On Gentzen's Structural Completeness Proof

by

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Abstract

In his very first publication, Gentzen introduced the structural rules of thinning and cut on sequents. He did not consider rules for logical operators. Gentzen provided a most interesting 'structural completeness proof', which it is the concern of this study to explain and clarify. We provide an improved (because more detailed) proof of Gentzen's completeness result. Then we reflect on the self-imposed limitations of this, Gentzen's earliest sequent-setting, and explore how his approach might have been generalized, even in the absence of logical operators, so as to cover cases involving sequents with empty antecedent or succedent, and logical consequences of infinite sets of sequents.

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1 Introduction and motivation

Gentzen's first publication was [2]. It introduced the structural rules of thinning (*Verdünnung*) and cut (*Schnitt*) on sequents (unhelpfully called *Sätze*). To call sequents *Sätze* is unhelpful because the main application, subsequently, is to be one where the sequents are made up of a set of sentences on the left, and a sentence on the right. Since the usual reading of *Sätze* is 'sentences', this could lead to confusion.

Gentzen did not consider any other rules—in particular, he gave no rules for logical operators. He provided, however, a most interesting 'structural completeness proof', which it is the concern of this study to explain and clarify. In §2 we set out notation and provide an improved (because more detailed) proof of Gentzen's completeness result. In §3 we reflect on the self-imposed limitations of this, Gentzen's earliest sequent-setting, and explore how his approach might have been generalized even in the absence of logical operators.

Our aim here is to re-cast Gentzen's structural completeness proof into a form that allows for ready generalization to cover cases of completeness that Gentzen himself did not consider: the cases involving sequents with empty antecedent or succedent, and logical consequences of infinite sets of sequents. In order to accomplish this, we break the proof down into proofs of more lemmas than Gentzen himself cared to isolate for separate statement and proof. The advantage of doing this is that one comes to appreciate better how 'all the bits fit together', as it were; and which ones have to be tweaked, or re-ordered, in order to effect the generalizations that are sought in this study.

In order to regiment Gentzen's reasoning more rigorously, we have cast into the form of a proof by induction (to be found in the proof of our Lemma 2) a crucial passage of his reasoning, which he presents very briefly and intuitively, and which relies on the conviction that a certain procedure, iterated sufficiently many times, will produce a certain result because of the way it eventually exhausts a finite set of possibilities (see footnote 7).[1]

[1] As Peter Schroeder-Heister notes ([4], at p. 261), here Gentzen, following Hertz [3], uses 'the fixed point construction which is now standard in the theory of logic programming'. The interest of Schroeder-Heister's paper is twofold. First, he provides a detailed comparison of the work of Gentzen with that of his predecessor Hertz, whom Schroeder-Heister credits with the invention of proof-trees. Secondly, Schroeder-Heister is concerned to reveal how the theory of SLD resolution (part of the modern theory of logic programming), when understood proof-theoretically, is actually a part of structural proof theory.

Our aim here is orthogonal to that of Schroeder-Heister. We seek to provide more detail about the logical structure of Gentzen's completeness proof than Schroeder-Heister was

For the average reader, this particular work of Gentzen is little known. Another service we try to render is to make Gentzen's definitions more perspicuous, by *parametrizing* them in a judicious way. Gentzen demanded a lot of his reader, by introducing arbitrary ciphers—single letters in unusual fonts—as cryptic abbreviations of concepts that contained a considerable amount of logical structure and involved more than one important parameter embedded within them. So we have tried to fashion slightly more expansive but still easily manipulable abbreviations that will obviate the need, as the exposition proceeds, to keep consulting earlier definitions of cryptic symbols.

2 Exposition of Gentzen's completeness results

We shall use here notational conventions preferred by the present author, which are more current in modern proof theory.[2]

Definition 1 *Gentzen's* sequents *are of the form*

$$\Delta : \psi,$$

where Δ *is a* non-empty*, finite set of 'elements'* (Elemente) *of the same kind as* ψ*.* Δ *is called the* antecedent *and* ψ *is called the* succedent *of the sequent.*

One can give the general form of a sequent as

$$\varphi_1, \ldots, \varphi_n : \psi,$$

on the understanding that the ordering of the elements on the left (i.e., before the colon) is of no consequence. Today, of course, we would think of the 'elements' as *sentences*, most probably of some formal language. As such, they could have internal logico-grammatical structure. Significant primitive expressions imparting that structure—such as logical connectives—would

able to provide, in his summary thereof, within the constraints of his wider comparative projects. The extra logical detail that we supply enables us to carry out the generalizations desired, to the case of sequents empty on the left or right, and the case of infinite premise-sets of sequents.

Perhaps by adapting an old metaphor we can make the contrast between [4] and the present study a little clearer. In [4], it is shown that modern logic-programming reinvents a certain proof-theoretical wheel. In the present study, we are concerned to supply some extra spokes for the original wheel, in order to attain a better understanding of how it worked.

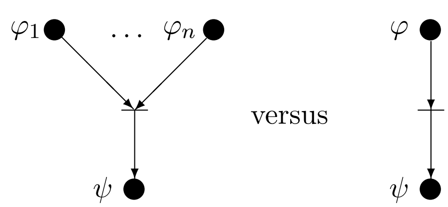
[2] Gentzen used '$\rightarrow$' where we use a colon, and preferred Latin variables to Greek ones.

have sequent-rules governing them specifically. But in his paper [2], Gentzen was not at all concerned with sentential structure. His ‘elements’ were indeed elemental. He did not inquire after their internal structure.

Nor did Gentzen seek to read a sequent $\varphi_1, \ldots, \varphi_n : \psi$ as making only a claim of logical consequence. That would be but one permissible reading—‘When the statements $\varphi_1, \ldots, \varphi_n$ are correct, so too is the statement ψ’. He gave examples of other possible readings for his sequents, such as ‘Any domain of elements that contains $\varphi_1, \ldots, \varphi_n$ also contains ψ’. He also treated the relation of logical consequence as holding among *sequents*, not among sentences, as premises and conclusions.

Definition 2 *A sequent $\varphi : \psi$ Gentzen called* linear*, and a sequent of the form $\psi : \psi$ he called* tautologous*. Any sequent $\Delta : \psi$ with ψ in Δ he called* trivial.

The reason why Gentzen calls a sequent of the form $\varphi : \psi$ linear is that if one thinks of making a downward inference from the premises of a sequent (in its antecedent) to its conclusion (i.e., its succedent), then with multiple premises $\varphi_1, \ldots, \varphi_n$ there would be *branching*, whereas with but a single premise φ there would not be:[3]



Gentzen considered two modes of inference involving sequents. The first, *thinning* (*Verdünnung*), allows one to put more elements on the left:

$$\text{THINNING} \qquad \frac{\Delta : \psi}{\Gamma, \Delta : \psi} \ .$$

(Read the comma as the sign for set union.) The second rule, which Gentzen called *cut* (*Schnitt*), allows one to avail oneself of the *transitivity* that is implicit in the two examples already given of how one might read a sequent:

[3]These will be the only diagrams we use in which tree-like arrays have their nodes labeled by *sentences*. In the rest of this discussion, we shall be considering only trees whose nodes are labeled by *sequents*.

$$\text{CUT} \qquad \frac{\Gamma : \varphi \qquad \Delta, \varphi : \psi}{\Gamma, \Delta : \psi} \quad .$$

(Assume that φ is not in Δ, and bear in mind that Δ could be empty.)

In both rules, the sequents above the line are called *premises*; the ones below the line are called *conclusions*. So: premises and conclusions are not single sentences; rather, they are sequents. In the case of CUT, $\Gamma : \varphi$ is called the *left* premise, and $\Delta, \varphi : \psi$ is called the *right* premise.

In applications of CUT, if both premises are linear, then so too is the conclusion:

$$\frac{\theta : \varphi \qquad \varphi : \psi}{\theta : \psi}$$

What Gentzen calls *proofs* (*Beweise*) may be built up *in linear fashion* using finitely many *starting sequents*, so as to reach an *end sequent*. Each individual *step* within a proof is an application of THINNING or of CUT. There are only finitely many steps in a proof. Here is how Gentzen defines proofs:

> Unter einem *Beweis* eines Satzes $\mathfrak{q}$ aus den Sätzen $\mathfrak{p}_1, \ldots, \mathfrak{p}_\nu$ ($\nu \geqq 0$) verstehen wir nunmehr *eine geordnete Anzahl von Schlüssen* (d.h. Verdünnungen und Schnitten[fn]), deren letzter $\mathfrak{q}$ als Konklusion besitzt, und in der jede Prämisse entweder zu den $\mathfrak{p}$ gehört oder tautologisch ist, oder mit einer vorangehenden Konklusion übereinstimmt. [Emphasis added—NT.]

Gentzen's phrase 'eine geordnete Anzahl von Schlüssen' is rather ambiguous in context. There are two interpretative possibilities.

1. The steps (Schlüssen) are to be represented in tree-like fashion. The sequent $\mathfrak{q}$ is at the root of the tree. Each upward furcation from a conclusion to one or to two premises corresponds to a distinct step (a thinning or a cut, respectively). Gentzen's definition of proof would imply that bifurcations could only ever take place when at least one premise-node is a *leaf* node of the proof-tree.

2. A proof is to be thought of as a Hilbert-like sequence of sequents (as opposed to sentences). That would impose the linearity that his phrase implies. But it would allow for one also to descry 'within the proof' bifurcations involving *two* complex subproofs. That is is to say, *both* premises of an application of CUT could stand as conclusions of complex subproofs.

Of these two possibilities, (2) is the less plausible, since Gentzen speaks of 'eine geordnete Anzahl von *Schlüssen*' [emphasis added—NT] rather than of 'eine geordnete Anzahl von *Sätze*'. Moreover, he only ever depicts a *Schluss* as a fragment of a tree, with each node labeled by a sequent,[4] as, for example, in the statements of THINNING and of CUT above.

Although Gentzen did not state his definition of proof in an inductive form, it is helpful to have it as an inductive definition. In giving the following definition, we are seeking to capture interpretation (1) above of what Gentzen intended.

Let us use $\mathfrak{p}$ and $\mathfrak{q}$ (as Gentzen did), with or without numerical subscripts, as sortal variables ranging over sequents. Let us also use $\mathfrak{P}$ and $\mathfrak{Q}$ for finite sets of sequents. First we define what we mean by a *tree of sequents.*

1. Any sequent $\mathfrak{p}$ counts as a tree of sequents.
2. If Π and Σ are finite trees of sequents, then so is

$$\frac{\Pi \quad \Sigma}{\mathfrak{p}}$$

   (This is the finite tree with root-node $\mathfrak{p}$ and immediate subtrees Π and Σ.)
3. (Closure) Every finite tree of sequents can be shown to be so by means of clauses (1) and (2).

We are now in a position to give our promised inductive definition of a *Gentzen-proof.* Clause (1) below is the basis clause, and clauses (2)–(4) are the inductive clauses. Clause (5) is the closure clause. Together these clauses define the ternary relation 'Π is a Gentzen-proof of the sequent $\mathfrak{q}$ from the (finite) set $\mathfrak{P}$ of sequents'.

Inductive definition of Gentzen-proof

1. Any non-tautologous sequent $\mathfrak{p}$ is a Gentzen-proof of $\mathfrak{p}$ from $\{\mathfrak{p}\}$; and any tautologous sequent $\mathfrak{p}$ is a Gentzen-proof of $\mathfrak{p}$ from $\emptyset$.

[4]If one is to be really careful in offering tree-representations of proofs, one might consider also treating each inference stroke as labelling a node lying above the conclusion-node, and below the relevant premise-nodes, of the inference in question.

2. If Π is a Gentzen-proof of the sequent $\Delta : \psi$ from the set $\mathfrak{P}$ of sequents, and Γ is a finite set of elements, then

   $$\frac{\Pi}{\Gamma, \Delta : \psi}$$

   is a Gentzen-proof of the sequent $\Gamma, \Delta : \psi$ from the set $\mathfrak{P}$ of sequents.
3. If Π is a Gentzen-proof of the sequent $\Gamma : \varphi$ from the set $\mathfrak{P}$ of sequents, and Δ is a finite set of elements other than φ, then

   $$\frac{\Pi \qquad \Delta, \varphi : \psi}{\Gamma, \Delta : \psi}$$

   is a Gentzen-proof of the sequent $\Gamma, \Delta : \psi$ from the set $\mathfrak{P} \cup \{\Delta, \varphi : \psi\}$ of sequents.
4. If Σ is a Gentzen-proof of the sequent $\Delta, \varphi : \psi$ ($\varphi \notin \Delta$) from the set $\mathfrak{Q}$ of sequents, and Γ is a finite set of elements, then

   $$\frac{\Gamma : \varphi \qquad \Sigma}{\Gamma, \Delta : \psi}$$

   is a Gentzen-proof of the sequent $\Gamma, \Delta : \psi$ from the set $\mathfrak{Q} \cup \{\Gamma : \varphi\}$ of sequents.
5. (Closure) Every Gentzen-proof can be shown to be so by means of clauses (1)–(4).

Note that whenever bifurcation is involved within a Gentzen-proof (i.e. whenever CUT is applied), at most one of the sub-trees is a *complex* tree, i.e. something other than a sequent. That is why the two clauses (3) and (4) are devoted to covering the possible forms that can be taken by applications of CUT in building up a Gentzen-proof.

Definition 3 *Gentzen called* normal *proofs of the forms*

$$\frac{\mathfrak{s}_0}{\mathfrak{q}}\ , \qquad \frac{\dfrac{\mathfrak{r}_0 \quad \mathfrak{s}_0}{\mathfrak{s}_1}}{\mathfrak{q}}\ , \qquad \frac{\dfrac{\mathfrak{r}_1 \qquad \dfrac{\mathfrak{r}_0 \quad \mathfrak{s}_0}{\mathfrak{s}_1}}{\mathfrak{s}_2}}{\mathfrak{q}}\ , \ \ldots, \qquad \frac{\dfrac{\mathfrak{r}_{\rho-1} \qquad \begin{array}{c} \dfrac{\mathfrak{r}_1 \qquad \dfrac{\mathfrak{r}_0 \quad \mathfrak{s}_0}{\mathfrak{s}_1}}{} \\ \vdots \\ \mathfrak{s}_{\rho-1} \end{array}}{\mathfrak{s}_\rho}}{\mathfrak{q}}\ , \ \ldots$$

Here the terminal single-premise steps are applications of THINNING, and the earlier double-premise steps are applications of CUT. The cut-element is always in the succedent of the $\mathfrak{r}$-sequent (hence in the antecedent of the corresponding $\mathfrak{s}$-sequent). Note that the $\mathfrak{s}$-sequents will all have the same succedent as $\mathfrak{q}$.

Observation 1 *For any trivial sequent $\mathfrak{s}$, there is a normal, one-step proof of $\mathfrak{s}$ from $\emptyset$.*

Proof. Suppose $\mathfrak{s}$ is $\Delta : \varphi$. Since $\mathfrak{s}$ is trivial, we have $\varphi \in \Delta$. The proof

$$\frac{\varphi : \varphi}{\Delta : \varphi}$$

begins with the tautologous sequent $\varphi : \varphi$, has one step of THINNING, and proves $\Delta : \varphi$ from $\emptyset$ in accordance with clauses (1) and (2) of the inductive definition of Gentzen-proof. Moreover, the proof is normal, since its form is that of the first in the series of forms listed in Definition 3.

Definition 4 *We go one further than Gentzen and call* super-normal *such proofs as are of the forms above except in so far as they do not contain the indicated terminal step of* THINNING.

Definition 5 *We say $\mathfrak{q}$ is* normal-deducible *from $\mathfrak{P} = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ (abbreviated: $\mathfrak{P} \vdash_N \mathfrak{q}$) if and only if there is a normal proof of $\mathfrak{q}$ from (some subset of) $\mathfrak{P}$.*

Definition 6 *We say $\mathfrak{q}$ is* super-normal-deducible *from $\mathfrak{P} = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ (abbreviated: $\mathfrak{P} \vdash_S \mathfrak{q}$) if and only if there is a super-normal proof of $\mathfrak{q}$ from (some subset of) $\mathfrak{P}$.*

Lemma 1 *Suppose $\Delta \subseteq \Gamma$ and $\mathfrak{P} \vdash_S \Delta : \psi$. Then $\mathfrak{P} \vdash_N \Gamma : \psi$.*

Proof. There is a super-normal proof of $\Delta : \psi$ from $\mathfrak{P}$. Call it Π. Then the normal proof

$$\begin{array}{c} \mathfrak{P} \\ \Pi \\ \dfrac{\Delta : \psi}{\Gamma : \psi} \end{array}$$

establishes that $\mathfrak{P} \vdash_N \Gamma : \psi$.

Corollary 1 *Suppose $\exists\Delta(\Delta \subseteq \Gamma \ \wedge \ \mathfrak{P} \vdash_S \Gamma : \psi)$. Then $\mathfrak{P} \vdash_N \Gamma : \psi$.*

Proof. Immediate from Lemma 1 by existential elimination.

Definition 7 *We define* $(\psi, \mathfrak{P})$-sequents *to be sequents* $\mathfrak{s}$ *such that*

> ψ *is the succedent of* $\mathfrak{s}$ *and there is a super-normal proof of* $\mathfrak{s}$ *from* $\mathfrak{P}$.

Definition 8 Γ *is* $(\psi, \mathfrak{P})$-weak *if and only if* $\neg\exists\Delta(\Delta \subseteq \Gamma \wedge \mathfrak{P} \vdash_S \Gamma : \psi)$— *that is, if and only if* no $(\psi, \mathfrak{P})$*-sequent has its antecedent included in* Γ.

Definition 9 *A (finite) set of elements is said to* undermine *a sequent* $\Delta : \varphi$ *just in case it contains all of* Δ *but does not contain* φ.

Definition 10 *A (finite) set of elements is said to* confirm *a sequent* $\Delta : \varphi$ *just in case it does not undermine* $\Delta : \varphi$.[5]

Observation 2 *A (finite) set of elements confirms a sequent* $\Delta : \varphi$ *if and only if it either lacks some member of* Δ *or contains* φ.

Observation 3 *Suppose* $\Gamma \subseteq \Theta$ *and* Θ *confirms* $\Gamma : \psi$. *Then* $\psi \in \Theta$.

Observation 4 *Suppose* Γ *is* $(\psi, \mathfrak{P})$*-weak. Then* Γ *does not undermine any* $(\psi, \mathfrak{P})$*-sequent. Hence,* Γ *confirms each* $(\psi, \mathfrak{P})$*-sequent.*

Definition 11 *We say that* $\mathfrak{q}$ *is a* consequence of $\mathfrak{P} = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ *(abbreviated:* $\mathfrak{P} \models \mathfrak{q}$*) just in case any set of elements involved in the sequents* $\mathfrak{p}_1, \ldots, \mathfrak{p}_n, \mathfrak{q}$ *that confirms* $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ *confirms* $\mathfrak{q}$*. Equivalently: ...just in case no set of elements involved in the sequents* $\mathfrak{p}_1, \ldots, \mathfrak{p}_n, \mathfrak{q}$ *confirms* $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ *but undermines* $\mathfrak{q}$.

For the rest of this section, consider the $\mathfrak{p}_i$ ($= \Xi_i : \varphi_i$, say) given in some fixed order. Collectively, we shall call them $\vec{\mathfrak{P}}$.

Definition 12 *The finite sequence*

$$\Gamma = \Gamma_1 \subseteq \Gamma_2 \subseteq \ldots \subseteq \Gamma_{n+1} = \Gamma^{\vec{\mathfrak{P}}}$$

[5] These are our terms of art, not Gentzen's. Gentzen used 'genügen' for 'confirm', and used its negation for 'undermine'.

of sets of elements, whose last member $\Gamma^{\vec{\mathfrak{P}}}$ *is called the* $\vec{\mathfrak{P}}$-completion of Γ, *is defined inductively as follows.*[6] *Let* Γ_i *be in hand. We define* Γ_{i+1} *by the dichotomous cases* (α) *and* (β)*:*

> (α) *For some* k Γ_i *undermines* $\mathfrak{p}_k$. *Let* j *be the least such* k.
>
> *Set* $\Gamma_{i+1} = \Gamma_i \cup \{\varphi_j\}$.
>
> (β) *For no* k *does* Γ_i *undermine* $\mathfrak{p}_k$.
>
> *Set* $\Gamma_{i+1} = \Gamma_i$.

Observation 5 $\Gamma \subseteq \Gamma^{\vec{\mathfrak{P}}}$.

Gentzen proves the following theorems.

Theorem 1 (Soundness) *If there is a Gentzen-proof of* $\mathfrak{q}$ *from* $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$, *then* $\mathfrak{q}$ *is a consequence of* $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$

Theorem 2 (Completeness) *If a non-trivial sequent* $\mathfrak{q}$ *is a consequence of* $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$, *then there is a normal proof of* $\mathfrak{q}$ *from* $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$.

Corollary 2 (Normalizability) *If there is a Gentzen-proof of a non-trivial sequent* $\mathfrak{q}$ *from* $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$, *then there is a normal proof of* $\mathfrak{q}$ *from* $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$.

Lemma 2 *For each* $i \geq 1$, *there are at least* i *distinct sequents* $\mathfrak{p}_{r_1}, \ldots, \mathfrak{p}_{r_i}$ *among the* $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ *such that* Γ_{i+1} *confirms each of* $\mathfrak{p}_{r_1}, \ldots, \mathfrak{p}_{r_i}$*; and, if* Γ_{i+1} *undermines at least one of* $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$, *then any set extending* Γ_{i+1} *by adding succedents of the sequents* $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ *confirms each of* $\mathfrak{p}_{r_1}, \ldots, \mathfrak{p}_{r_i}$.

Proof. By induction.
Basis ($i = 1$). We need to show that

> there is at least one sequent $\mathfrak{p}_{r_1}$ among the $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ such that Γ_2 confirms $\mathfrak{p}_{r_1}$; and, if Γ_2 undermines at least one of $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$, then any set extending Γ_2 by adding succedents of the sequents $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ confirms $\mathfrak{p}_{r_1}$.

[6] Here we supply a little more detail than Gentzen did, so as to enable a proper proof by mathematical induction of the desired properties of the set $\Gamma^{\vec{\mathfrak{P}}}$ being constructed. Our method has the advantage of being more readily generalizable to the infinite case.

For the reader interested in comparing our proof with that of Gentzen: he wrote u for our φ, v for our ψ, L for our Γ, M_i for our Γ_i, and N for our $\Gamma^{\vec{\mathfrak{P}}}$.

Now we reason by the dichotomous cases (α) and (β) for $i = 1$.

(α) For some k Γ_1 undermines $\mathfrak{p}_k$ ($= \Xi_k : \varphi_k$, say). Let m be the least such k. So $\Xi_m \subseteq \Gamma_1$ but $\varphi_m \notin \Gamma_1$. Also by definition $\Gamma_2 = \Gamma_1 \cup \{\varphi_m\}$. Hence Γ_2 (and any set extending Γ_2) confirms $\Xi_m : \varphi_m$. So Γ_2 confirms at least one of the sequents $\mathfrak{p}_1, \dots, \mathfrak{p}_n$; and, if Γ_2 undermines at least one of $\mathfrak{p}_1, \dots, \mathfrak{p}_n$, then any set extending Γ_2 by adding succedents of the sequents $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ confirms $\Xi_m : \varphi_m$.

(β) For no k does Γ_1 undermine $\mathfrak{p}_k$. So Γ_1 confirms $\Xi_j : \varphi_j$, $1 \leq j \leq n$. Also by definition $\Gamma_2 = \Gamma_1$. Hence Γ_2 confirms $\Xi_j : \varphi_j$, $1 \leq j \leq n$. So Γ_2 confirms at least one of the sequents $\mathfrak{p}_1, \dots, \mathfrak{p}_n$; and, if Γ_2 undermines at least one of $\mathfrak{p}_1, \dots, \mathfrak{p}_n$, then any set extending Γ_2 by adding succedents of the sequents $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ confirms that sequent. (The last conjunct is vacuously true, since its antecedent, *ex hypothesi*, is false.)

Inductive Hypothesis. There are at least i sequents $\mathfrak{p}_{r_1}, \dots, \mathfrak{p}_{r_i}$ among the $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ such that Γ_{i+1} confirms each of $\mathfrak{p}_{r_1}, \dots, \mathfrak{p}_{r_i}$; and, if Γ_{i+1} undermines at least one of $\mathfrak{p}_1, \dots, \mathfrak{p}_n$, then any set extending Γ_{i+1} by adding succedents of the sequents $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ confirms each of $\mathfrak{p}_{r_1}, \dots, \mathfrak{p}_{r_i}$.

Inductive Step. We need to show that

> There are at least $i+1$ sequents $\mathfrak{p}_{r_1}, \dots, \mathfrak{p}_{r_{i+1}}$ among the $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ such that Γ_{i+2} confirms each of $\mathfrak{p}_{r_1}, \dots, \mathfrak{p}_{r_{i+1}}$; and, if Γ_{i+2} undermines at least one of $\mathfrak{p}_1, \dots, \mathfrak{p}_n$, then any set extending Γ_{i+2} by adding succedents of the sequents $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ confirms each of $\mathfrak{p}_{r_1}, \dots, \mathfrak{p}_{r_{i+1}}$.

Once again we reason by the dichotomous cases (α) and (β).

(α) For some k Γ_{i+1} undermines $\mathfrak{p}_k$ ($= \Xi_k : \varphi_k$, say). Let m be the least k such that Γ_{i+1} undermines $\mathfrak{p}_k$. So $\Xi_m \subseteq \Gamma_{i+1}$ but $\varphi_m \notin \Gamma_{i+1}$. Also by definition $\Gamma_{i+2} = \Gamma_{i+1} \cup \{\varphi_m\}$. So Γ_{i+2} confirms $\Xi_m : \varphi_m$.

Let $\mathfrak{p}_{r_1}, \dots, \mathfrak{p}_{r_i}$ be as in IH. By IH, Γ_{i+2} confirms $\mathfrak{p}_{r_1}, \dots, \mathfrak{p}_{r_i}$. But $\Xi_m : \varphi_m$ cannot be among these sequents. So Γ_{i+2} confirms the sequents $\mathfrak{p}_{r_1}, \dots, \mathfrak{p}_{r_i}$, as well as the sequent $\Xi_m : \varphi_m$, which we can now take for $\mathfrak{p}_{r_{i+1}}$. Moreover, if Γ_{i+2} undermines at least one of $\mathfrak{p}_1, \dots, \mathfrak{p}_n$, then by IH any set extending Γ_{i+2} (hence extending Γ_{i+1}) by adding succedents of the sequents $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ confirms all of $\mathfrak{p}_{r_1}, \dots, \mathfrak{p}_{r_i}$; and also confirms $\mathfrak{p}_{r_{i+1}}$, because the succedent of this sequent is in Γ_{i+2}.

(β) For no k does Γ_{i+1} undermine $\mathfrak{p}_k$. So Γ_{i+1} confirms every one of the sequents $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$. By definition $\Gamma_{i+2} = \Gamma_{i+1}$. Hence Γ_{i+2} confirms every one of the sequents $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$. The result follows.

Corollary 3 *$\Gamma^{\vec{\mathfrak{P}}}$, which is Γ_{n+1}, confirms $\mathfrak{p}_j$, $1 \leq j \leq n$.*[7]

Lemma 3 *Suppose Γ is $(\psi, \mathfrak{P})$-weak. Then each Γ_k confirms every $(\psi, \mathfrak{P})$-sequent.*

Proof. By induction.
Basis step. By Observation 4, Γ $(= \Gamma_1)$ confirms every $(\psi, \mathfrak{P})$-sequent.

Inductive Hypothesis. Suppose that Γ_i confirms every $(\psi, \mathfrak{P})$-sequent.

Inductive Step. Show that Γ_{i+1} confirms every $(\psi, \mathfrak{P})$-sequent.
If $\Gamma_{i+1} = \Gamma_i$, then by IH we are done.
If $\Gamma_{i+1} \neq \Gamma_i$, we argue as follows.
Suppose that in the construction of the Γ-sequence,

$\mathfrak{p}^i$ $(= \Xi^i : \varphi^i$, say) is the first sequent among
the sequents $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ that is undermined by Γ_i

—so that

$$\Xi^i \subseteq \Gamma_i \tag{1}$$

and

$$\varphi^i \notin \Gamma_i, \tag{2}$$

—and in accordance with the definition we set

$$\Gamma_{i+1} = \Gamma_i \cup \{\varphi^i\}. \tag{3}$$

Now suppose for *reductio* that Γ_{i+1} undermines some $(\psi, \mathfrak{P})$-sequent $\mathfrak{s}$, say. By the definition of $(\psi, \mathfrak{P})$-sequent, $\mathfrak{s}$ is of the form $\Delta : \psi$.

There are now two cases to consider: either

[7] We claim here some improvement on Gentzen's terse claim (p. 336)

> Offenbar tritt nach endlich vielen Hinzufügungen der Fall ein, daß der letzte Komplex [$\Gamma^{\vec{\mathfrak{P}}}$] allen $\mathfrak{p}$ genügt. Denn die $\mathfrak{p}$ haben nur endlich viele Elemente, und der Komplex aller dieser Elemente genügt sicher jedem $\mathfrak{p}$.

Gentzen's reasoning here is actually fallacious. This is because at each stage of construction of the Γ sets one can add only the *succedent* of some $\mathfrak{p}_i$. So there is no guarantee that one would ever use up *all* of the elements in the sequents $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$. Gentzen should have limited his remark by saying that the set of all *succedents* of $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ confirms all of $\mathfrak{p}_1, \ldots \mathfrak{p}_n$. He could have done this by replacing the first occurrence of 'Elemente' in the last quote with the plural of 'Sukzedens'.

(i) $\varphi^i \in \Delta$, or

(ii) $\varphi^i \notin \Delta$.

Ad (i): $\varphi^i \in \Delta$. Let Δ' be $\Delta \setminus \varphi^i$. So $\mathfrak{s}$ is of the form $\Delta', \varphi^i : \psi$. Moreover, since $\mathfrak{s}$ is a $(\psi, \mathfrak{P})$-sequent, it has a super-normal proof, Π say, from (finitely many) members of $\mathfrak{P}$:

$$\begin{array}{c} \mathfrak{P} \\ \Pi \\ \Delta', \varphi^i : \psi \ (= \mathfrak{s}) \end{array}$$

We are supposing for *reductio* that Γ_{i+1} undermines $\Delta', \varphi^i : \psi$. It follows from this supposition that $(\Delta', \varphi^i) \subseteq \Gamma_{i+1}$. By (3), $\Gamma_{i+1} = \Gamma^i \cup \{\varphi^i\}$, where by (2) $\varphi^i \notin \Gamma_i$. So we can conclude that

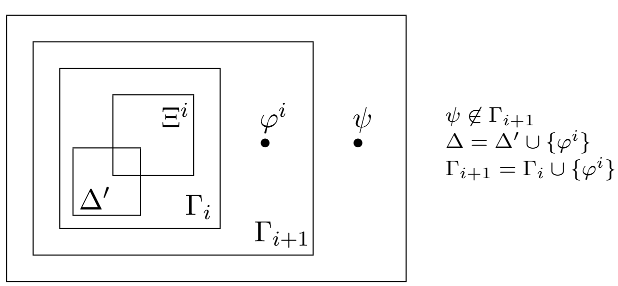
$$\Delta' \subseteq \Gamma_i. \tag{4}$$

Consider now the super-normal proof

$$\Sigma \quad = \quad \dfrac{\Xi^i : \varphi^i \ (= \mathfrak{p}^i) \qquad \begin{array}{c} \mathfrak{P} \\ \Pi \\ \Delta', \varphi^i : \psi \ (= \mathfrak{s}) \end{array}}{\Xi^i, \Delta' : \psi}$$

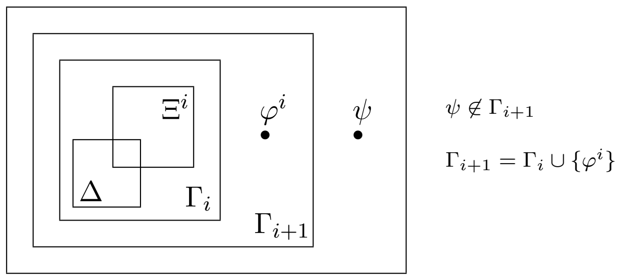
Since $\mathfrak{p}^i \in \mathfrak{P}$, Σ is a super-normal proof of the sequent $\Xi^i, \Delta' : \psi$ from $\mathfrak{P}$. So $\Xi^i, \Delta' : \psi$ is a $(\psi, \mathfrak{P})$-sequent. By (1) we have $\Xi^i \subseteq \Gamma_i$; by (4) we have $\Delta' \subseteq \Gamma_i$; and we are assuming for this *reductio* that Γ_{i+1} undermines $\Delta', \varphi^i : \psi$, whence $\psi \notin \Gamma_{i+1}$, whence in turn $\psi \notin \Gamma_i$. Hence Γ_i undermines the $(\psi, \mathfrak{P})$-sequent $\Xi^i, \Delta' : \psi$. This contradicts IH.

The picture in case (i) is this:



Ad (ii): $\varphi^i \notin \Delta$. We are supposing that Γ_{i+1} undermines $\Delta : \psi$. It follows from this supposition that $\Delta \subseteq \Gamma_{i+1}$ and $\psi \notin \Gamma_{i+1}$. By (3), $\Gamma_{i+1} = \Gamma_i \cup \{\varphi^i\}$, where $\varphi^i \notin \Gamma_i$. So $\Delta \subseteq \Gamma_i \cup \{\varphi^i\}$. But we are supposing that $\varphi^i \notin \Delta$. Thus $\Delta \subseteq \Gamma_i$. By (1) we have $\Xi^i \subseteq \Gamma_i$. So Γ_i undermines the $(\psi, \mathfrak{P})$-sequent $\Xi^i, \Delta : \psi$. Once again this contradicts IH.

The picture in case (ii) is this:



We have now reduced to absurdity the assumption that Γ_{i+1} undermines some $(\psi, \mathfrak{P})$-sequent. We conclude that Γ_{i+1} confirms every $(\psi, \mathfrak{P})$-sequent.

Lemma 4 *Suppose* Γ *is* $(\psi, \mathfrak{P})$*-weak. Then each* Γ_k *does not contain* ψ*.*

Proof. By induction.
Basis step. By initial supposition $\mathfrak{q}$ is non-trivial, that is, Γ does not contain ψ. So Γ_1 does not contain ψ.

Inductive Hypothesis. Suppose that $\psi \notin \Gamma_i$.

Inductive Step. Show that $\psi \notin \Gamma_{i+1}$.
If $\Gamma_{i+1} = \Gamma_i$, then by IH we are done.
If $\Gamma_{i+1} \neq \Gamma_i$, we argue as follows.
Suppose that in the construction of the Γ-sequence,

$$\mathfrak{p}^i \ (= \Xi^i : \varphi^i, \text{ say}) \text{ is the first sequent among}$$
$$\text{the sequents } \mathfrak{p}_1, \ldots, \mathfrak{p}_n \text{ that is undermined by } \Gamma_i$$

—so that

$$\Xi^i \subseteq \Gamma_i \tag{5}$$

—and in accordance with the definition we set

$$\Gamma_{i+1} = \Gamma_i \cup \{\varphi^i\}. \tag{6}$$

By IH, $\psi \notin \Gamma_i$. Hence, given (5), we have that

$$\Gamma_i \text{ undermines } \Xi^i : \psi. \tag{7}$$

Now suppose for *reductio* that $\psi \in \Gamma_{i+1}$. By (6) either $\psi \in \Gamma_i$ or $\varphi^i = \psi$. By IH, $\psi \notin \Gamma_i$. So $\varphi^i = \psi$. Substituting φ^i for ψ in (7), we infer that

$$\Gamma_i \text{ undermines } \Xi^i : \varphi^i.$$

But $\Xi^i : \varphi^i$ is $\mathfrak{p}^i$, which, since $\varphi^i = \psi$, is a $(\psi, \mathfrak{P})$-sequent. So Γ_i undermines a $(\psi, \mathfrak{P})$-sequent.

This contradicts Lemma 3. So $\psi \notin \Gamma_{i+1}$.

Corollary 4 *Suppose* Γ *is* $(\psi, \mathfrak{P})$*-weak. Then* $\psi \notin \Gamma^{\vec{\mathfrak{P}}}$.

Proof. Immediate by Lemma 4, since $\Gamma^{\vec{\mathfrak{P}}}$ is Γ_{n+1}.

Proof of Theorem 2.

Suppose that $\mathfrak{P} \models \Gamma : \psi$. By Corollary 3, $\Gamma^{\vec{\mathfrak{P}}}$ confirms $\mathfrak{P}$. Hence $\Gamma^{\vec{\mathfrak{P}}}$ confirms $\Gamma : \psi$. By Observation 5, $\Gamma \subseteq \Gamma^{\vec{\mathfrak{P}}}$. Hence $\psi \in \Gamma^{\vec{\mathfrak{P}}}$.

Suppose for *reductio* that Γ is $(\psi, \mathfrak{P})$-weak, i.e. $\neg\exists\Delta(\Delta \subseteq \Gamma \wedge \mathfrak{P} \vdash_S \Delta : \psi)$. By Corollary 4, $\psi \notin \Gamma^{\vec{\mathfrak{P}}}$. Contradiction.

So, by *classical reductio*, $\exists\Delta(\Delta \subseteq \Gamma \wedge \mathfrak{P} \vdash_S \Delta : \psi)$. By Corollary 1, $\mathfrak{P} \vdash_N \Gamma : \psi$. *QED*

This argument has the formal structure

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\text{O5:}\\ \Gamma \subseteq \Gamma^{\vec{\mathfrak{P}}}\end{array} \quad \dfrac{\begin{array}{c}C3:\\ \Gamma^{\vec{\mathfrak{P}}} \text{ confirms } \mathfrak{P}\end{array} \quad \mathfrak{P} \models \Gamma : \psi}{\Gamma^{\vec{\mathfrak{P}}} \text{ confirms } \Gamma : \psi}}{\psi \in \Gamma^{\vec{\mathfrak{P}}}}\text{(O3)} \qquad \dfrac{\dfrac{}{\neg\exists\Delta(\Delta \subseteq \Gamma \wedge \mathfrak{P} \vdash_S \Delta : \psi)}\text{(1)}}{\psi \notin \Gamma^{\vec{\mathfrak{P}}}}\text{(C4)}}{\dfrac{\perp}{\exists\Delta(\Delta \subseteq \Gamma \wedge \mathfrak{P} \vdash_S \Delta : \psi)}\text{(1)}}}{\mathfrak{P} \vdash_N \Gamma : \psi}\text{(C1)}$$

3 Generalizing: sequents empty on the left or right

Observation 1 and Theorem 2 together yield the fuller completeness result

Theorem 3 *If a sequent* $\mathfrak{q}$ *is a consequence of* $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$*, then there is a normal proof of* $\mathfrak{q}$ *from* $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$*.*

In this statement of completeness, $\mathfrak{q}$ is not required to be non-trivial.

The real interest, however, lies in the case where $\mathfrak{q}$ *is* non-trivial, since (by Observation 1) trivial $\mathfrak{q}$ has such trivial proof! Note that even in the statement of Theorem 2, with its restriction to non-trivial $\mathfrak{q}$, there is no mention of any corresponding requirement on the premise-sequents $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$. This invites further reflection. Did Gentzen miss some opportunity here? Could he have stated and proved a sharper, more informative result?

In Gentzen's normal proofs, the rule of THINNING is applied only once, if at all, and then only at the terminal step. He *does* leave open, however, the possibility that among the premise-sequents $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ could be some trivial ones. (His formal definition of proof rules out mention only of *tautologous* sequents, but not trivial ones. And this is true also of normal proofs.)

Yet a trivial premise-sequent really makes no contribution at all to the result supposedly being proved. A trivial sequent is confirmed by *every* set of elements, and undermined by *none.* So, if $\mathfrak{Q}$ is a set of non-trivial sequents, $\mathfrak{P}$ a set of trivial sequents, and $\mathfrak{q}$ a non-trivial sequent, we have

$$\mathfrak{q} \text{ is a consequence of } \mathfrak{P} \cup \mathfrak{Q} \text{ if and only if } \mathfrak{q} \text{ is a consequence of } \mathfrak{Q}.$$

Clearly, in light of this, the theoretical focus should be only on consequences involving sets $\mathfrak{Q}$ of *non-trivial* sequents.

Further reflection on the proof-theoretical side backs this up. There really is no point in ever beginning a proof with an application of a rule of inference to a trivial sequent. This is borne out by the following lemmas. Their combined effect is to guarantee that even without increasing the number of steps in a proof, one never needs a non-tautologous but trivial sequent at a leaf-node of any Gentzen-proof tree.

Lemma 5 *Any thinning of a trivial premise can be replaced by a thinning of a tautologous premise:*

$$(\textit{where } \varphi \in \Delta\ :) \qquad \frac{\Delta : \varphi}{\Delta, \Gamma : \varphi} \quad \mapsto \quad \frac{\varphi : \varphi}{\Delta, \Gamma : \varphi}$$

Lemma 6 *Any cut with trivial left premise can be replaced by a thinning of its right premise:*

$$(\textit{where } \varphi \in \Delta\ :) \qquad \frac{\Delta : \varphi \quad \Gamma, \varphi : \psi}{\Delta, \Gamma : \psi} \quad \mapsto \quad \frac{\Gamma, \varphi : \psi}{\Delta, \Gamma : \psi}$$

Lemma 7 *Any cut whose right premise is trivial and has the cut element as succedent can be replaced by a thinning of its left premise:*

$$\frac{\Delta : \psi \quad \Gamma, \psi : \psi}{\Delta, \Gamma : \psi} \quad \mapsto \quad \frac{\Delta : \psi}{\Delta, \Gamma : \psi}$$

Lemma 8 *Any cut whose right premise is trivial and does not have the cut element as succedent has a trivial conclusion, which could just as well have been derived from a tautologous sequent by a thinning:*

$$(\textit{where } \psi \in \Gamma\ :) \quad \frac{\Delta : \varphi \quad \Gamma, \varphi : \psi}{\Delta, \Gamma : \psi} \quad \mapsto \quad \frac{\Gamma, \varphi : \psi}{\Delta, \Gamma : \psi} \quad \mapsto \quad \frac{\psi : \psi}{\Delta, \Gamma : \psi}$$

The upshot is clear: we can limit our attention to Gentzen-proofs of non-trivial sequents from non-trivial sequents (just as we could for the semantic relation of consequence among sequents).

Theorem 2 of course still holds under this limitation to non-trivial premise-sequences. Now, what is most remarkable about Theorem 2 is the very constrained form of the normal proofs that it affords for consequences among sequents. As noted earlier, in a normal proof the rule of THINNING is applied only once, if at all, and then only at the terminal step.

Hence the *penultimate* sequent of a normal proof is the strongest statement (of consequence) that one can glean from the proof. And it can be obtained (perhaps not surprisingly, given that THINNING can only weaken a claim of consequence) by means of cuts alone, making up what we called a *super-normal* proof.

CUT is the crucible in which optimally strong statements of consequence among (non-trivial) sequents can be forged. In the limited context of this investigation by Gentzen, however, the *elements* of these (non-trivial) sequents behave like propositional variables. They are assumed to be able to take their semantic values quite independently of each other. The underlying thought appears to be that logical relations among elements of sequents (such as, say, contariety or mutual inconsistency) would need to be considered only when logical operators (such as negation) are introduced. At that point, sequents would no longer consist of unstructured *elements* on the right and on the left, but would consist, rather, of *sentences* in the new formal language that provides not only for atomic sentences but also for logically structured ones.

This postponement to complex languages of possible consideration of logical relations is, however, theoretically short-sighted. We need to examine what might happen with the structural relations of consequence and deducibility among sequents if we allow for the possibility that logical relations are entered into, and logical properties enjoyed, by even the *unstructured elements* that we have been considering thus far. Naturally this calls for a conception of the elements as corresponding more to propositional constants than to propositional variables. They will be behaving more like *propositional variables under an interpretation.* But—and this is the crucial feature—such behavior will not be the result of such logical form as would be bestowed by logical operators occurring within them. For, *ex hypothesi*, they will be operator-free.

We are arguing here, in effect, for consideration of what modern proof-theorists call *atomic rules of inference.* Take, for example, the two atomic sentences 'this is red' (ρ) and 'this is colored' (γ). The meaning-connection between them is that the former entails the latter. This can be registered by the sequent

$$\rho : \gamma.$$

One might think *'Well and good; so, let's just allow this sequent $\rho : \gamma$ always to be available for use as a (non-trivial) premise-sequent within Gentzen-proofs. What's the problem?'*

The problem becomes apparent only when we consider two atomic sentences such as 'this is red' (ρ) and 'this is blue' (β). These are *contraries.* They cannot be true together. Modern proof-theorists have at their disposal the following sequent-expression of contrariety:

$$\rho, \beta : \emptyset,$$

or, more simply,

$$\rho, \beta : \quad .$$

This is a sequent with *empty succedent.* We may find it convenient to write

$$\rho, \beta : \perp,$$

in order to have a symbol that will both emphasize the fact that the succedent is empty, and remind one of the semantical significance of that fact.

Gentzen, however, in his 1932 study, *did not allow for sequents with empty succedent*, i.e. for explicit statements to the effect that the antecedent in question is unsatisfiable (or inconsistent).[8]

The reader will see also that Gentzen *did not allow for sequents with empty antecedent*.[9] Thus he would have been unable to express the fact that certain atomic sentences are *logically true*, such as '0=0'. The sequent that expresses this is

$$: 0 = 0.$$

One might legitimately wish to express the polar opposite, by being able to say that the atomic sentence '0=1' is *logically false*. The sequent that expresses this is

$$0 = 1 : \quad .$$

The temptation is clear, and irresistible: the 'elemental sequent-theorist' should be able to provide a treatment that allows for these expressive possibilities involving 'logically unstructured' sentences, even without considering any logical operators such as connectives and quantifiers.

Suppose then that we modify Definition 1 as follows.

Definition 13 Extended sequents *are of the form* $\Delta : \psi$ *or* $\Delta : \perp$, *where* Δ *is a (possibly empty) finite set of elements and* ψ *is an element.*

[8]See Definition 1 above, which is faithful to Gentzen's text at p. 330:

> Ein *Satz* hat die Form
>
> $$u_1 u_2 \ldots u_\nu \to v \qquad (\nu \geqq 1).$$
>
> Die u und v heißen *Elemente*.
>
> (A *sequent* has the form
>
> $$u_1 u_2 \ldots u_\nu \to v \qquad (\nu \geqq 1).$$
>
> The u and v are called *elements*.)

[9]The closest that Gentzen could have come to expressing that λ is logically true would be to have available as a premise each and every non-trivial sequent of the form $\Delta : \lambda$. If, however, there are only finitely many sets Δ, this falls short of saying that λ is logically true. Indeed, even if there are *infinitely* many sets Δ, this would still fall short of saying that λ is logically true. For the language could be extended by new elements not yet involved in any of these (infinitely many) sets Δ.

If we now had to consider *extended* sequents instead of Gentzen's original sequents (i.e., those of just the first of the three permitted forms above), what would happen to Gentzen's main results proved above?

First we would have to inquire after the forms that might now be taken by the rules of THINNING and CUT. The temptation would be strong to have THINNING cover inferences of the following form:

$$\frac{\Delta :}{\Delta : \psi}$$

Likewise, one would be tempted to allow CUT to apply when the right premise has empty succedent so as to yield a conclusion with empty succedent:[10]

$$\frac{\Gamma : \varphi \qquad \Delta, \varphi :}{\Gamma, \Delta :}$$

We now seek to go beyond Gentzen's results in his 1932 paper. Fortunately we can get by with minor adaptations of our exposition of his completeness proof above, by yielding to these two temptations.

Theorem 4 *If a non-trivial extended sequent $\mathfrak{q}$ is a consequence of non-trivial extended sequents $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$, then there is a normal proof of $\mathfrak{q}$ from $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$.*

Definition 14 *We liberalize the definition of* super-normal *proofs so as to allow them to have a terminal step of* THINNING on the right*.*

We liberalize also the notion of $(\sigma, \mathfrak{P})$-sequent, so that it is understood by reference to the preceding more liberal notion of super-normal proof.

Definition 15 *We define* $(\sigma, \mathfrak{P})$-sequents *(where σ is either $\perp$ or ψ) to be sequents $\mathfrak{s}$ such that*

> *σ is the succedent of $\mathfrak{s}$ and there is a super-normal proof of $\mathfrak{s}$ from sequents in the set $\mathfrak{P}$.*

Our proposed generalization to extended sequents brings with it a need to revisit the notion of a set of elements *undermining* or *confirming* a sequent. This is because we can now have empty antecedents or succedents. (Remember that we use $\perp$ to indicate an empty succedent.)

[10] As Peter Schroeder-Heister has pointed out (personal correspondence), this cut rule with empty succedent is the SLD resolution rule used in logic programming (in the propositional case, i.e. without substitution).

Our earlier definition (on Gentzen's behalf) of the notion of *undermining* (Definition 9), though speaking blandly of membership of an arbitrary set (the set that does the undermining), really turned on the idea that undermining is a matter of *making all the elements on the left* **true**, while *making (all) the element(s) on the right* **false**. If we had thought of the succedent as a *set*, it would have been a singleton (since, as we have pointed out, Gentzen did not allow for empty succedents). So, even at that early stage, one could have symmetrized the expression of the definition of undermining, so that it could have read

> Θ undermines $\Delta : \sigma$ if and only if Θ contains every element of Δ and Θ lacks every element of σ.

(Clearly, the motivating idea here is that Θ is the set of truths on some interpretation.) With extended sequents, we can now have empty succedents σ. This symmetrized definition of undermining will now serve our purposes perfectly.

Definition 16 *Suppose $\Delta : \sigma$ is an extended sequent. Then Θ undermines $\Delta : \sigma$ if and only if $\Delta \subseteq \Theta$ and $\Theta \cap \sigma = \emptyset$.*

Observation 6 *A (finite) set Θ of elements undermines an extended sequent $\mathfrak{s}$ if and only if:*

1. *$\mathfrak{s}$ is of the form $\Delta : \varphi$, and $\Delta \subseteq \Theta$ but $\varphi \notin \Theta$; or*
2. *$\mathfrak{s}$ is of the form $\Delta : \bot$, and $\Delta \subseteq \Theta$.*

Definition 17 *A (finite) set of elements is said to* confirm *a sequent $\Delta : \varphi$ just in case it does not undermine $\Delta : \varphi$.*

Observation 7 *Suppose $\Gamma \subseteq \Theta$ and Θ confirms $\Gamma : \psi$. Then $\psi \in \Theta$.*

Three new possibilities arise for extended sequents, which did not obtain for Gentzen's sequents.

First new possibility. Given certain set $\mathfrak{P}$ of extended sequents as premises, it can be shown that a set Θ of elements is incoherent, in the sense that there is a (normal) proof, from $\mathfrak{P}$, of the extended sequent $\Theta : \bot$. By way of illustration, take for $\mathfrak{P}$ the set consisting of just the extended sequent $\theta : \bot$, and for the proof take just that sequent on its own! This example is of course very degenerate, but serves our purposes. A less degenerate example would be the following.

Example. Consider the extended sequents

$$a : b \ ; \ a : c \ ; \ b, c : \bot.$$

The (normal) proof

$$\frac{a : c \qquad \dfrac{a : b \quad b, c : \bot}{a, c : \bot}}{a : \bot}$$

shows that $\{a\}$ is inconsistent.

Second new possibility. Definition 13 of extended sequents allows the *empty* sequent $\emptyset : \emptyset$ (or $\emptyset : \bot$) to count as an extended sequent. The empty sequent is undermined by every set of elements, hence confirmed by none. Some collections of extended sequents allow one to construct a (normal) proof of the empty sequent. The simplest example of this would be the premise-collection

$$\emptyset : a \ ; \ a : \bot \,,$$

used in the proof

$$\frac{\emptyset : a \quad a : \bot}{\emptyset : \bot} \ .$$

Such a proof shows that the premise-collection (of *sequents*) is *incoherent.* It is impossible for any set of elements to confirm both its members.

In the restricted context of Gentzen's sequents, which could not be empty on the left or on the right, no premise-set of sequents allows one to deduce the empty sequent as a conclusion. For every sequent at a leaf-node in a Gentzen proof-tree (even if tautologous) has at least one element on the left, and at least one element on the right. And each rule (THINNING or CUT) preserves that property from its premise(s) to its conclusion. So, by induction, the conclusion of any Gentzen sequent-proof has at least one element on the left, and at least one element on the right.

Not so, however, for sequent-proofs involving extended sequents, as is seen from our last example.

It is important that one's sequent calculus be able to reveal the incoherence of a set of premise-sequents that indeed cannot have all its members confirmed by any set of elements. This in effect becomes a new requirement of completeness on a sequent calculus.

Our first new possibility above concerned the 'incoherence' of certain sets Θ of elements being demonstrable on the basis of a set $\mathfrak{P}$ of extended sequents. The demonstration consists in a deduction of the sequent $\Theta : \perp$ from $\mathfrak{P}$.

Third new possibility. The third new possibility that arises for extended sequents concerns the other logical extreme. It is now possible to demonstrate, on the basis of a set $\mathfrak{P}$ of extended sequents, that a certain element θ 'must be true'. Such a demonstration consists in a deduction of the sequent $\emptyset : \theta$ from $\mathfrak{P}$. By way of illustration, take for $\mathfrak{P}$ the set consisting of just the extended sequent $\emptyset : \theta$, and for the proof take just that sequent on its own! This example is of course very degenerate, but serves our purposes. A less degenerate example would be the following.

Example. $\mathfrak{P}$ is the set consisting of just the extended sequents

$$\emptyset : a \quad ; \quad \emptyset : b \quad ; \quad a, b : \theta$$

and the (normal) proof showing that θ 'must be true' (given the foregoing sequents) is

$$\dfrac{\emptyset : b \qquad \dfrac{\emptyset : a \qquad a, b : \theta}{b : \theta}}{\emptyset : \theta}$$

Observation 8 *Suppose* Γ *is* $(\psi, \mathfrak{P})$*-weak. Then* Γ *does not undermine any* $(\psi, \mathfrak{P})$*-sequent. Hence,* Γ *confirms each* $(\psi, \mathfrak{P})$*-sequent.*

Definition 18 *We say that* $\mathfrak{q}$ *is a* consequence of $\mathfrak{P} = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ *(abbreviated:* $\mathfrak{P} \models \mathfrak{q}$*), where all the sequents involved are* extended *sequents, just in case any set of elements involved in the sequents* $\mathfrak{p}_1, \ldots, \mathfrak{p}_n, \mathfrak{q}$ *that confirms* $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ *confirms* $\mathfrak{q}$*. Equivalently: ...just in case no set of elements involved in the sequents* $\mathfrak{p}_1, \ldots, \mathfrak{p}_n, \mathfrak{q}$ *confirms* $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ *but undermines* $\mathfrak{q}$*.*

We are considering the $\mathfrak{p}_i$ ($= \Xi_i : \sigma_i$, say) given in some fixed order. We are calling the collection $\vec{\mathfrak{P}}$, in order to emphasize the underlying ordering. In each case the succedent σ_i is either $\perp$ or some element φ_i.

Definition 19 *The finite sequence*

$$\Gamma = \Gamma_1 \subseteq \Gamma_2 \subseteq \ldots \subseteq \Gamma_{n+1} = \Gamma^{\vec{\mathfrak{P}}}$$

of sets of elements, whose last member $\Gamma^{\vec{\mathfrak{P}}}$ *is called the* $\vec{\mathfrak{P}}$-completion of Γ, *is defined inductively. Let* Γ_i *be in hand.* Γ_{i+1} *is then constructed in the trichotomous cases* (α.1), (α.2) *and* (β) *as follows.*

(α) *For some* k Γ_i *undermines* $\mathfrak{p}_k$. *Let* j *be the least such* k.

(1) $\mathfrak{p}_j$ *is of the form* $\Xi_j : \varphi_j$. *Set* $\Gamma_{i+1} = \Gamma_i \cup \{\varphi_j\}$.
(2) $\mathfrak{p}_j$ *is of the form* $\Xi_j : \bot$. *Set* $\Gamma_{i+1} = \Gamma_i$.

(β) *For no* k *does* Γ_i *undermine* $\mathfrak{p}_k$. *Set* $\Gamma_{i+1} = \Gamma_i$.

Observation 9 $\Gamma \subseteq \Gamma^{\vec{\mathfrak{P}}}$.

Lemma 9 *(for extended sequents). Suppose* Γ *is* $(\psi, \mathfrak{P})$*-weak. Then each* Γ_k *confirms every* $(\psi, \mathfrak{P})$*-sequent.*

Proof. Exactly as for Lemma 3, except that in the case where $\Gamma_{i+1} \neq \Gamma_i$, the premise-sequent $\mathfrak{p}^i$ ($= \Xi^i : \varphi^i$, say) is chosen to be the first sequent among the sequents $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ *that does not have succedent* $\bot$ and is undermined by Γ_i. From that point on, the proof goes through unchanged.

Lemma 10 *(for extended sequents). Suppose* Γ *is* $(\bot, \mathfrak{P})$*-weak. Then each* Γ_k *confirms every* $(\bot, \mathfrak{P})$*-sequent.*

Proof. By induction.

Basis step. By Observation 8, Γ ($= \Gamma_1$) confirms every $(\bot, \mathfrak{P})$-sequent.

Inductive Hypothesis. Suppose that Γ_i confirms every $(\bot, \mathfrak{P})$-sequent.

Inductive Step. Show that Γ_{i+1} confirms every $(\bot, \mathfrak{P})$-sequent.
If $\Gamma_{i+1} = \Gamma_i$, then by IH we are done.
If $\Gamma_{i+1} \neq \Gamma_i$, we argue as follows.
Suppose that in the construction of the Γ-sequence,

> $\mathfrak{p}^i$ ($= \Xi^i : \varphi^i$, say) is the first sequent among the sequents $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ that does not have succedent $\bot$ and is undermined by Γ_i

—so that
$$\Xi^i \subseteq \Gamma_i \tag{8}$$
and
$$\varphi^i \notin \Gamma_i, \tag{9}$$
—and in accordance with the definition we set
$$\Gamma_{i+1} = \Gamma_i \cup \{\varphi^i\}. \tag{10}$$

Now suppose for *reductio* that Γ_{i+1} undermines some $(\perp, \mathfrak{P})$-sequent $\mathfrak{s}$, say. There are now two cases to consider:

(i) $\varphi^i \in \Delta$; or

(ii) $\varphi^i \notin \Delta$.

Ad (i): $\varphi^i \in \Delta$. Let Δ' be $\Delta \setminus \varphi^i$. Since $\mathfrak{s}$ is a $(\perp, \mathfrak{P})$-sequent, it has a super-normal proof, Π say, from (finitely many) members of $\mathfrak{P}$:

$$\begin{array}{c} \mathfrak{P} \\ \Pi \\ \Delta', \varphi^i : \perp\ (= \mathfrak{s}) \end{array}$$

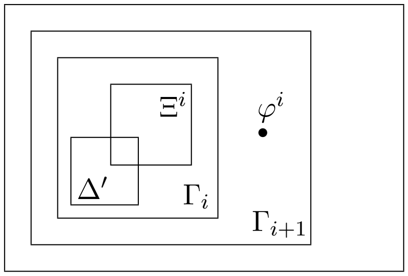
We are supposing for *reductio* that Γ_{i+1} undermines $\Delta', \varphi^i : \perp$. It follows from this supposition that $(\Delta', \varphi^i) \subseteq \Gamma_{i+1}$. By (10), $\Gamma_{i+1} = \Gamma^i \cup \{\varphi^i\}$, where by (9) $\varphi^i \notin \Gamma_i$. So we can conclude that
$$\Delta' \subseteq \Gamma_i. \tag{11}$$

Consider now the super-normal proof

$$\Sigma \quad = \quad \dfrac{\Xi^i : \varphi^i\ (= \mathfrak{p}^i) \qquad \begin{array}{c} \mathfrak{P} \\ \Pi \\ \Delta', \varphi^i : \perp\ (= \mathfrak{s}) \end{array}}{\Xi^i, \Delta' : \perp}$$

Since $\mathfrak{p}^i \in \mathfrak{P}$, Σ is a super-normal proof of the sequent $\Xi^i, \Delta' : \perp$ from $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$. So $\Xi^i, \Delta' : \perp$ is a $(\perp, \mathfrak{P})$-sequent. By (8) we have $\Xi^i \subseteq \Gamma_i$; by (11) we have $\Delta' \subseteq \Gamma_i$. Hence Γ_i undermines the $(\perp, \mathfrak{P})$-sequent $\Xi^i, \Delta' : \perp$. This contradicts IH.

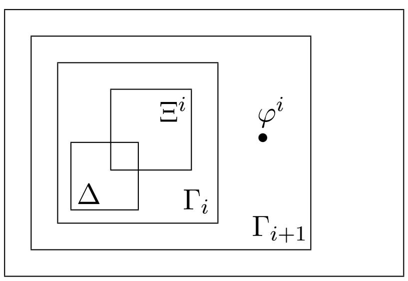
The picture in case (i) is this:



$\Delta = \Delta' \cup \{\varphi^i\}$
$\Gamma_{i+1} = \Gamma_i \cup \{\varphi^i\}$

Ad (ii): $\varphi^i \notin \Delta$. We are supposing that Γ_{i+1} undermines $\Delta : \bot$. It follows from this supposition that $\Delta \subseteq \Gamma_{i+1}$. By (10), $\Gamma_{i+1} = \Gamma_i \cup \{\varphi^i\}$, where $\varphi^i \notin \Gamma_i$. So $\Delta \subseteq \Gamma_i \cup \{\varphi^i\}$. But we are supposing that $\varphi^i \notin \Delta$. Thus $\Delta \subseteq \Gamma_i$. By (8) we have $\Xi^i \subseteq \Gamma_i$. So Γ_i undermines the $(\bot, \mathfrak{P})$-sequent $\Xi^i, \Delta : \bot$. Once again this contradicts IH.

The picture in case (ii) is this:



$\Gamma_{i+1} = \Gamma_i \cup \{\varphi^i\}$

We have now reduced to absurdity the assumption that Γ_{i+1} undermines some $(\bot, \mathfrak{P})$-sequent. We conclude that Γ_{i+1} confirms every $(\bot, \mathfrak{P})$-sequent.

Lemma 11 *Suppose* Γ *is* $(\sigma, \mathfrak{P})$*-weak. Then each* Γ_k *does not contain* ψ*, if* $\sigma = \psi$.

Proof. By induction.
Basis step. Suppose $\sigma = \psi$. By initial supposition $\mathfrak{q}$ is non-trivial, that is, Γ does not contain ψ. So Γ_1 does not contain ψ, if $\sigma = \psi$.

Inductive Hypothesis. Suppose that $\psi \notin \Gamma_i$, if $\sigma = \psi$.

Inductive Step. Show that $\psi \notin \Gamma_{i+1}$, if $\sigma = \psi$.
If $\Gamma_{i+1} = \Gamma_i$, then by IH we are done.

If $\Gamma_{i+1} \neq \Gamma_i$, we argue as follows.

Suppose that in the construction of the Γ-sequence,

> $\mathfrak{p}^i$ ($= \Xi^i : \varphi^i$, say) is the first sequent among the sequents $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ that does not have succedent $\bot$ and is undermined by Γ_i

—so that

$$\Xi^i \subseteq \Gamma_i \tag{12}$$

—and in accordance with the definition we set

$$\Gamma_{i+1} = \Gamma_i \cup \{\varphi^i\}. \tag{13}$$

By IH, $\psi \notin \Gamma_i$, if $\sigma = \psi$. Hence, given (12), we have that

$$\Gamma_i \text{ undermines } \Xi^i : \sigma. \tag{14}$$

Suppose $\sigma = \psi$. Now suppose for *reductio* that $\psi \in \Gamma_{i+1}$. By (13) either $\psi \in \Gamma_i$ or $\varphi^i = \psi$. By IH, $\psi \notin \Gamma_i$. So $\varphi^i = \psi$. Hence $\varphi^i = \sigma$. Substituting φ^i for σ in (14), we infer that

$$\Gamma_i \text{ undermines } \Xi^i : \varphi^i.$$

But $\Xi^i : \varphi^i$ is $\mathfrak{p}^i$, which, since $\varphi^i = \psi$, is a $(\psi, \mathfrak{P})$-sequent. So Γ_i undermines a $(\psi, \mathfrak{P})$-sequent.

This contradicts Lemma 9. So $\psi \notin \Gamma_{i+1}$.

All this was on the supposition that $\sigma = \psi$. So $\psi \notin \Gamma_{i+1}$ if $\sigma = \psi$.

Lemma 12 *Suppose Γ is $(\sigma, \mathfrak{P})$-weak. Then for each $i \geq 1$, there are at least i distinct sequents $\mathfrak{p}_{r_1}, \ldots, \mathfrak{p}_{r_i}$ among the $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ such that Γ_{i+1} confirms each of $\mathfrak{p}_{r_1}, \ldots, \mathfrak{p}_{r_i}$; and, if Γ_{i+1} undermines at least one of $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$, then any set extending Γ_{i+1} by adding succedents of the sequents $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ confirms each of $\mathfrak{p}_{r_1}, \ldots, \mathfrak{p}_{r_i}$.*

Basis ($i = 1$). We need to show that

> there is at least one sequent $\mathfrak{p}_{r_1}$ among the $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ such that Γ_2 confirms $\mathfrak{p}_{r_1}$; and, if Γ_2 undermines at least one of $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$, then any set extending Γ_2 by adding succedents of the sequents $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ confirms $\mathfrak{p}_{r_1}$.

Now we reason by the trichotomous cases (α.1), (α.2) and (β) for $i = 1$.

(α.1) For some k Γ_1 undermines $\mathfrak{p}_k$; j is the least such k; and $\mathfrak{p}_j$ is of the form $\Xi_j : \varphi_j$.

> By definition $\Gamma_2 = \Gamma_1 \cup \{\varphi_j\}$. Hence Γ_2 (and any set extending Γ_2) confirms $\Xi_j : \varphi_j$. So Γ_2 confirms at least one of the sequents $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$; and, if Γ_2 undermines at least one of $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$, then any set extending Γ_2 by adding succedents of the sequents $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ confirms $\Xi_j : \varphi_j$.

(α.2) For some k Γ_1 undermines $\mathfrak{p}_k$; j is the least such k; and $\mathfrak{p}_j$ is of the form $\Xi_j : \perp$. Remember that $\Gamma_1 = \Gamma$. So Γ undermines $\mathfrak{p}_j$ $(= \Xi_j : \perp)$. Hence $\Xi_j \subseteq \Gamma$. Moreover $\Xi_j : \sigma$ is a $(\sigma, \mathfrak{P})$-sequent. This is immediate if $\sigma = \perp$. But if $\sigma = \psi$, the proof

$$\frac{\Xi_j : \perp}{\Xi_j : \psi}\ ,$$

which is super-normal (by Definition 14), shows that $\Xi_j : \sigma$ is a $(\sigma, \mathfrak{P})$-sequent. This contradicts the main supposition that Γ is $(\sigma, \mathfrak{P})$-weak—i.e., that *no* $(\sigma, \mathfrak{P})$-sequent has its antecedent included in Γ. So this case is impossible.

(β) For no k does Γ_1 undermine $\mathfrak{p}_k$. So Γ_1 confirms $\mathfrak{p}_k$ for every k. Also by definition $\Gamma_2 = \Gamma_1$. Hence Γ_2 confirms $\mathfrak{p}_k$ for every k. So Γ_2 confirms at least one of the sequents $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$; and, if Γ_2 undermines at least one of $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$, then any set extending Γ_2 by adding succedents of the sequents $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ confirms that sequent. (The last conjunct is vacuously true, since its antecedent, *ex hypothesi*, is false.)

Inductive Hypothesis. There are at least i sequents $\mathfrak{p}_{r_1}, \ldots, \mathfrak{p}_{r_i}$ among the $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ such that Γ_{i+1} confirms each of $\mathfrak{p}_{r_1}, \ldots, \mathfrak{p}_{r_i}$; and, if Γ_{i+1} undermines at least one of $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$, then any set extending Γ_{i+1} by adding succedents of the sequents $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ confirms each of $\mathfrak{p}_{r_1}, \ldots, \mathfrak{p}_{r_i}$.

Inductive Step. We need to show that

> there are at least $i+1$ sequents $\mathfrak{p}_{r_1}, \ldots, \mathfrak{p}_{r_{i+1}}$ among the $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ such that Γ_{i+2} confirms each of $\mathfrak{p}_{r_1}, \ldots, \mathfrak{p}_{r_{i+1}}$; and, if Γ_{i+2} undermines at least one of $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$, then any set extending Γ_{i+2} by adding succedents of the sequents $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ confirms each of $\mathfrak{p}_{r_1}, \ldots, \mathfrak{p}_{r_{i+1}}$.

We reason again by the trichotomous cases (α.1), (α.2) and (β).

(α.1) For some k Γ_{i+1} undermines $\mathfrak{p}_k$; j is the least such k; and $\mathfrak{p}_j$ is of the form $\Xi_j : \varphi_j$.

By definition $\Gamma_{i+2} = \Gamma_{i+1} \cup \{\varphi_j\}$. So Γ_{i+2} confirms $\Xi_j : \varphi_j$. Let $\mathfrak{p}_{r_1}, \ldots, \mathfrak{p}_{r_i}$ be as in IH. By IH, Γ_{i+2} confirms the sequents $\mathfrak{p}_{r_1}, \ldots, \mathfrak{p}_{r_i}$ that Γ_{i+1} confirms. But $\Xi_j : \varphi_j$ cannot be among these. So Γ_{i+2} confirms the sequents $\mathfrak{p}_{r_1}, \ldots, \mathfrak{p}_{r_i}$, as well as the sequent $\Xi_j : \varphi_j$, which we can now take for $\mathfrak{p}_{r_{i+1}}$. Moreover, if Γ_{i+2} undermines at least one of $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$, then by IH any set extending Γ_{i+2} (hence extending Γ_{i+1}) by adding succedents of the sequents $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ confirms all of $\mathfrak{p}_{r_1}, \ldots, \mathfrak{p}_{r_i}$; and also confirms $\mathfrak{p}_{r_{i+1}}$, because the succedent of this sequent is in Γ_{i+2}.

(α.2) For some k, Γ_{i+1} undermines $\mathfrak{p}_k$; j is the least such k ; and $\mathfrak{p}_j$ is of the form $\Xi_j : \bot$.

Suppose that σ is $\bot$, so that our main supposition is to the effect that Γ is $(\bot, \mathfrak{P})$-weak. Then $\mathfrak{p}_j$ $(= \Xi_j : \bot)$ is a $(\bot, \mathfrak{P})$-sequent that is undermined by Γ_{i+1}. This contradicts Lemma 10.

Now suppose that σ is ψ, so that our main supposition is to the effect that Γ is $(\psi, \mathfrak{P})$-weak. Then the super-normal proof

$$\frac{\Xi_j : \bot}{\Xi_j : \psi}$$

shows that $\mathfrak{p}_j$ $(= \Xi_j : \psi)$ is a $(\psi, \mathfrak{P})$-sequent. Moreover, by Lemma 11, Γ_{i+1} does not contain ψ. So Γ_{i+1} undermines $\Xi_j : \psi$. This contradicts Lemma 9. So this case is impossible.

(β) For no k does Γ_{i+1} undermine $\mathfrak{p}_k$.

In this case Γ_{i+1} confirms every one of the sequents $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$. Also by definition $\Gamma_{i+2} = \Gamma_{i+1}$. Hence Γ_{i+2} confirms every one of the sequents $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$. The result follows.

Comment. Note that we have appealed to Lemma 9 and Lemma 10 in proving Lemma 12. This is unlike the situation with Gentzen's original results. When the sequents are not allowed to be empty on the left or the right, the proof of Lemma 2 (the analogue of Lemma 12) did *not* appeal to Lemma 3 (which has split into the two analogues Lemma 9 and Lemma 10 for extended sequents).

Corollary 5 *Suppose* Γ *is* $(\sigma, \mathfrak{P})$*-weak. Then* $\Gamma^{\vec{\mathfrak{P}}}$ *confirms* $\mathfrak{p}_j$*,* $1 \leq j \leq n$*.*

Proof. Immediate by Lemma 12, since $\Gamma^{\vec{\mathfrak{P}}}$ is Γ_{n+1}.

Corollary 6 *Suppose* Γ *is* $(\psi, \mathfrak{P})$*-weak. Then* $\psi \notin \Gamma^{\vec{\mathfrak{P}}}$*.*

Proof. Immediate by Lemma 11, since $\Gamma^{\vec{\mathfrak{P}}}$ is Γ_{n+1}.

Proof of Theorem 4.

Suppose that $\mathfrak{P} \models \Gamma : \psi$. By Corollary 5, $\Gamma^{\vec{\mathfrak{P}}}$ confirms $\mathfrak{P}$. Hence $\Gamma^{\vec{\mathfrak{P}}}$ confirms $\Gamma : \psi$. By Observation 9, $\Gamma \subseteq \Gamma^{\vec{\mathfrak{P}}}$. Hence $\psi \in \Gamma^{\vec{\mathfrak{P}}}$.

Suppose for *reductio* that Γ is $(\psi, \mathfrak{P})$-weak, i.e. $\neg\exists\Delta(\Delta \subseteq \Gamma \wedge \mathfrak{P} \vdash_S \Delta : \psi)$. By Corollary 11, $\psi \notin \Gamma^{\vec{\mathfrak{P}}}$. Contradiction.

So, by *classical reductio*, $\exists\Delta(\Delta \subseteq \Gamma \wedge \mathfrak{P} \vdash_S \Delta : \psi)$. By Corollary 1, $\mathfrak{P} \vdash_N \Gamma : \psi$. *QED*

This argument has the formal structure

$$
\dfrac{\dfrac{\text{O9: } \Gamma \subseteq \Gamma^{\vec{\mathfrak{P}}} \qquad \dfrac{\dfrac{\dfrac{}{\neg\exists\Delta(\Delta \subseteq \Gamma \wedge \mathfrak{P} \vdash_S \Delta : \psi)}{\scriptstyle(1)}}{\Gamma^{\vec{\mathfrak{P}}} \text{ confirms } \mathfrak{P}}{\scriptstyle(\mathrm{C5})} \qquad \mathfrak{P} \models \Gamma : \psi}{\Gamma^{\vec{\mathfrak{P}}} \text{ confirms } \Gamma : \psi}}{\psi \in \Gamma^{\vec{\mathfrak{P}}}}{\scriptstyle(\mathrm{O7})} \qquad \dfrac{\dfrac{}{\neg\exists\Delta(\Delta \subseteq \Gamma \wedge \mathfrak{P} \vdash_S \Delta : \psi)}{\scriptstyle(1)}}{\psi \notin \Gamma^{\vec{\mathfrak{P}}}}{\scriptstyle(\mathrm{C6})}}{\dfrac{\dfrac{\bot}{\exists\Delta(\Delta \subseteq \Gamma \wedge \mathfrak{P} \vdash_S \Delta : \psi)}{\scriptstyle(1)}}{\mathfrak{P} \vdash_N \Gamma : \psi}{\scriptstyle(\mathrm{C1})}}
$$

Note here how Corollary 5 has as its hypothesis the *reductio* assumption $\neg\exists\Delta(\Delta \subseteq \Gamma \wedge \mathfrak{P} \vdash_S \Delta : \psi)$. This has been occasioned by the need to accommodate extended sequents. In our regimentation of Gentzen's simpler completeness result in §2, the analogue of Corollary 5, namely Corollary 3, did not need $\neg\exists\Delta(\Delta \subseteq \Gamma \wedge \mathfrak{P} \vdash_S \Delta : \psi)$ as an hypothesis.

4 Generalizing: the infinite case

Thus far we have been considering only finite sets of premise-sequents. But the infinite case merits attention.[11] We shall consider infinitely many premise-sequents $\mathfrak{p}_i$ ($= \Xi_i : \sigma_i$, say) given in some fixed order. We shall call the collection $\vec{\mathfrak{P}}$, so as to emphasize its underlying ordering. In each $\mathfrak{p}_i$ the succedent σ_i is either $\perp$ or some element φ_i. Moreover, each antecedent Ξ_i is still finite. The conclusion-sequent $\mathfrak{q}$ likewise has a finite antecedent, which is called Γ.

Definition 20 *Let $\mathfrak{P}$ be any set of extended sequents, possibly infinite. We say that an extended sequent $\mathfrak{q}$ is a* consequence *of $\mathfrak{P}$ just in case any set of elements involved in sequents in $\mathfrak{P}$ or involved in $\mathfrak{q}$ that confirms every sequent in $\mathfrak{P}$ confirms $\mathfrak{q}$. Equivalently: ...just in case no such set of elements confirms every sequent in $\mathfrak{P}$ but undermines $\mathfrak{q}$.*

The construction of $\Gamma^{\vec{\mathfrak{P}}}$ requires an extra degree of delicacy in the infinite case, so as to ensure the desired result that $\Gamma^{\vec{\mathfrak{P}}}$ confirms every sequent in $\vec{\mathfrak{P}}$ (see Lemma 17 below). $\vec{\mathfrak{P}}$ is an infinite, ordered set $\{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n, \ldots\}$. We shall denote by $\mathfrak{P}_n$ its 'initial segment' $\{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n\}$. $\mathfrak{P}_0$ is accordingly $\emptyset$; whence $\Gamma^{\mathfrak{P}_0}$ is $\Gamma^{\emptyset}$. And $\Gamma^{\emptyset}$ is of course Γ itself.

Definition 21 (the $\vec{\mathfrak{P}}$-completion of Γ, where $\vec{\mathfrak{P}}$ is infinite) *We set*

$$\Gamma^0 = \Gamma^{\mathfrak{P}_0};$$
$$\Gamma^1 = (\Gamma^0)^{\mathfrak{P}_1};$$
$$\Gamma^2 = (\Gamma^1)^{\mathfrak{P}_2};$$
$$\vdots$$
$$\Gamma^{i+1} = (\Gamma^i)^{\mathfrak{P}_{i+1}};$$
$$\vdots$$

Finally, we set

$$\bigcup_i \Gamma^i = \Gamma^{\vec{\mathfrak{P}}}$$

[11] Schroeder-Heister remarks ([4], p. 247), that 'Structural rules may be looked upon as axiomatizing a consequence relation in Tarski's sense (for the *finite case*, of course).' [Emphasis added—NT.] Here we are concerned to cancel any implicature that this remark may carry to the effect that a consequence relation in Tarski's sense that is defined on *infinite* sets of premise-sequents could not be axiomatized in exactly the same way.

Note that in order to determine Γ^{n+1} from Γ^n, one takes Γ^n as one's 'initial set' for the *earlier* kind of completion-construction, and one carries out the earlier method of construction with respect to the finite ordered set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_{n+1}\}$, so as to obtain the $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_{n+1}\}$-completion of Γ^n, denoted more succinctly as $(\Gamma^n)^{\mathfrak{P}_{n+1}}$.

This means that the construction at each finite stage is as it was earlier. For given n, and for $1 \leq i \leq n$, let $(\Gamma^n)_i$ be in hand. $(\Gamma^n)_{i+1}$ is then constructed in the trichotomous cases (α.1), (α.2) and (β) as follows.

> (α) $(\Gamma^n)_i$ undermines at least one of $\mathfrak{p}_1, \ldots, \mathfrak{p}_{n+1}$. Let j be the index of the first one.
>
> > (1) $\mathfrak{p}_j$ is of the form $\Xi_j : \varphi_j$. Set $(\Gamma^n)_{i+1} = (\Gamma^n)_i \cup \{\varphi_j\}$.
> > (2) $\mathfrak{p}_j$ is of the form $\Xi_j : \perp$. Set $(\Gamma^n)_{i+1} = (\Gamma^n)_i$.
>
> (β) $(\Gamma^n)_i$ undermines none of $\mathfrak{p}_1, \ldots, \mathfrak{p}_{n+1}$. Set $(\Gamma^n)_{i+1} = (\Gamma^n)_i$.

Observation 10

$$
\begin{aligned}
&\Gamma = \\[1em]
&\Gamma^0 = \\[1em]
&(\Gamma^0)_1 \subseteq (\Gamma^0)_2 = (\Gamma^0)^{\mathfrak{P}_1} = \\[1em]
&\Gamma^1 = \\[1em]
&(\Gamma^1)_1 \subseteq (\Gamma^1)_2 \subseteq (\Gamma^1)_3 = (\Gamma^1)^{\mathfrak{P}_2} = \\[1em]
&\Gamma^2 \\
&\vdots \\
&\Gamma^n = \\[1em]
&(\Gamma^n)_1 \subseteq (\Gamma^n)_2 \subseteq (\Gamma^n)_3 \subseteq \ldots \subseteq (\Gamma^n)_{n+2} = (\Gamma^n)^{\mathfrak{P}_{n+1}} = \\[1em]
&\Gamma^{n+1} \\
&\vdots \\
&\subseteq \Gamma^{\vec{\mathfrak{P}}}.
\end{aligned}
$$

Each subset-chain in Observation 10 is produced in accordance with our earlier Definition 19, with respect to the initial segment $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_{n+1}\}$ of $\vec{\mathfrak{P}}$.

Lemma 13 *(for infinitely many extended sequents). Suppose Γ is $(\psi, \mathfrak{P})$-weak. Then each Γ^k confirms every $(\psi, \mathfrak{P})$-sequent.*

Proof. Exactly as for Lemma 9.

Corollary 7 *(for infinitely many extended sequents). Suppose Γ is $(\psi, \mathfrak{P})$-weak. Then $\Gamma^{\vec{\mathfrak{P}}}$ confirms every $(\psi, \mathfrak{P})$-sequent.*

Proof. Immediate by Lemma 13, since $\Gamma^{\vec{\mathfrak{P}}}$ is $\bigcup_i \Gamma^i$.

Lemma 14 *(for infinitely many extended sequents). Suppose Γ is $(\perp, \mathfrak{P})$-weak. Then each Γ^k confirms every $(\perp, \mathfrak{P})$-sequent.*

Proof. Exactly as for Lemma 10.

Lemma 15 *(for infinitely many extended sequents) Let σ be either $\perp$ or ψ. Suppose Γ is $(\sigma, \mathfrak{P})$-weak. Then $\Gamma^{\vec{\mathfrak{P}}}$ confirms every $(\sigma, \mathfrak{P})$-sequent.*

Proof. Let $\mathfrak{s}$ be any $(\sigma, \mathfrak{P})$-sequent. Suppose for *reductio* that $\Gamma^{\vec{\mathfrak{P}}}$ (which we defined to be $\bigcup_i \Gamma^i$) undermines $\mathfrak{s}$. Then for some k, Γ^k undermines $\mathfrak{s}$. This contradicts Lemma 14 if σ is $\perp$, and contradicts Lemma 13 if σ is ψ. Hence $\Gamma^{\vec{\mathfrak{P}}}$ confirms $\mathfrak{s}$. But $\mathfrak{s}$ was an arbitrary $(\sigma, \mathfrak{P})$-sequent. Hence $\Gamma^{\vec{\mathfrak{P}}}$ confirms every $(\sigma, \mathfrak{P})$-sequent.

Lemma 16 *Suppose Γ is $(\sigma, \mathfrak{P})$-weak. Then each Γ^k does not contain ψ, if $\sigma = \psi$.*

Proof. Exactly as for Lemma 11.

Corollary 8 *(for infinitely many extended sequents) Suppose Γ is $(\psi, \mathfrak{P})$-weak. Then $\psi \notin \Gamma^{\vec{\mathfrak{P}}}$.*

Proof. Suppose for *reductio* that ψ is in $\Gamma^{\vec{\mathfrak{P}}}$. $\Gamma^{\vec{\mathfrak{P}}}$ is $\bigcup_i \Gamma^i$. Thus for some i, Γ^i contains ψ. This contradicts Lemma 16. Thus $\psi \notin \Gamma^{\vec{\mathfrak{P}}}$.

Lemma 17 *Suppose Γ is $(\sigma, \mathfrak{P})$-weak. Then for each $n \geq 0$, and for $1 \leq i \leq (n+1)$, there are at least i distinct sequents $\mathfrak{p}_{r_1}, \ldots, \mathfrak{p}_{r_i}$ among $\mathfrak{p}_1, \ldots, \mathfrak{p}_{n+1}$ such that $(\Gamma^n)_{i+1}$ confirms each of $\mathfrak{p}_{r_1}, \ldots, \mathfrak{p}_{r_i}$; and, if $(\Gamma^n)_{i+1}$ undermines at least one of $\mathfrak{p}_1, \ldots, \mathfrak{p}_{n+1}$, then any set extending $(\Gamma^n)_{i+1}$ by adding succedents of the sequents $\mathfrak{p}_1, \ldots, \mathfrak{p}_{n+1}$ confirms each of $\mathfrak{p}_{r_1}, \ldots, \mathfrak{p}_{r_i}$.*

Proof. For fixed n, the induction on i is as in the proof of Lemma 12. The very first basis step, for $n = 0$ and $i = 1$, is obvious. By Corollary 5, we thereby obtain, for successive values of n, the result that $(\Gamma^n)^{\vec{\mathfrak{P}}_{n+1}}$ confirms $\mathfrak{p}_j$, $1 \leq j \leq n+1$.

Corollary 9 *Suppose* Γ *is* $(\sigma, \mathfrak{P})$*-weak. Then* $\Gamma^{\vec{\mathfrak{P}}}$ *confirms every member of* $\mathfrak{P}$.

Proof. Immediate by Lemma 17, since $\Gamma^{\vec{\mathfrak{P}}}$ is $\bigcup_i \Gamma^i$.

Theorem 5 *If a non-trivial extended sequent* $\mathfrak{q}$ *is a consequence of the set* $\mathfrak{P}$ *of non-trivial extended sequents, then there is a normal proof of* $\mathfrak{q}$ *from sequents in* $\mathfrak{P}$.

Proof of Theorem 5.

Suppose that $\mathfrak{P} \models \Gamma : \psi$. By Corollary 9, $\Gamma^{\vec{\mathfrak{P}}}$ confirms $\mathfrak{P}$. Hence $\Gamma^{\vec{\mathfrak{P}}}$ confirms $\Gamma : \psi$. By Observation 10, $\Gamma \subseteq \Gamma^{\vec{\mathfrak{P}}}$. Hence $\psi \in \Gamma^{\vec{\mathfrak{P}}}$.

Suppose for *reductio* that Γ is $(\psi, \mathfrak{P})$-weak, i.e. $\neg\exists\Delta(\Delta \subseteq \Gamma \wedge \mathfrak{P} \vdash_S \Delta : \psi)$. By Corollary 8, $\psi \notin \Gamma^{\vec{\mathfrak{P}}}$. Contradiction.

So, by *classical reductio*, $\exists\Delta(\Delta \subseteq \Gamma \wedge \mathfrak{P} \vdash_S \Delta : \psi)$. By Corollary 1, $\mathfrak{P} \vdash_N \Gamma : \psi$. *QED*

This argument has the formal structure

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{O9:}\quad \Gamma \subseteq \Gamma^{\vec{\mathfrak{P}}} \qquad \dfrac{\dfrac{\dfrac{}{\neg\exists\Delta(\Delta \subseteq \Gamma \wedge \mathfrak{P} \vdash_S \Delta : \psi)}(1)}{\Gamma^{\vec{\mathfrak{P}}} \text{ confirms } \mathfrak{P}}(\text{C9}) \qquad \mathfrak{P} \models \Gamma : \psi}{\Gamma^{\vec{\mathfrak{P}}} \text{ confirms } \Gamma : \psi}}{\psi \in \Gamma^{\vec{\mathfrak{P}}}}(\text{O7}) \qquad \dfrac{\dfrac{}{\neg\exists\Delta(\Delta \subseteq \Gamma \wedge \mathfrak{P} \vdash_S \Delta : \psi)}(1)}{\psi \notin \Gamma^{\vec{\mathfrak{P}}}}(\text{C8})}{\dfrac{\bot}{\exists\Delta(\Delta \subseteq \Gamma \wedge \mathfrak{P} \vdash_S \Delta : \psi)}(1)}}{\mathfrak{P} \vdash_N \Gamma : \psi}(\text{C1})$$

Corollary 10 *Logical consequence among non-trivial extended sequents is compact.*

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