\in a}}\qquad \dfrac{\forall x(x\in a\leftrightarrow(x=u\wedge\psi))\quad \exists!u}{u\in a\leftrightarrow(u=u\wedge\psi)}}{u=u\wedge\psi}}{\psi}$$

With Σ_1 and Σ_2 in hand, one is able to foreshadow a more general form of excluded-middle conclusion, drawn *constructively and relevantly* from the principles stated so far:

$$\underbrace{a=\{u\}\vee\neg a=\{u\}\ ,\ \forall x\chi(x)\ ,\ \exists!u\ ,\ \exists!a}_{\substack{\Sigma_3\\ \psi\vee\neg\psi}} \quad : \quad \dfrac{a=\{u\}\vee\neg a=\{u\}\qquad \dfrac{\underbrace{\overset{(1)}{\overline{a=\{u\}}}\ ,\ \forall x\chi(x)\ ,\ \exists!u}_{\substack{\Sigma_2\\ \psi}}}{\psi\vee\neg\psi}\qquad \dfrac{\dfrac{\dfrac{\underbrace{\overset{(2)}{\overline{\psi}}\ ,\ \exists!a\ ,\ \forall x\chi(x)}_{\substack{\Sigma_1\\ a=\{u\}}}\qquad \overset{(1)}{\overline{\neg a=\{u\}}}}{\bot}}{\neg\psi}\,{\scriptstyle (2)}}{\psi\vee\neg\psi}}{\psi\vee\neg\psi}\,{\scriptstyle (1)}$$

Note that none of the proofs Σ_0, Σ_1, Σ_2, or Σ_3 involves $\neg 0=1$ as an implicit premise.

3.1.3. *Choice*

Two familiar and widely used abbreviations will now be introduced. Let t be any term. (It need not stand for a type. It could be a parameter or a set-abstraction term.) Let us use

$$(\forall x \in t)\varphi x$$

as short for

$$\forall x(x \in t \rightarrow \varphi x);$$

and use

$$(\exists x \in t)\varphi x$$

as short for

$$\exists x(x \in t \wedge \varphi x).$$

The abbreviatory notation

$$\phi : t \mapsto u$$

will shorten whatever sentence in primitive notation renders the notion 'ϕ is a function that assigns to each member of t a value in u'. If a is a member of t, then ϕa is the member of u that ϕ assigns to a.

Consider next how to formulate (full) Choice as a rule of inference. The following rule is submitted for consideration.

$$\mathrm{C} \qquad \frac{(\forall x \in t)(\exists y \in u)R(x,y)}{(\exists \phi : t \mapsto u)(\forall x \in t)R(x,\phi x)}\ .$$

This rule tells us that if the binary relation R relates each member of t to some member of u, then there is a 'choice function' ϕ that assigns to each member of t exactly one R-image (from among those that are available in u). Compare the rule form given in [Aczel, 1982], p. 11 *infra*.

A point on notation: One speaks of Choice 'on t', where t is the set mentioned in the initial quantifier prefix $(\forall x \in t)$ in the premise of the rule above. We shall have occasion, when attention needs to be drawn to t, to use the label 'C^{t}' (or 'AC^{t}') for that particular form of Choice.

The *axiomatic* version of Choice as a *single formal sentence scheme* rather than as the foregoing rule of inference is of course the conditional

$$(\forall x \in t)(\exists y \in u)R(x,y) \ \rightarrow\ (\exists \phi : t \mapsto u)(\forall x \in t)R(x,\phi x).$$

We call this a sentence *scheme* because t and u are placeholders for singular terms (including parameters), and R is a placeholder for binary formulae. One can also universally quantify into the places occupied by t and u, to get the result

$$\forall z \forall w[(\forall x \in z)(\exists y \in w)R(x,y) \ \rightarrow\ (\exists \phi : t \mapsto w)(\forall x \in z)R(x,\phi x)].$$

The following special instance of this:

$$\text{AC}^\omega \qquad \forall w[(\forall x \in \omega)(\exists y \in w)R(x,y) \;\rightarrow\; (\exists \phi : t \mapsto w)(\forall x \in \omega)R(x, \phi x)]$$

is called the Axiom of Countable Choice (where ω is the set of all finite von Neumann ordinals, whose existence would have to have been secured by postulation). It could also be called 'Choice on ω', since ω is the domain of the relation R and of the resulting choice function ϕ.

An aside on nomenclature. In general, the spot marked by t in the Choice scheme above allows us to think of 'Choice on t' as the form of Choice involved. In a context where the substituends for t are so restricted as to denote sets only of kind Φ, one can speak of 'Choice on Φs'. This corresponds to allowing the universally quantified version of Choice to be instantiated only on Φs:

$$\forall z(\Phi z \rightarrow \forall w[(\forall x \in z)(\exists y \in w)R(x,y) \;\rightarrow\; (\exists \phi : t \mapsto w)(\forall x \in z)R(x, \phi x)]).$$

If Φz is 'z is finite', we have Choice on Finite Sets. If Φz is 'z has at most two members', we have Choice on Doubletons. And so on. Only if Φz is '$z = z$' (*i.e.*, z is arbitrary) do we have full Choice.
End of aside.

McCarty [1986, p. 213] states (Theorems 3.1 and 3.2 on pp. 158–159) that the 'realizability universe' $V(Kl)$ is a model of IZF and, among other principles, AC^ω, the Axiom of Choice over ω (also known as Countable Choice).

For the detailed proofs, McCarty refers the reader to his Oxford D.Phil. dissertation, [McCarty, 1984]. This work contains the following important results.

> **Theorem 7.1** (p. 96): $\text{V}(A) \models \text{IZF}$, where $\text{V}(A)$ is the cumulative realizability structure over any 'applicative structure' A.
>
> **Theorem 2.3** (p. 103): *Kl* is an applicative structure.
>
> **Theorem 2.4** (p. 103): V*(Kl)* is a model of IZF.
>
> **Theorem 4.1** (p. 111): If ψ is closed and if $\text{IZF} \vdash \psi \rightarrow \forall x(\phi \vee \neg\phi)$ for arbitrary ϕ (or even arbitrary number-theoretic Σ^0_1 ϕ), then $\text{V}(Kl) \models \neg\psi$ and ψ is independent of IZF.
>
> **Corollary 4.4** (p. 112): TND [*i.e.*, LEM] is independent of IZF.
>
> **Theorem 5.1** (p. 119): $\text{V}(Kl) \models \text{AC}^{\omega,\omega}$.

In the proof of Theorem 7.1, dealing in turn with each of the axioms and axiom schemes of IZF, special interest will attach to the case of Separation, dealt with on p. 97. The reasoning is terse, and much is left to the reader to fill in (for example, where the author writes 'Starting with the soundness of identity, ... and using induction in the metalanguage, one can prove that there is a $j_\phi \in |A|$ such that:

$$j_\phi \Vdash\!\!- \;\; (z \in a \wedge \phi[x/z] \wedge z = y) \rightarrow (y \in a \wedge \phi[x/y]).\text{'}).$$

A similar challenge arises in the proof of Theorem 5.1, where McCarty constructs a particular choice function $\overline{g_1}$, and gives 'a proof that $\overline{g_1}$ does the "choosing" ', but 'leave[s] the proof of $\overline{g_1}$'s functionality to the reader'.

So it would appear that Choice in the limited form AC^ω is unable, *à la* Goodman and Myhill, to wreak its alleged LEM mischief.

One can expect, then, to find that upon properly detailed regimentation of the Goodman–Myhill proof the application of Choice that is called for therein is one of *full* Choice (AC) — or at least, some very strong version of Choice — and cannot be construed as an application of the more modest AC^ω, or indeed of any other of the weaker forms of Choice currently countenanced by constructivists, such as the principle of Dependent Choice. We shall be seeing, however, that — contrary to this expectation — Choice on (doubletons of!) the naturals appears to be enough to precipitate the Goodman–Myhill result. Since this is in tension with McCarty's results about $V(Kl)$ modeling IZF, *and* Choice on the naturals, but invalidating Excluded Middle, the whole question seems to call for closer examination. Our regimentation of the Goodman–Myhill proof is a first tentative step towards this.

McCarty has observed (personal correspondence) that the Brouwer–Heyting–Kolmogorov reading that makes Choice plausible for the constructivist (see below) is more controversial for the real numbers than it is for the naturals. If, however, it were to turn out that Choice (in conjunction with Separation, in constructive set theory) poses problems even for the constructivist who focuses on just the naturals, then McCarty's observation might need to be amended so as to read 'is even more controversial' (assuming, of course, that Separation is not being held responsible).

On the BHK interpretation of the constructivist's logical operators $\rightarrow$, $\exists$, and $\forall$, any warrant for the assertion of the antecedent

$$(\forall x \in t)(\exists y \in u)R(x, y)$$

of the scheme conditional ought to furnish, for any member x of t, a demonstrable or constructible witness w (in u) such that $R(x, w)$.[6] Such a warrant,

[6] There is the subtle question whether one is to read the antecedent's prefix '$(\forall x \in t)$' as 'given any object x in t' or as 'given any object x *along with a proof that it is in (i.e., is of type)* t'. The latter reading, which governs within Martin-Löf type theory, creates an intensional context out of what follows. Here, however, only the more literal, former reading of the antecedent is being considered, which makes the context that follows it extensional.

on the BHK interpretation, encodes a method that will determine a(n extensional) function (call it ϕ) that will produce the value w on the argument x. The Skolemizing consequent

$$(\exists \phi : t \mapsto u)(\forall x \in t)R(x, \phi x)$$

of the conditional seems merely to make this requirement on such a warrant explicit. On this analysis, well known from the influential work of Dummett [1977, pp. 52–53], Choice seems eminently plausible for the constructivist, on the BHK interpretation, even in this extensional setting. Dummett's argument for the intuitionistic acceptability of Choice of course preceded the Goodman–Myhill result. But it should also be pointed out that Dummett did not venture to formulate any axioms for constructive set theory. Nor did Bishop; but he seemed unperturbed about having Choice within his informal framework for constructive analysis. (The reader is here reminded of the quote from Bishop in §2.3.)

It would therefore be unsettling to learn (if this is indeed the case) that by adopting Choice the constructivist will incur commitment, *by constructive reasoning*, to the Law of Excluded Middle. The aim in what follows will be to examine how convincing a case is made for this by Goodman and Myhill.

There is an objection to consider against Dummett's argument for Choice on the BHK interpretation, along the following lines.[7]

> Dummett's argument for Choice at the 'object level' uses Choice at the *meta*level. Therefore, the argument does not really provide the constructivist with adequate justification for adopting Choice.

One can respond to this objection, however, even if one were to concede its initial claim that the argument itself uses Choice at the metalevel. One can simply point out that this is the kind of predicament in which theorists would find themselves with respect to any attempt to justify a fundamental rule of inference, whether in logic proper or in mathematics. Every logical rule of natural deduction, for example, can be 'justified' as truth-preserving only by invoking that very rule in one's metalinguistic reasoning.[8] It is just a mark of fundamentality — recognition of which might even be *in favor* of Choice, as a fundamental rule in mathematics.

Let us return now to the task of formalizing the Goodman–Myhill reasoning, by examining exactly how it invokes Choice, and, more to the point, just how weak a form of Choice that is.

The following formula, in which a and b are our familiar (proof-theoretic) parameters, is an interesting substituend for Rxy in the Choice rule C:

$$(x = a \wedge y = 0) \vee (x = b \wedge y = 1).$$

[7] This objection was raised by an anonymous referee.

[8] On this point, see [Tennant, 1978, p. 74].

Along with this selection for Rxy in the premise of rule C, take for t therein the set term

$$\{z|z=a \vee z=b\}, \textit{i.e.}, \{a, b\}$$

and for u therein the set term

$$\{z|z=0 \vee z=1\}, \textit{i.e.}, \{0, 1\}.$$

One can prove the corresponding instance of the premise

$$(\forall x \in t)(\exists y \in u)R(x, y)$$

of the rule C, namely

$$(\forall x \in \{a, b\})(\exists y \in \{0, 1\})(x=a \wedge y=0) \vee (x=b \wedge y=1).$$

Here is the proof, which will be called Σ_4. Note that Σ_4 makes no use of the implicit premise $\neg\, 0=1$.

$$\Sigma_4 \ : \quad \dfrac{\dfrac{\{\,\}\mathrm{E}_3\dfrac{{}^{(2)}\overline{\;c\in\{a,b\}\;}}{c=a\vee c=b} \quad \dfrac{\dfrac{\dfrac{{}^{(1)}\overline{\;c=a\;} \quad \dfrac{\exists!0}{0=0}}{c=a\wedge 0=0}}{(c=a\wedge 0=0)\vee(c=b\wedge 0=1)} \quad \begin{matrix}\underbrace{\exists!0,\text{Pairs}}_{\Xi[t/0;\,u/1]}\\ 0\in\{0,1\}\end{matrix}}{(\exists y\in\{0,1\})(c=a\wedge y=0)\vee(c=b\wedge y=1)} \quad \dfrac{\dfrac{\dfrac{{}^{(1)}\overline{\;c=b\;} \quad \dfrac{\exists!1}{1=1}}{c=b\wedge 1=1}}{(c=a\wedge 1=0)\vee(c=b\wedge 1=1)} \quad \begin{matrix}\underbrace{\exists!1,\text{Pairs}}_{\Xi'[t/0;\,u/1]}\\ 1\in\{0,1\}\end{matrix}}{(\exists y\in\{0,1\})(c=a\wedge y=0)\vee(c=b\wedge y=1)}}{(\exists y\in\{0,1\})(c=a\wedge y=0)\vee(c=b\wedge y=1)}{\scriptstyle(1)}}{(\forall x\in\{a,b\})(\exists y\in\{0,1\})(x=a\wedge y=0)\vee(x=b\wedge y=1)}{\scriptstyle(2)}$$

This proof Σ_4 can be extended by an application of the rule C to form the proof Σ_5, still making no use of the implicit premise $\neg 0=1$.

$$\Sigma_5 \ : \quad \dfrac{\begin{matrix}\underbrace{\exists!0, \exists!1, \text{Pairs}}\\ \Sigma_4\\ (\forall x\in\{a,b\})(\exists y\in\{0,1\})(x=a \wedge y=0) \vee (x=b \wedge y=1)\end{matrix}}{(\exists\phi\colon\{a,b\}\mapsto\{0,1\})(\forall x\in\{a,b\})(x=a \wedge \phi x=0) \vee (x=b \wedge \phi x=1)}{\scriptstyle(\mathrm{C})}$$
$$\text{i.e., } (\exists\phi\colon\{a,b\}\mapsto\{0,1\})(\forall x\in\{a,b\})\,\xi^{\phi}(x)$$

N.B.: Σ_5 uses Choice! But note also that we are, in effect, appealing here only to Choice on Doubletons $\{a, b\}$. We shall have to assess carefully, when the task of regimentation is complete, just ‘how much’ of Choice that turns out to be.

The reasoning can now proceed further by applying the rule of existential elimination to this last conclusion, using the function parameter f in place of the bound function variable ϕ.[9] The resulting proof will be called Π_6. Because

[9] One says ‘function variable’, though there is nothing higher-order about it. Within the setting of first-order set theory, one would be using only ordinary individual variables ranging over sets, and interpreting ‘there is a function ϕ such that $\Phi(\phi)$’ by means of an appropriately defined first-order predicate $\mathrm{Fn}(x)$ meaning ‘x is a function’: $\exists x(\mathrm{Fn}(x) \wedge \Phi(x))$. Typically one takes for x a certain set of ordered pairs.

it contains Π_5 as a subproof, the proof Π_6 *does* involve $\neg\, 0=1$ as an implicit premise. So Π_6 has the overall form

$$\begin{array}{c}\underbrace{\text{Choice}, \neg\, 0=1, \exists!0, \exists!1, \text{Pairs}}\\ \Pi_6 \\ a=b\vee\neg a=b\end{array},$$

inheriting dependency on Choice and on the existence of both 0 and 1 from its subproof Σ_5, and inheriting its dependency on $\neg\, 0 = 1$ from its subproof Π_5:

$$\Pi_6 : \quad \dfrac{\begin{array}{c}\underbrace{\text{Choice}, \exists!0, \exists!1, \text{Pairs}}\\ \Sigma_5 \\ (\exists\phi : \{a,b\} \mapsto \{0,1\})(\forall x \in \{a,b\})\xi^{\phi}(x)\end{array} \quad \dfrac{\overset{(2)}{\overline{(\forall x \in \{a,b\})\xi^{f}(x)}} \quad \begin{array}{c}\underbrace{\overset{(1)}{\overline{\xi^{f}(a)}}\ ,\ \overset{(1)}{\overline{\xi^{f}(b)}}}\\ \Pi_5 \\ a=b\vee\neg a=b\end{array}}{a=b\vee\neg a=b}{\scriptstyle(1)}}{a=b\vee\neg a=b}{\scriptstyle(2)}$$

Let us pause to consider how important this result is. It is the culmination of the first of the two stages mentioned earlier. The parameters a and b in the conclusion of Π_6 do not occur in any of the premises of Π_6. So the following is a *universal* result:

$$\Pi_6^{\forall\forall} : \quad \dfrac{\dfrac{\begin{array}{c}\underbrace{\text{Choice}, \text{Pairs}, \exists!0, \exists!1, \neg 0=1}\\ \Pi_6 \\ a=b\vee\neg a=b\end{array}}{\forall y(a=y\vee\neg a=y)}}{\forall x\forall y(x=y\vee\neg x=y)}\ .$$

The universal decidability of identity on the domain poses particular problems for both the Brouwerian intuitionist and the Bishop-style constructivist. For it allows one to define, and prove the existence of, *discontinuous functions on the reals*,[10] once one has adopted sufficiently strong further axioms to get the reals (and functions defined on them) into the ontological picture. One is already certain of this, on the basis of proof $\Pi_6^{\forall\forall}$, even before adopting such further axioms. Note also that $\Pi_6^{\forall\forall}$ uses no instance of Separation as a premise. Choice alone is doing the dirty work — albeit only Choice on (arbitrary) Doubletons.

Discontinuous functions on the reals are unacceptable not only to the Brouwerian intuitionist, but also to the Bishop-style constructivist. The former even goes so far as to claim to have proved that *every function on the reals is continuous*; whereas the latter refrains from asserting such a result, while nevertheless also refraining from asserting the existence of any *dis*continuous function. The Bishop-style constructivist seeks to use mathematical axioms that the classical mathematician would accept, but which are in a form congenial to the

[10] On this point, discussion with Stewart Shapiro has been most helpful.

constructivist; and to use only constructive deductive reasoning when proving theorems from those axioms.

It is clear, then, that the constructive set theorist cannot adopt 'full Choice' — that is, Choice in the form considered above; or *even* just Choice on (arbitrary) Doubletons — without seriously compromising both Bishop-style constructivists and Brouwerian intuitionists as they venture further into their treatment of real analysis. So the search would be on for less ambitious forms of Choice — ones that will enable the derivation of enough constructive mathematics, while also not being powerful enough to imply the decidability of identity. It is this project, for example, that is pursued by Aczel and Rathjen [2010]. As Rathjen puts it (personal communication)

> [I]n the constructive context, it is usually no problem to adopt some forms of choice such as countable choice, dependent choice and the presentation axiom. The former two are the ones used by Bishop and many other constructivists. The "bad" form of choice (that gives you LEM) usually arises from the form where you can pick a representative from each equivalence class. The latter is never used by Bishop.

3.2. The Second Stage

Let us proceed now to the second stage of the promised regimentation of the Goodman–Myhill proof, whose aim is to precipitate the *Law of Excluded Middle* $\psi \vee \neg\psi$. This result is of course even more sweeping than the universal decidability of identity, which was the conclusion of the first stage. Moreover, in order to secure it, as will be seen, Goodman and Myhill availed themselves of a little more in the way of premises. They appealed to an abstraction principle that is an instance of the Axiom Scheme of Separation. There will be reason for disquiet about the latter. The form of Choice that Goodman and Myhill employ, plus this instance of Separation constructively imply, *modulo* a few more innocuous assumptions, the worst result possible for the constructivist. We have chosen our words carefully here, in writing of the form of Choice 'that Goodman and Myhill employ'. Part of the interest of complete regimentation lies in determining exactly 'how much' Choice this really turns out to be. If it turns out to be a mere smidgeon of Choice — a form of Choice *acceptable* to the constructivist — then the ground will have been laid for greater preparedness to blame Separation, rather than Choice, for precipitating Excluded Middle.

The question then arises: if constructivists were prepared to 'bite the bullet' on the decidability of identity, could they persevere with (full) Choice and nevertheless avoid the utter disaster of Excluded Middle, by carefully restricting the admissible forms of Separation? This appears to be a line of inquiry not considered by any of the constructive set theorists who have set themselves the goal of providing a set theory in which to formalize Bishop's constructive mathematics. This is understandable, in light of the first stage just discussed, given their refusal to recognize the existence of any discontinuous functions on the reals (once they have adopted whatever further axioms will furnish the reals for their purposes in real analysis).

But what if the project were to be liberalized slightly? What about seeking a constructive set theory for a possible *broadening* of Bishop's constructive mathematics by admitting the existence of discontinuous functions *while yet* not endorsing Excluded Middle across the board? The reader should bear this possibility in mind in examining the second stage of the Goodman–Myhill proof (or, rather, of the regimentation about to be provided for it). But the reader should also bear in mind that neither the first stage nor the second stage of the regimentation under way of the Goodman–Myhill result involves any of those further axioms that would be needed in order to furnish the reals for the purposes of real analysis. The possible broadening suggested here would have the consequence, of course, that functions on the reals could no longer be warranted, in general, to allow one to approximate their values by numerical computation. It would merit further investigation whether this concession would make such functions inherently non-constructive. The bruited concession might end up being a better bargain for the constructivist than the admission of Excluded Middle.

So, back to formalities: applying universal elimination (twice) to the conclusion of $\Pi_6^{\forall\forall}$ would result in a proof not in normal form. So one works instead with its parametric subproof Π_6, and makes the desired substitutions. In Π_6 substitute now $\{u\}$ for b:

$$\begin{array}{c} b \\ \downarrow \\ \{u\} \end{array}$$

This substitution results in a proof of the following overall form (making all premises explicit):

$$\begin{array}{c}\underbrace{\text{Choice}, \text{Pairs}, \exists!0, \exists!1, \neg 0=1}\\ \Pi_6[b/\{u\}]\\ a=\{u\}\vee\neg a=\{u\}\end{array} .$$

Recall the proof Σ_3:

$$\begin{array}{c}\underbrace{a=\{u\}\vee\neg a=\{u\}\ ,\ \forall x\chi(x)\ ,\ \exists!u\ ,\ \exists!a}\\ \Sigma_3\\ \psi\vee\neg\psi\end{array} \quad : \quad \dfrac{a=\{u\}\vee\neg a=\{u\} \qquad \dfrac{\begin{array}{c}\underbrace{\overset{(1)}{\overline{a=\{u\}}}\ ,\ \forall x\chi(x)\ ,\ \exists!u}\\ \Sigma_2\\ \psi\end{array}}{\psi\vee\neg\psi} \qquad \dfrac{\dfrac{\dfrac{\begin{array}{c}\underbrace{\overset{(2)}{\overline{\psi}}\ ,\ \exists!a\ ,\ \forall x\chi(x)}\\ \Sigma_1\\ a=\{u\}\end{array} \quad \overline{\neg a=\{u\}}^{\,(1)}}{\bot}\,(2)}{\neg\psi}}{\psi\vee\neg\psi}}{\psi\vee\neg\psi}\,(1) \quad .$$

With the appeal to Choice, and the substitution just mentioned, one has secured via $\Pi_6[b/\{u\}]$ the major premise

$$a = \{u\} \vee \neg a = \{u\}$$

for the terminal step of $\vee$E in the proof Σ_3. And by so doing, one has made the conclusion $\psi \vee \neg\psi$ depend both on the implicit premise $\neg\, 0 = 1$ and on Choice (on Doubletons). Let us call Π_7 the proof thus composed out of Σ_3 and $\Pi_6[b/\{u\}]$.

$$\Pi_7 \;:\; \underbrace{\overset{\overbrace{\text{Choice}, \exists!0, \exists!1, \text{Pairs}}^{\textstyle \Pi_6[b/\{u\}]}}{a = \{u\} \vee \neg a = \{u\}} \;,\; \forall x \chi(x) \;,\; \exists! u \;,\; \exists! a}_{\textstyle \begin{array}{c}\Sigma_3 \\ \psi \vee \neg\psi\end{array}}$$

Now all that is needed are proofs of Σ_3's remaining premises

$$\forall x \chi(x)$$

— that is,

$$\forall x (x \in a \leftrightarrow (x = u \wedge \psi));$$

and

$$\exists! u \; .$$

The latter premise is easy — for u take 0 (for it has been secured somehow that 0 exists). That is, employ now the further substitution

$$\begin{array}{c} u \\ \downarrow \\ 0 \end{array}$$

This means the former premise now takes the more specific form

$$\forall x (x \in a \leftrightarrow (x = 0 \wedge \psi)).$$

The proof — $\Pi_7[u/0]$ — that has now been reached is of the following form, with all premises stated explicitly. Let us now call this proof Θ, for short.

$$\Theta \; (= \Pi_7[u/0]) : \underbrace{\neg\, 0 = 1,\; \text{Choice},\; \forall x (x \in a \leftrightarrow (x = 0 \wedge \psi)),\; \exists! 0\,,\; \exists! 1\,,\; \exists! a\,,\; \text{Pairs}}_{\textstyle \begin{array}{c}\Theta \\ \psi \vee \neg\psi\end{array}} \; .$$

Note that the parameter a in Θ can always be chosen so as not to occur in ψ. With four of the premises of Θ, namely $\neg 0=1$, $\exists!0$, $\exists!1$, and Pairs there is no quarrel. And one can acknowledge that Choice, so plainly declared as a premise of Θ, has had a large question mark placed over it in the recent tradition.

Nota bene, however, that at this point in our path to full regimentation of the Goodman–Myhill proof of Excluded Middle, we have performed the substitutions just mentioned, with the result that we are now in a position to assess more carefully the form of Choice that Goodman and Myhill actually employ (or that we are actually employing on their behalves, via our regimentation) in the second stage of their proof. That form — call it GM-Choice — is the result of substituting $\{0\}$ for b for the step called Choice back in the proof Σ_5. For the reader's convenience we recall that proof here.

$$\Sigma_5 \;:\quad \cfrac{\cfrac{\underbrace{\exists!0, \exists!1, \text{Pairs}}_{\Sigma_4}}{(\forall x\in\{a,b\})(\exists y\in\{0,1\})(x=a\wedge y=0)\vee(x=b\wedge y=1)}}{(\exists\phi:\{a,b\}\mapsto\{0,1\})(\forall x\in\{a,b\})(x=a\wedge \phi x=0)\vee(x=b\wedge\phi x=1)}\,(\mathrm{C})$$

$$i.e.,\ (\exists\phi:\{a,b\}\mapsto\{0,1\})(\forall x\in\{a,b\})\,\xi^{\phi}(x)$$

The important point is that with the substitution of $\{0\}$ for b therein, GM-Choice (that is, this particular instance of Choice) is revealed to be, more specifically,

$$\frac{(\forall x\in\{a,\{0\}\})(\exists y\in\{0,1\})(x=a\wedge y=0)\vee(x=\{0\}\wedge y=1)}{(\exists\phi:\{a,\{0\}\}\mapsto\{0,1\})(\forall x\in\{a,\{0\}\})(x=a\wedge\phi x=0)\vee(x=\{0\}\wedge\phi x=1)}\,(\text{GM-C})$$

The only proof-theoretic parameter here is a. Moreover, this instance of Choice has its (Boolean, quantifier-free!) relation R fully specified, using as atomic formulae only identities involving a, 0, $\{0\}$, 1, x and y. Bear in mind, also, that we are working within a theoretical context that determines that

$$1 \;=_{df}\; \{0\}, \quad \text{and} \quad \neg\, 0=1.$$

This means that the instance of Choice actually employed by Goodman and Myhill in their proof of generalized LEM can be simplified even further:

$$\frac{(\forall x\in\{a,\{0\}\})(\exists y\in\{0,\{0\}\})(x=a\wedge y=0)\vee(x=\{0\}\wedge y=\{0\})}{(\exists\phi:\{a,\{0\}\}\mapsto\{0,\{0\}\})(\forall x\in\{a,\{0\}\})(x=a\wedge\phi x=0)\vee(x=\{0\}\wedge\phi x=\{0\})}\,(\text{GM-C})\ .$$

The general rule form of (GM-C), using $R(x,y)$ as characterized above, and still with a as sole parameter, is therefore

$$\frac{(\forall x\in\{a,\{0\}\})(\exists y\in\{0,\{0\}\})R(x,y)}{(\exists\phi:\{a,\{0\}\}\mapsto\{0,\{0\}\})(\forall x\in\{a,\{0\}\})R(x,\phi x)}\,(\text{GM-C})\ .$$

The *axiomatic* form of (GM-C) (again, for $R(x, y)$ as characterized above) is

$$\forall z((\forall x \in \{z, \{0\}\})(\exists y \in \{0, \{0\}\})R(x, y) \rightarrow (\exists \phi : \{z, \{0\}\} \mapsto \{0, \{0\}\})(\forall x \in \{z, \{0\}\})R(x, \phi x)).$$

Let us turn our attention for a moment back to the rule-form of GM-C, expressed by means of the parameter a (rather than the bound variable z of the axiomatic form). If the general setting were to secure the understanding that a is any *natural number* — or any member of an accessible domain, such as the hereditarily finite pure sets, each of which is canonically denotable — then it would be very difficult to raise any constructivist objections to GM-C. We shall be returning in §6 to this possibility of a constructively acceptable construal, within an appropriate setting, of the amount of Choice actually used in the second stage of the Goodman–Myhill proof of Excluded Middle.

Despite its possible constructive acceptability, the smidgeon of Choice we are calling GM-C manages (constructively) to precipitate the result

$$\psi \vee \neg\psi,$$

where ψ is an *arbitrary* sentence of the language about the domain in question.

It is therefore appropriate, at this point, to examine more closely the remaining premise $\forall x(x \in a \leftrightarrow (x=0 \wedge \psi))$ of Θ, on which too the untoward conclusion $\psi \vee \neg\psi$ has been made to rest.

3.2.1. How Does One Secure the Main Remaining Assumption on Which Excluded Middle Has Been Shown to Rest?

What axiomatic principle will now afford us an 'outright' proof of $\psi \vee \neg\psi$? (By 'outright' here, it is not meant that the prover should be deprived of the two obviously uncontroversial premises $\neg\, 0 = 1$ and $\exists!0$.) There are only two possibilities.

The first possibility for the axiomatic principle is the following abstraction principle for sets:

$$\exists y \forall x(x \in y \leftrightarrow (x=0 \wedge \psi)).$$

The second possibility is to note that

$$t=0 \dashv\vdash t \in \{0\}$$

and to rephrase the principle, accordingly, as

$$\exists y \forall x(x \in y \leftrightarrow (x \in \{0\} \wedge \psi)),$$

thereby 'seeing it as' deduced by $\forall$-Elimination from the following substitution instance of the Axiom Scheme of Separation:

$$\forall z \exists y \forall x(x \in y \leftrightarrow (x \in z \wedge \psi)),$$

with ψ as the separating formula.

Whichever way one construes the principle in question, by adopting it one would have the following proof — call it Π_8 — whose final step is an application of Existential Elimination (using parameter a):[11]

Π_8 :

$$\dfrac{\begin{array}{c}\dfrac{(L3)\dfrac{\exists!0}{\exists!\{0\}}\qquad \begin{array}{c}\text{Separation:}\\ \forall z\exists y\forall x(x\in y\leftrightarrow(x\in z\wedge\psi))\end{array}}{\exists y\forall x(x\in y\leftrightarrow(x\in\{0\}\wedge\psi))}\\ \Upsilon\\ \exists y\forall x(x\in y\leftrightarrow(x=0\wedge\psi))\end{array}\qquad \begin{array}{c}\underbrace{\neg 0=1\,,\,\text{GM-C}\,,\overline{\forall x(x\in a\leftrightarrow(x=0\wedge\psi))}^{(1)}\,,\,\exists!0\,,\,\exists!1\,,\,\text{Pairs}}\\ \Theta\\ \psi\vee\neg\psi\end{array}}{\psi\vee\neg\psi}\,{\scriptstyle(1)}$$

This was the path taken by Goodman and Myhill [1978]. In effect, they simply laid down as a *postulate* the abstraction (or comprehension) principle just stated — with its placeholder for arbitrary open or closed sentences ψ — in order to secure the 'more general-looking' conclusion

$$\psi \vee \neg\psi.$$

The title of their one-page paper — 'Choice implies Excluded Middle' — greatly overstates, however, the significance of their result. As they themselves say, they prove that Choice implies Excluded Middle

> *for every formula ψ such that ...*
>
> $$\exists y\forall x(x \in y \leftrightarrow (x=0 \wedge \psi)) \ .$$
>
> [Emphasis added. For uniformity of notation the bound variable y is used instead of the upper-case A in the original.]

If, however, one is leery of this abstraction principle (in so far as it potentially concerns *undecidable* sentences ψ), then the constructivist has little to be concerned about when presented with (a closer analysis of) Goodman and Myhill's result. This is what will be argued in Part II.

4. THE RECEPTION OF THE GOODMAN–MYHILL RESULT

It is worth quickly reviewing how the Goodman–Myhill result has been received (in its original form as published, not in the foregoing more fully regimented form).

4.1. Martin-Löf

That the result does commit the constructive set theorist to Excluded Middle is a claim that appears to be endorsed by Per Martin-Löf, who writes

[11] The reader should note that within the (sub)proof Θ we shall have been able to replace every occurrence of 1 with an occurrence of $\{0\}$, so as to be able to have GM-Choice take on the last, simplest possible, form just discussed.

> ... by Diaconescu's theorem [[1975] — Author] *as transferred to constructive set theory by Goodman and Myhill*, the law of excluded middle follows from the axiom of choice in the context of constructive set theory. [Martin-Löf, 2006, p. 349, emphasis added]

The immediate context for this endorsement of the Goodman–Myhill result was Martin-Löf's discussion of the strengthening of constructive set theory by the *extensional* Axiom of Choice (equivalently, Zermelo's Axiom of Choice). Martin-Löf continues

> What Zermelo wrote ... about the omnipresent, and often subconscious, use of the axiom of choice in mathematical proofs is incontrovertible, but it concerns the constructive, or *intensional*, version of it, which follows almost immediately from the strong rule of existential elimination. It cannot be taken as a justification of his own version of the axiom of choice, including as it does the extensionality of the choice function. [*ibid.*, pp. 349–350, emphasis added.]

According to Martin-Löf, it is only the use of an *extensional* form of the Axiom of Choice that precipitates the unwanted result, for the constructive set theorist, that Excluded Middle holds. If one were to use, instead, an appropriate *intensional* form of Choice, commitment to Excluded Middle could be avoided. And it is moreover only the intensional form, not any extensional one, that appears to be intuitively compelling for the constructivist.

§3 provided a regimentation of the proof of the Goodman–Myhill result within the first-order formalism for constructive set theory furnished in §2. Its fundamental rules (for introducing and eliminating the set-abstraction operator) ensure the overall *extensionality* of one's theorizing about sets. An investigation was conducted into the consequences of postulating Choice in the very form that makes it intuitively compelling for the constructivist (the form that is often thought of as the 'intensional' form).[12] And reason will presently be found to question whether that form of Choice is to be blamed (within the context of the overall Goodman–Myhill argument) for precipitating Excluded Middle. One hesitates to call this form of Choice 'intensional', because, as just explained, the overall treatment of set theory provided here is extensional. This study is not engaging the problem of logical assessment within the dialectical space provided by Martin-Löf's type theory, which enables one to distinguish extensional and intensional forms of Choice.

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[12] Martin-Löf, of course, would say that such a treatment remains blind to the distinction between extensional and intensional choice functions. As he wrote

> ... the problem with Zermelo's Axiom of Choice is not the existence of the choice function but its extensionality, *and this is not visible within an extensional framework*, like Zermelo-Fraenkel set theory, where all functions are by definition extensional. [*ibid.*, p. 349, emphasis added.]

4.2. Bell

John Bell is another writer who has appealed to Goodman and Myhill's result. Two of his works are worth examining. In the first of these he presents the Goodman–Myhill result in a new setting, without any critique of whether they really are entitled to regard Choice as the culprit principle precipitating Excluded Middle. In the second work he advances to a diagnosis in agreement with ours, as to which underlying principle (other than Choice) is the real culprit in the proof that 'Choice implies Excluded Middle'.

4.2.1. Bell 2008

In [Bell, 2008], at p. 200, he remarks that one of his results (Theorem 1(b) on p. 196) 'is based on that given in [Goodman and Myhill, 1978]'. Bell's result can be explained as follows.

First, Bell defines a 'restricted' formula φ as one in which every one of its quantifications can be expressed in one of the explicitly restricted forms

$$\exists x(x \in a \wedge \ldots x \ldots);$$
$$\forall x(x \in a \rightarrow \ldots x \ldots).$$

He lays down axioms for a weak constructive set theory, called **WST**. He identifies in addition a particular extensionality principle, which he calls **Extsub(2)**, and a particular 'weak' form of Choice, which he calls **WAC(2)**. (This is, in effect, what we called Choice on Doubletons, in §3.1.3.)

Bell's result is that for restricted formulae φ, the constructivist can derive $\varphi \vee \neg\varphi$ from **WST+Extsub(2)+WAC(2)**. The definitional details do not matter very much; what is important in this context is simply that Bell follows the line of argument given by Goodman and Myhill but without considering whether the 'blame' for Excluded Middle being derivable is really to be attributed to (his form of) *Choice*, rather than, say, one of the other principles involved in his proof — one which, perhaps, he thinks is innocuous, and which happens to be nestling within **WST**.

The principle in question is the axiom that Bell calls **Restricted Subsets**, which in notation chosen so as to be consistent with the rest of this study can be written

$$\exists y \forall x(x \in y \leftrightarrow x \in a \wedge \psi).$$

§3.1.3 showed how a premise tantamount to this axiom of **WST** is crucially at work in the reasoning of Goodman and Myhill, on which Bell said he had based his own.

4.2.2. Bell 2009

In [Bell, 2009],[13] his diagnosis of the underlying offender (among the basic premises) in the Goodman–Myhill results has changed quite significantly, to the point where it is in broad agreement with the diagnosis offered in this study (and which was arrived at independently).[14] The underlying principles Bell mentions, in their order of presentation in his 2009 book and with the culprit one reddened, are as follows. (Note: **PA** is the power sort of the individual sort **A**.)

- **Binary Sort Principle** (p. 102)
  There is a sort **2** and constants **0:2**, **1:2** subject to the axioms $\mathbf{0}{\neq}\mathbf{1}$ and $\forall x\!:\!\mathbf{2}[x{=}\mathbf{0} \vee x{=}\mathbf{1}]$
- **Predicative Comprehension Principle** (p. 103)
  $\exists X\mathbf{:PA}\,\forall x\!:\!\mathbf{A}[X(x) \Leftrightarrow \varphi(x)]$, where φ has at most the free variable x and contains no bound *function or predicate* variables. [Emphasis added — Author]
- **Principle of Extensionality of Functions** (p. 104)
  $\forall F : \mathbf{PA}{\rightarrow}\mathbf{A}\,\forall X, Y\mathbf{:PA}[X \approx Y \Rightarrow FX = FY]$, where $X \approx Y$ is an abbreviation for $\forall x{:}\mathbf{A}[X(x) \Leftrightarrow Y(x)]$, that is, X and Y are *extensionally equivalent.*

Compare the culprit principle here with Bell's earlier principle **Restricted Subsets**, to which §4.2.1 drew attention as a potential culprit:

$$\exists y \forall x(x \in y \leftrightarrow x \in a \wedge \psi).$$

The formula $\varphi(x)$ in Bell's **Predicative Comprehension Principle** is of course allowed to contain bound *individual* variables, and could therefore take the form

$$x \in a \wedge \psi$$

[13] Thanks are owed to Bell for this reference to his 2009 book, in response to an earlier draft sent out for comments. That draft did not contain this section (§4.2.2), which has been added in response to Bell's helpful comments.

[14] As Bell put it (personal communication)

> I agree with your analysis, and in fact I made a similar observation myself in my book *The Axiom of Choice* (College [Publications], 2009), which perhaps you have not come across [which, I regret to say, I had not — Author]. On pp. 103 *et seq.* you will find a derivation of LEM from AC together with what I term the Predicative Comprehension Principle, the Principle of Extensionality of Functions, and the Binary Sort Principle. Just as you give reasons for questioning the constructive validity of the axiom of separation, I go on to point out that the Predicative Comprehension Principle is not constructively justified.

on the right-hand side of the biconditional in **Restricted Subsets**, with ψ permitted to be a first-order *sentence* (containing no free variables) destined to feature in a conclusion of the LEM form $\psi \vee \neg\psi$.

Bell's **Predicative Comprehension Principle** simply makes explicit the sorts **PA** and **A** over which the variables X and x respectively range.

The regimentation here (in §3) of the Goodman–Myhill reasoning has been furnished by operating wholly within the language of first-order set theory, without the complication of types or sorts, and concomitant typing of variables. This approach allows one to distill the essential structure of the reasoning 'from Choice' to LEM, and to bring an alternative culprit (other than Choice) into clear view. Since both of Bell's studies have proceeded at the same level of *informal* rigor as did Goodman and Myhill, it should be emphasized here that this novel feature of *complete regimentation* is a further aid to a keener understanding of this deep, important and motivating question: does Choice *really* constructively imply Excluded Middle? To echo a remark of Zermelo [1908, p. 262]:[15]

> ...I hope to have done at least some useful spadework hereby for subsequent investigations in such deeper problems.

5. GOODMAN AND MYHILL'S CURIOUS OMISSION OF CONTEXT FOR THEIR RESULT

In their paper [Goodman and Myhill, 1978], which is the motivating focus of this study, the authors prove their result but do not supply it with any investigative context. They did not link it to any preceding studies, or reflect on the methodological problem that they had thereby created for the wider field of constructive set theory as it related to Bishop-style constructive analysis.

As it happens, however, they had both written earlier singly authored works which, in hindsight, place their joint result in a much richer historical context. Its ramifications within that context will be the topic of this brief section. Indeed, it will be seen that the singly authored papers touch on the central issue that occupies this study, namely whether it is Choice or Separation that is to blame for the untoward precipitation of Excluded Middle.

The relevant earlier papers are [Myhill, 1975] and [Goodman, 1976]. The former is discussed in Part II, §2.1, where it emerges that Myhill's conditions on substituends in the Axiom Scheme of Separation allow for separating formulae that will not be effectively decidable — thereby making LEM on *those* formulae disastrous for the constructivist. Although Myhill [1975] had Extensionality axioms, he did not, however, propose Choice as an axiom of his constructive set theory. So LEM is not actually visited on that particular system of his.

[15] 'Für spätere Untersuchungen, welche sich mit solchen tiefer liegenden Problemen beschäftigen, möchte ich hiermit wenigstens eine nützliche Vorarbeit liefern.' The English translation here is by Stefan Bauer-Mengelberg, in [van Heijenoort, 1967, p. 201]. The English translations to be used of other quotations from Zermelo are from the same source, unless otherwise indicated.

[Goodman, 1976] (whose main result was proved again, by a different method, in [Goodman, 1978]) showed that Heyting Arithmetic in all finite types (called HA^{ω}), when extended by both full Choice and Relativized Dependent Choice, conservatively extends HA. (This had actually been conjectured in [Goodman and Myhill, 1972, p. 90 *infra*].) Note that the type theory remains intensional. It would have been a germane observation for Goodman and Myhill to have made, that *something extra* in the (constructive) set-theoretic context must be responsible for the constructive *non-conservativity* that is established by showing that *Excluded Middle* is provable once Choice is adopted. What could this extra ingredient be? There was one obvious candidate (if one were to remain unsuspecting of Separation) — namely, the *extensionality* of the set theory in question. This should then have raised the question whether adding Extensionality to HA^{ω}+AC+RDC could possibly result in a *non*-conservative extension of HA.

As it happened Goodman's result that HA^{ω}+AC+RDC conservatively extends HA was strengthened by Beeson [1979], *so as to include Extensionality* in the conservatively extending theory. The immediate reflection, given the joint paper of Goodman and Myhill the preceding year, would have been that *something other than* Extensionality (in the set-theoretical setting) must provide the deeper reason behind the provability of Excluded Middle. And that extra ingredient would be *Separation* — which type theories such as HA^{ω}+AC+RDC importantly *lack*.[16] These ruminations depend on the assumption that notions such as Choice and Extensionality are invariant enough across different foundational frameworks (such as type theory, category theory and set theory) to make this conceptual arbitraging possible.

6. MACRO-REFLECTIONS ON THE FORMALIZATION OF THE GOODMAN–MYHILL ARGUMENT

Table 1 shows which principles or axioms or rules of inference (listed in the top row) find application in which proofs (listed in the leftmost column).
Figure 1 shows which proofs (in our regimentation) occur as subproofs of which other proofs; and which result from certain others by substitution of terms.

The conclusion of Π_6 is parametric in a and b, in that these two parameters do not occur in any undischarged assumption of Π_6. And Π_6 does not use Separation. The only problematic premise of Π_6 appears to be Choice on Doubletons. The conclusion of the resulting proof $\Pi_6^{\forall\forall}$ (the culmination of the 'first stage' of the Goodman–Myhill proof) is Excluded Middle just on identities:

$$\forall x \forall y (x = y \vee \neg\, x = y).$$

[16] This crucial observation is owed to Laura Crosilla (personal correspondence).

TABLE 1. *What rules or axioms each proof uses*

	{ }I	{ }E$_1$	{ }E$_3$	$\exists!0$	$\exists!1$	$\neg\, 0=1$	Pairs	$\mathrm{C}^{\{a,b\}}$	$\mathrm{C}^{\{a,\{u\}\}}$	$\mathrm{C}^{\{a,\{0\}\}}$	$\mathrm{Sep}^{\underline{\mathrm{n}}}$
Ω							●				
Λ_1			●				●				
Λ_2							●				
Λ_3							●				
Υ											●
Ξ		●					●				
Ξ'		●					●				
Σ_0		●									
Σ_1	●										
Σ_2		●									
Σ_3	●	●									
Σ_4		●	●	●	●		●				
Σ_5		●	●	●	●		●	●			
Π_1						●					
Π_2						●					
Π_3						●					
Π_4						●					
Π_5						●					
Π_6			●	●	●	●	●	●			
$\Pi_6^{\forall\forall}$			●	●	●	●	●	●			
Π_7	●	●	●	●	●	●	●		●		
Θ	●	●	●	●	●	●	●			●	
Π_8	●	●	●	●	●	●	●			●	●

As the diagram of subproof containments makes clear, there is another way that Π_6 gets exploited, as one embarks on the 'second stage' of the Goodman–Myhill proof. This is by substituting the singleton term $\{u\}$ for the parameter b therein, to obtain Π_7; and thereafter substituting 0 for u in Π_7, which combines with Σ_3 (similarly modified by these substitutions) to yield the proof Θ. This proof Θ then features as the parametric subproof for $\exists$-Elimination (involving parameter a) at the final step of the proof Π_8. The major premise for that elimination is the conclusion of Υ, one of whose premises is the instance

$$\forall z \exists y \forall x (x \in y \leftrightarrow (x \in z \wedge \psi))$$

of Separation. Here, ψ is completely arbitrary — it need not even have x free; and the conclusion of Π_8 is

$$\psi \vee \neg\psi.$$

So with Π_8 we finally have Excluded Middle in its starkest form — on *every* sentence, not just on identities. This alarming extension of the reach of Excluded

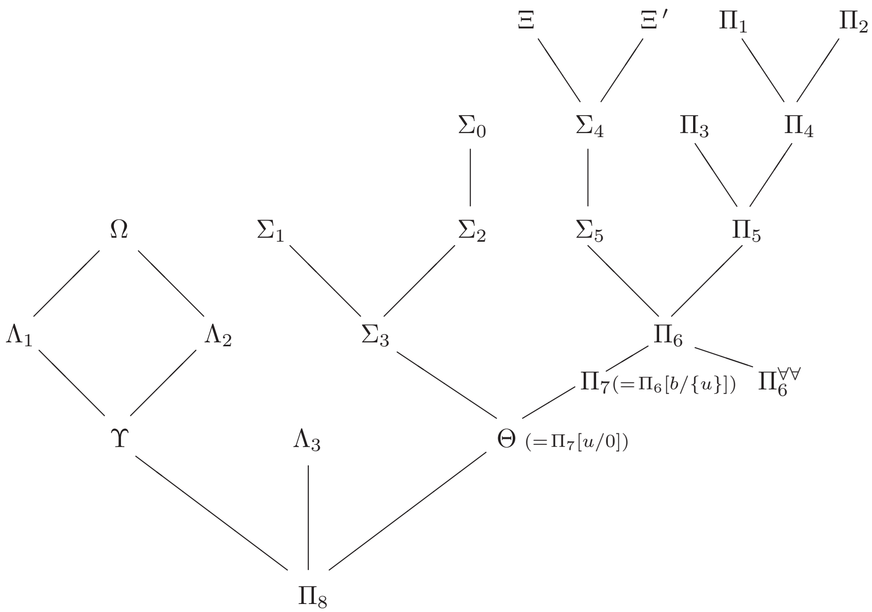

Fig. 1. The tree of subproof embeddings

Middle is owing to the extra contribution made by the proof Υ, which invokes Separation to furnish that major premise for $\exists$-Elimination.

For the constructivist, something has gone wrong. From the sidelines, one can see that Excluded Middle has been derived by means of the proof Π_8 whose only possible problematic premises are (GM-)Choice and (the displayed instance of) Separation.

The question arises, however: *'how much'* of Choice does Π_8 actually use? — that is, what exactly *is* GM-Choice? The answer appears to be: *very little.* This answer was obtained earlier (see §3.2) by reflecting on what happens when one substitutes the term $\{0\}$ for every parametric occurrence of b within the subproof Π_6 of Π_8, and then substitutes $\{0\}$ for 1. The substitutions percolate all the way up the foregoing tree of subproof containments: every proof on any branch leading down to Π_6 undergoes the overall modification called for here, *i.e.*, substitution of $\{0\}$ for every parametric occurrence of b, and for every occurrence of 1. Let us indicate any such modified result by adding an asterisk to the name of the unmodified proof.

Upon performing these substitutions, we discovered that the form of Choice actually used at the final step of Σ_5^* (the form that we are calling GM-Choice) is the following:

$$\frac{(\forall x \in \{a,\{0\}\})(\exists y \in \{0,\{0\}\})(x = a \wedge y = 0) \vee (x = \{0\} \wedge y = \{0\})}{(\exists \phi : \{a,\{0\}\} \mapsto \{0,\{0\}\})(\forall x \in \{a,\{0\}\})(x = a \wedge \phi x = 0) \vee (x = \{0\} \wedge \phi x = \{0\})}\text{(GM-C)} \quad .$$

One finds also that the conclusion of the similarly modified Π_6^* would be the more specific LEM-instance

$$a = \{0\} \vee \neg\, a = \{0\},$$

rather than the completely general

$$a = b \vee \neg\, a = b.$$

Yet the *more specific* LEM-instance suffices for the further deductive passage to the absolutely general form

$$\psi \vee \neg\psi$$

of Excluded Middle, upon combining Σ_3^* with Π_6^* to get Θ (which needs no asterisk!). Finally, the a-involving undischarged assumption

$$\forall x(x \in a \leftrightarrow (x = 0 \wedge \psi))$$

of Θ gets discharged by $\exists$-Elimination, with major premise

$$\exists y \forall x(x \in y \leftrightarrow (x = 0 \wedge \psi)),$$

which is the conclusion of the proof Υ, *whose sole suspect premise is the instance*

$$\forall z \exists y \forall x(x \in y \leftrightarrow (x \in z \wedge \psi))$$

of Separation.

Is the modest amount of Choice pinpointed above constructively justifiable? There is good reason for an affirmative answer. Being able to 'make the choice' of either 0 or $\{0\}$ as the value of the function called for requires no more than an ability (in principle) to discern whether an arbitrarily given member a of the domain is identical to $\{0\}$. Consider a constructivist who wishes to acknowledge the existence only of *hereditarily finite pure sets*, and who is happy to identify 0 with the empty set $\emptyset$ and 1 with the singleton $\{\emptyset\}$. (One could call such a constructivist a 'Kroneckerian Von Neumannite'.) Surely she would claim to have the in-principle ability just mentioned. *Yet*, by espousing a form of Choice as weak as $C^{\{a,1\}}$, she would have Excluded Middle visited upon her (by Π_8), should she also happen to adopt the aforementioned instance of Separation.

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