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LANGUAGE GAMES AND INTUITIONISM*

I

In his paper 'The Philosophical Basis of Intuitionistic Logic'[1], Dummett argues towards a justification of intuitionistic logic from a broad Wittgensteinian thesis about meaning: that every aspect of meaning must be such that one's grasp thereof is capable of being manifested eventually and implicitly in observable behaviour. His argument is intricate; it would take us too far afield here even to sketch its general route.[2] I am concerned rather with charting a different passage from the broad thesis just stated to the claim that it is intuitionistic logic which correctly reflects graspable meanings of the logical operators.

This new route takes as its point of departure the game theoretic semantics for first order languages which has recently been made better known by Hintikka in exegetical application to what Wittgenstein said about language games.[3] I shall first try to improve upon Hintikka's explanation of the (classical) language game for first order interpreted languages; then examine his claim that these serve a Wittgensteinian purpose in capturing or conveying the forces of the logical operators; and finally describe a modified game which, given the broad meaning thesis above, serves this purpose better and confers upon the logical operators intuitionistic rather than classical meanings.

II

I shall call the whole, consisting of language and the actions into which it is woven, the 'language game'. (para. 7)

Our clear and simple language games are not preparatory studies for a future regularization of language – as it were first approximations, ignoring friction and air-resistance. The language games are rather set up as *objects of comparison* which are meant to throw light on the facts of our language by way not only of similarities but also of dis-similarities (para. 150) (Wittgenstein: *Philosophical Investigations*, Part I)

The game is between two players, each intent on winning. It is played with respect to a first order sentence ϕ against the background of an interpretation M of the non-logical expressions occurring in ϕ. There are two 'roles' in the game, each occupied by one of the players who may, however, have occasion to swap these roles in the course of play. Let us simply call these roles T and F.[4]

Before play begins, the players decide who shall be T at the outset and who F. Intuitively, he who starts as T may be thought of as associated with the claim that ϕ is true in M, and prepared to move accordingly in vindication of that claim; and he who starts as F may be thought of as out to show that ϕ is false in M. As play proceeds, the players will, in accordance with the rules of the game, make various choices intent on vindicating their claims: choices of conjuncts or disjuncts with which to proceed, and choices of individuals from M as   vindicating   instances   or   counterinstances   to quantifications. Play proceeds with respect to the subformula remaining after each such choice, until finally after finitely many moves an atomic formula is reached. By this stage various individuals will in the course of play have been assigned to the variables occurring in the atomic formula. If this assignment satisfies the atomic formula (in M) he who happens to occupy role T at that final stage wins; if not, he who happens to occupy role F wins.

The rules governing moves are as follow:

(i) At $\sim \psi$ roles have to be exchanged. Play proceeds with respect to ψ.

(ii) At $\psi \vee \theta$ and $\psi \& \theta$ T and F respectively choose the subformula ψ or θ with which play is to proceed.

(iii) At $(\exists x)\psi$, $(\forall x)\psi$ T and F respectively choose an individual in M and assign it to the variable x. Play proceeds with respect to ψ.

(iv) At an atomic formula play stops. If the assignment constructed satisfies this formula, he who happens to be T wins; if not, he who happens to be F wins.

After setting out this explanation,[5] Hintikka then observes that ϕ is true in M if and only if he who starts as T has a winning strategy; and ϕ is false in M if and only if he who starts as F has a winning strategy. (Contrast 'having a winning strategy' with 'happening to win a particular play'.)

It is in this language game, and especially in the activities of seeking and finding individuals at moves marked by quantifiers, that Hintikka claims to find a source of philosophical and semantical illumination of quantification and the other logical operations. Here we have language woven into certain actions: quantifier prefixes signal searches for individuals, explorations of the world; their meanings reside in these missions.

III

I wish to inquire first into the novelty and independence of this semantical account from the classical one given by Tarski. In the course of so doing, I shall provide an equivalent reformulation of the language game described by Hintikka, by attending more carefully to clues in Wittgenstein concerning the constitution of language games.

Hintikka's description of the game seems to make it something played *on* bits of language rather than *with* them or *by using* them. In the *Brown Book* Wittgenstein said

... We are not ... regarding the language games which we describe as incomplete parts of a language, but ... as complete systems of human communication. To keep this point of view in mind, it very often is useful to imagine such a simple language to be the entire system of communication of a tribe in a primitive state of society.

What *system of communication* does Hintikka's game provide? To communicate that ϕ by *playing the game* I cannot simply utter ϕ and rely on my audience's grasp of its meaning via conversancy with the rules of the game; for that is not to play the game at all. Rather, I must demonstrate that if I start as T in any play of the game with respect to ϕ I shall win – that is, I must demonstrate that I have a winning strategy. It may take many plays to do so convincingly. This makes the game different from Wittgenstein's language games, in which communication is effected by each move in the course of play. (The builder calls "Slab!"; his assistant shakes his head.) In fact, there is evidence for a view stressed by Dummett (*Frege*, ch. 1) that a

sentence corresponds to a *move in a language game*. At para. 22 of the *Investigations* Wittgenstein says almost exactly this:

But 'that such-and-such is the case' is *not* a sentence in our language – so far it is not a *move* in the language game.

I think that considerable clarification of the game theoretic semantics given above can be achieved by attending more closely to the correlation between *moves* and *sentences*: or, better, between *moves* and *complete assertions*. The way I propose to effect this correlation is to construe the players of the game as associated at every state of play with an assertion and its denial. A play of the game can then be construed as a protracted dispute between the players;[6] losing in several plays beginning with the same sentence may convince a player that his opponent was correct in asserting the original sentence. This way of looking at the game leads to its recursive reconstruction, as will emerge below. This recursive reconstruction is the equivalent reformulation promised above, and provides a re-appraisal of the distinctiveness of the game theoretical approach to classical logical semantics.

As has already been remarked, in the course of play on the way to a final 'win-lose' stage involving an atomic formula, various instantiations will be made. An instantiation may be understood as the fresh co-ordination of an individual with a variable freed by the removal of a quantifier. Thus an assignment of values to variables is a set of instantiations. Now instead of thinking of just one assertion being contested at the outset of a play, let us mark the following: *at each state of play he who happens to be (in role) T is associated with the assertion that the remaining subformula is satisfied by the assignment of values to variables which play has so far produced; while he who happens to be (in role) F is out to show that this is not the case.* This explains the active choices made by the players: at a disjunction he who happens to be T at that stage, associated as he is then with the assertion that the assignment thus far obtained does satisfy the disjunction, chooses what he thinks is a disjunct which that assignment satisfies. At a conjunction he who happens to be F at

that stage, associated as he is then with the denial of the assertion that the assignment thus far obtained satisfies the conjunction, chooses a conjunct which he thinks that assignment does not satisfy. At an existential quantification, he who happens to be T at that stage chooses an individual which he thinks is a vindicating instance of the quantification, thereby committing himself to the relational claim made by the unquantified sentence: so now the resulting assignment, enlarged or modified by the coordination of that individual to the variable bound by the existential quantifier concerned, is asserted by T to satisfy the unquantified sentence. *Mutatis mutandis* for universal quantification, player F and denial.

IV

At any state of play of the following is determined:

(a) which player happens then to occupy role T and which role F;

(b) which individuals have so far been assigned to the free variables of

(c) whatever subformula has been reached with respect to which play will proceed.

A *state of play*, therefore, is determined by three components: a role-assignment R, assignment s of individuals to variables, and a formula ϕ. If R maps T to a and F to b (where $a \neq b$), then $\bar{R}$ by definition maps T to b and F to a. $s(x/\alpha)$ is that assignment which differs from s, if at all, by assigning α to x.

What states of play, according to the rules of the game, can follow immediately after a given state (R, s, ϕ)?

A.   (i) If ϕ is atomic, none; the play is over.

     (ii) If ϕ is $\sim \psi$, then $(\bar{R}, s, \psi)$ must follow.

     (iii) If ϕ is $\psi \vee \theta$, then either (R, s, ψ) or (R, s, θ) may follow; R(T) chooses which.

     (iv) If ϕ is $\psi \,\&\, \theta$, then either (R, s, ψ) or (R, s, θ) may follow; R(F) chooses which.

     (v) If ϕ is $(\exists x)\psi$, then $(R, s, (x/\alpha), \psi)$ may follow, for any α in M. R(T) chooses which.

(vi) If ϕ is $(\exists x)\psi$, then $(R, s(x/\alpha), \psi)$ may follow, for any α in M. $R(F)$ chooses which.

The three components R, s, ϕ are, so to speak, intrinsic to states of play. There is another question, extrinsic insofar as it depends crucially on the background interpretation M, which is also settled at each state of play: the question who, if any, possesses a winning strategy at that state. The classical game so far described satisfies the following condition: at any state of play, exactly one player has a winning strategy. In other words, if he exercises his choices with omniscient astuteness in the subsequent course of play, he cannot but win, regardless of the choices his opponent may make. So if the possessor of a winning strategy makes a move in the execution of one of those strategies (i.e. chooses a disjunct or conjunct, or an individual from the domain) he thereby preserves his status as the possessor of a winning strategy. If he makes an ill-judged move then he ceases to be, and his opponent becomes, the possessor of a winning strategy. Thus derivatively and degenerately, the possessor of a winning strategy at the close of play is he who wins.

The foregoing suggests how to identify the possessor of a winning strategy at state of play (R, s, ϕ): we define $P(R, s, \phi)$ by recursion on the well-founded relation 'state S' could, according to the rules of the game, follow immediately after state S in a play of the game'. The details are as follow:

B.   (i) If ϕ is atomic: $P(R, s, \phi) = R(T)$ if s satisfies ϕ in M; $= R(F)$ otherwise;

(ii) If ϕ is $\sim\psi$: $P(R, s, \phi) = P(\bar{R}, s, \psi)$;

(iii) If ϕ is $\psi \vee \theta$: $P(R, s, \phi) = R(T)$ if either $P(R, s, \psi) = R(T)$ or $P(R, s, \theta) = R(T)$; $= R(F)$ otherwise;

(iv) If ϕ is $\psi \& \theta$: $P(R, s, \phi) = R(F)$ if either $P(R, s, \psi) = R(F)$ or $P(R, s, \theta) = R(F)$; $= R(T)$ otherwise;

(v) If ϕ is $(\exists x)\psi$: $P(R, s, \phi) = R(T)$ if for some α in M $P(R, s(x/\alpha), \psi) = R(T)$; $= R(F)$ otherwise;

(vi) If ϕ is $(\forall x)\psi$: $P(R, s, \phi) = R(F)$ if for some α in M $P(R, s(x/\alpha), \psi) = R(F)$; $= R(T)$ otherwise.

Now if we make the following correlation:

$$P(R,s,\phi) = R(T) \longleftrightarrow s \text{ satisfies } \phi \text{ in } M$$

$$P(R,s,\phi) = R(F) \longleftrightarrow s \text{ does not satisfy } \phi \text{ in } M$$

we see that B (i)–(vi) translate directly into the clauses of a Tarskian definition of satisfaction in a model. Thus by a trivial induction the following theorem holds:

> R(T) has a winning strategy at (R, s, ϕ) iff s satisfies ϕ in M; and

> R(F) has a winning strategy at (R, s, ϕ) iff s does not satisfy ϕ in M.

V

The immediacy of this equivalence between the game theoretic account and the standard Tarskian account must raise the question whether the former provides any new and genuine insight into the nature of the logical operators. The only place in that account where such an insight is likely to be found would appear to be over the question of what the execution and possession of a winning strategy consists in. We have a horizontal correlation between clauses of a Tarskian definition on the one hand, and, on the other, clauses which describe conditions under which winning strategies are retained (or executed) and forfeited. What we now need to develop further is the account of the vertical line of development in a game: of the sustained execution of a strategy with intent to win.

As remarked earlier, Hintikka emphasizes seeking and finding as the characteristic forms of activity associated with the language game for classical logic. For presumably a well-considered choice – one made in the knowing execution of a winning strategy – must be one which is made only after a process of seeking or searching for, and thereby finding an individual of the right kind. The connotations of these activities are primarily physical or spatio-temporal. Wittgenstein might perhaps have been indicating a more thoroughgoing set of ambiguities when he remarked in the *Blue Book* that "(some philosophizing mathematicians) are not aware of the different *mean-*

ings of the word 'discovery' when in one case we talk of the discovery of the construction of the pentagon and in the other case of the discovery of the South Pole." Perhaps we would be well advised to bear this difference in mind, and talk in mathematical contexts of constructing, exhibiting or defining our sought individuals – though even these seem to be terms prone to the sort of catachresis to which Wittgenstein was alluding. Informal mathematical discourse suggestively abounds with locutions like "One can find a number n such that ...". But in classical reasoning from an existentially quantified premiss $(\exists x)Fx$ we usually say "Let a be such an object (i.e. one with property F)... " or, "So, suppose a is such an object..."; we very rarely talk, in these classical contexts, of being able to *find* or *construct* such an object. In those rare cases when we do, it is usually because there is some underlying well-ordering, whose existence is assumed or proven, which can make the choice of some particular such object conspicuously natural. Seeking and finding seem misplaced activities when their purported objects are sheerly classical, constructively inaccessible existents. So perhaps there is after all an underlying pattern or structure to the activities of seeking, searching and finding in both the physical and mathematical realms; one which (*pace* Wittgenstein) can be made manifest in behaviour, and which is the basis of the widespread sharing of terminology in informal quantifying phrases.

What is it, then, about seeking and finding, searching for and locating, tracking down and trapping, sorting through and extracting, scanning for and spotting, which makes the outcomes the results of such activities as can confer any graspable semantical force upon the quantifiers whose occurrence prompts them? Why cannot we without impropriety add to the foregoing list casting around blindly for and, lo and behold! – coming up with; or grabbing the first thing at hand; or quoting the first number that comes to mind? The answer must surely be that seeing, searching for, tracking down, sorting through and scanning for are kinds of behaviour undertaken with and providing evidence for the intention to effect a certain strategy. And, without play on words, I maintain that one can behave in such a way as to manifest an intention to effect a certain strategy *only if* the strategy in

question is *effective*. (Here I use 'effective' in the broad sense, the narrower number theoretic sense of which is captured by 'recursive' according to Church's Thesis.) Merely being on the lookout for a certain kind of object, without assurance that every object in the domain will eventually be encountered; or casting around blindly for, or randomly grabbing are not activities in which one diagnoses the execution of a strategy calling for careful and methodical inspection and selection.

Yet with the classical game just described there are plays of the game – involving certain sentences against the background of certain infinite, undecidable interpretations – in which the winning strategies whose existence correlates with classical truth are, considered as functions mapping one state of play to another, set theoretic entities not specifiable by any effective set of rules. No person could apply these functions in a way which exhibits strategic intent. Nothing the 'possessor' of such a strategy can do could be construed as behaviour manifesting his grasp of such meanings of the quantifiers as call for the execution of such strategies.[7]

We have arrived, then, at the following position: in accordance with the Wittgensteinian thesis that one's grasp of meaning must be capable of being manifested eventually and implicitly in observable behaviour, we require the strategies in our game to be effective. The meanings conferred upon the logical operators by the game theoretic semantics modified by this constraint will not be classical. For classical meaning determines each sentence ϕ as either true or false of *M*. Since we are now correlating truth of ϕ with the existence of an *effective* strategy for the player starting as T, and falsity of ϕ with the existence of an *effective* strategy for the player starting as F, bivalence will fail: play might become 'effectively deadlocked', with neither player possessing an *effective* winning strategy. I shall show below that the logic which reflects and respects the meanings conferred upon the logical operators by the game satisfying this constraint is intuitionistic logic.

VI

The new game is played with respect to a finite *set* of formulae rather than a single formula. We also alter our conception of the 'back-

ground model' M, regarding it now as an *atomic base* in Prawitz's sense.[8] Thus M can contain inference rules and axioms involving only atomic formulae. From M we can generate open or closed atomic arguments. The closed atomic arguments provide winning strategies for player T on their atomic conclusions. The open atomic arguments provide effective methods for transforming winning strategies on their assumptions into winning strategies on their conclusions.

Each state of play is now fully characterized by some finite set of formulae φ. Roles are never exchanged, so we may dispense with R; and since only such individuals are chosen as can be referred to by terms of the language we no longer need s, the assignment of values to variables. Player T is associated with the assertion of each formula in φ against the background M; and player F is associated with the claim that *that* leads to absurdity.

Each player, when making a move, is free to invoke a consistent extension of the base M forming the background to play. A winning strategy at any state of play must therefore take into account all consistent extensions of the background at that state. At first we consider the language with $\vee$, $\&$, $\exists$ and $\forall$ as its only logical operators. We must stipulate anew what states of play may immediately succeed a given state of play.

(i)   If every member of Φ is atomic, none; the play is over.

(ii) If Φ is of the form $\{\ldots\psi \vee \theta\ldots\}$, then either $\{\ldots\psi\ldots\}$ or $\{\ldots\theta\ldots\}$ may follow; T chooses which.

(iii) If Φ is of the form $\{\ldots\psi \& \theta\ldots\}$, then $\{\ldots\psi, \theta\ldots\}$ may follow.

(iv) If φ is of the form $\{\ldots\exists x\psi\ldots\}$, then $\{\ldots\psi(x/t)\ldots\}$ may follow, for any term t denoting an individual in M. T chooses which.

(v)   If Φ is of the form $\{\ldots\forall x\psi\ldots\}$, then $\{\ldots\psi(x/t_1)\ldots\psi(x/t_n)\ldots\}$ may follow, for any terms $t_1, \ldots, t_n$ denoting individuals in M. F chooses which.

In the course of play it is up to F to specify, at each state of play Φ, which $\phi \in \Phi$ is to be considered for the application of one of the succession rules above. This is necessary because we are now dealing

in general with a set of formulae rather than a single formula at each state of play.

Let us now look more closely at winning strategies. Succession between states of play is still a well-founded relation, so we may still attempt a recursive definition of $P(\Phi)$ – the possessor of a winning strategy at state of play Φ. The crucial difference now is that – except in the case of finite and decidable $M - P$ may be partial, i.e. not everywhere defined. In a state of play Φ at which $P(\Phi)$ is not defined, neither player possesses a winning strategy. Play is then 'effectively deadlocked'.

(i) $P\{A_1...A_n\} = T$ if T has a closed atomic argument for each A_i valid with respect to M; $= F$ if F has an argument for Λ (absurdity) from assumptions $A_1...A_n$ valid with respect to M; is undefined otherwise.

(ii) $P\{...\psi \lor \theta...\} = T$ if T has an effective method for determining a winning strategy at $\{...\psi...\}$ or a winning strategy at $\{...\theta...\}$; $= F$ if F has an effective method for determining a winning strategy at $\{...\psi...\}$ and a winning strategy at $\{...\theta...\}$; is undefined otherwise.

(iii) $P\{...\psi \& \theta...\} = T$ if T has an effective method for determining a winning strategy at $\{...\psi, \theta...\}$; $= F$ if F has an effective method for determining a winning strategy at $\{...\psi, \theta...\}$; is undefined otherwise.

(iv) $P\{...\exists x\psi...\} = T$ if T has an effective method for determining t in M and a winning strategy at $\{...\psi(x/t)...\}$; $= F$ if F has an effective method for determining, for every t in M, a winning strategy at $\{...\psi(x/t)...\}$; is undefined otherwise.

(v) $P\{...\forall x\psi...\} = T$ if T has an effective method for determining, for every $t_1,...,t_n$ in M, a winning strategy at $\{...\psi(x/t_1)...\psi(x/t_n)...\}$; $= F$ if F has an effective method for determining $t_1...t_n$ in M and a winning strategy at $\{...\psi(x/t_1)...\psi(x/t_n)...\}$; is undefined otherwise.

It should be clear that the principle behind the formulation of these clauses is that a winning strategy must be effective, and is retained so long as it is properly executed.

I shall assume the following plausible generalization of Prawitz's completeness conjecture[9]: Suppose there is an effective operation f such that for any consistent extension M' of M and any closed canonical arguments π_i for ψ_i valid relative to M' ($1 \le i \le n$), $f(\pi_1, \ldots, \pi_n)$ is a closed canonical argument for θ valid relative to M'. Then there is an argument for θ from $\psi_1, \ldots, \psi_n$ using only intuitionistic rules and rules in M. (Prawitz's conjecture concerns the special case where M is empty.)

We can then prove by induction the following result about the language game here defined:

THEOREM: $P(\Phi) = T$ iff for each $\phi \in \Phi$ T has a closed canonical argument for ϕ valid with respect to M.

$P(\Phi) = F$ iff F has an argument for Λ from assumptions Φ valid with respect to M.

Example: Let M contain $\dfrac{0 = 1}{\Lambda}$, $\dfrac{A(t) \, A(u)}{t = u}$; $\Phi = \{\forall x A x\}$.

Then $P(\Phi) = F$ by virtue of the argument

$$\dfrac{\dfrac{\forall x A x}{A(0)} \quad \dfrac{\forall x A x}{A(1)}}{\dfrac{0 = 1}{\Lambda}}$$

for Λ from Φ valid with respect to M. Note that any closed canonical argument for $\forall x A x$:

$$\dfrac{\dfrac{\Sigma(a)}{A(a)}}{\forall x A(x)}$$

can be transformed to a closed canonical argument for Λ by $\forall$-reduction:

$$\dfrac{\dfrac{\dfrac{\Sigma(a)}{A(a)}}{\forall x A x}{A(0)} \quad \dfrac{\dfrac{\Sigma(a)}{A(a)}}{\forall x A x}{A(1)}}{\dfrac{0 = 1}{\Lambda}} \quad \longrightarrow \quad \dfrac{\dfrac{\Sigma(0)}{A(0)} \quad \dfrac{\Sigma(1)}{A(1)}}{\dfrac{0 = 1}{\Lambda}}$$

So in this example player F's winning strategy on Φ consists in an effective method for transforming closed canonical arguments for members of Φ to a closed canonical argument for Λ. This is what, in general, a winning strategy for F on Φ consists in. We shall return to this point later.

For each $\phi \in \Phi$ ground for convincement in ϕ (with respect to M) is that one possesses a winning strategy on $\{\phi\}$ – or, equivalently, that one possesses a closed canonical argument for ϕ (valid with respect to M). Now if one is convinced that ψ is a logical consequence of Φ, then the grounds one may have for conviction in each $\phi \in \Phi$ must automatically provide ground for conviction in ψ. That is, given winning strategies for T on all the members of Φ with respect to any M, one should be able to devise a winning strategy for T on ψ with respect to M. So there would have to be a tight connection between behaviour (in the language game) in advocacy of the premisses and behaviour in advocacy of the conclusion. That one's grasp of logical consequence should mediate and modify behaviour in this way is once again required by the Wittgensteinian thesis about meaning. This observation suggests the following definition of logical consequence:

$\phi_1...\phi_n \models \psi$ iff there is an affective method for transforming any winning strategies for T on all the ϕ_i against the background of any M to a winning strategy for T on ψ against the background of M.

Now Prawitz[9] has put forward a conception of intuitionistic consequence as consisting in the existence of an effective method for transforming closed canonical arguments for the ϕ_i valid with respect to any atomic base M to a closed canonical argument for ψ valid with respect to M. The parallel with the definition above is immediate, when winning strategies for T on ϕ are correlated with closed canonical arguments for ϕ.

Our result above establishes that a winning strategy for T on ϕ is no more nor less than a closed canonical argument for ϕ (with respect to M). Such a proof is a recipe for invincibility in the language game. It reveals which disjunct to favour; how to defend either conjunct of a conjunction; how to defend any instance of a universal

quantification; and how to adduce a verifying instance of an existential quantification. It would be composed entirely of applications of introduction rules after, perhaps, some initial applications of atomic rules from M.

A winning strategy for F on Φ, on the other hand, is a canonical argument for Λ from Φ (with respect to M). It is a recipe for inexorable *reductio* of any attempted justification of the members of Φ. It reveals which conjuncts to seize upon; how to refute either disjunct of a disjunction; how to oppose any alleged justifying instance of an existential quantification; and shows which instances of a universal quanitfication to select for joint refutation (modulo the rest of Φ). It would be composed entirely of applications of elimination rules followed by some terminal applications of atomic rules from M to yield Λ as the final conclusion.

Such an argument is easily seen to provide an effective method for *transforming closed canonical arguments* for members of Φ to a closed atomic argument for Λ. Each of the former arguments would have the form

$$\text{Intro} \begin{cases} \pi_i \\ \phi_i \end{cases}$$

whereas the argument for Λ from $\phi_1 \ldots \phi_n$ would have the form

$$\text{Elim} \begin{cases} \dfrac{\phi_1 \ldots \phi_n}{\Sigma} \\ \Lambda \end{cases}$$

Obviously successive reductions will transform the argument

$$\text{Intro} \begin{cases} \pi_1 \ldots \pi_n \\ \phi_1 \ldots \phi_n \end{cases} \text{Intro}$$
$$\begin{matrix} \Sigma \\ \Lambda \end{matrix} \Bigg\} \text{Elim}$$

to a closed *atomic* argument for Λ involving no applications of rules for the logical operators.

We are now ready to consider the extension of our game to a language involving implication as a connective. Ground for conviction in $\phi \supset \psi$ (with respect to M) would be provided by any method by

which, given a winning strategy for T on ψ with respect to M, one could devise a winning strategy for T on ψ with respect to M. So we have the clause

(vi) $P\{\ldots\psi \supset \theta \ldots\} = T$ if there is an effective method for transforming any winning strategy for T on ψ (against the background of M) to one on θ, and $P\{\ldots\} = T$; $= F$ if there is an effective method for transforming any such effective method as the one above into a winning strategy for F on $\{\ldots\}$ (against the background of M); is undefined otherwise.

This clause is different from its predecessors in that it introduces effective methods for operating on effective methods – 'higher order' effective methods, one might say. And here it becomes difficult to see how the possession of a higher order method can be diagnosed from or convincingly disclosed by behaviour in the language game. So with an implication the game can not very well proceed with immediate subformulae as happens in the case of formulae of the forms $\psi \vee \theta$, $\psi \& \theta$, $\exists x \psi$ and $\forall x \psi$. The players will not play the game out to the atomic subformulae of an implication. This is an important difference from the earlier game. There the winning strategy is revealed by repeated plays to the bitter end of atomic subformulae. Now the player with the winning strategy will have to reveal the relevant part of it to his opponent by exhibiting a surveyable argument. Thus in order to show you that I can turn any winning strategy on ψ into one on θ I show you an argument for for θ from ψ, Having done so, we continue the game with respect to the set of whatever formulae remain. But if *you* reveal a winning strategy the game ends and you win.

The considerations two paragraphs earlier of winning strategies for F suggest an obvious way of subsuming negation into the present treatment. Since a winning strategy for T on $\sim\phi$ should be a winning strategy for F on ϕ, those considerations show that $\sim\phi$ is best taken as $\phi \supset \Lambda$. We therefore have a language game for the full language of first order logic.

By induction on the implicational complexity of Φ our theorem above generalizes to that language. But the constitutive lacuna (introduced by clause (vi)) which arises from the possibility of wholesale

strategic disclosures in the course of play, sealing off possible corridors of contention in the sequel, is of course infectious. It is only a short step from there to the dialectic economy of substituting words for actions throughout. Instead of convincing you that ϕ by repeatedly winnings plays of the game on ϕ, I simply save myself the effort by revealing my winning strategy: an argument for ϕ. I no longer play the game, but instead divulge my game plan. This synopsis heralds the mature move of the most evolved language game: assertion backed by proof. Proof precludes play: but only when it is intuitionistic proof.

VII

If M is decidable and finite, the classical game does not differ from the intuitionistic game as far as the players' strategic arsenals are concerned. Effective search through the domain is always possible, as is effective determination of whether a given predicate holds of a given tuple of individuals. If M is construed then as a system of atomic axioms, the intuitionistic game is applicable. We readily see that it cannot become 'effectively deadlocked'. Intuitionistic truth coincides with classical truth, since at every state of play S against the background of such M P(S) is always defined. So if we consider only finite M with decidable atomic relations the resulting classical and intuitionistic consequence relations coincide in extension. (This result can be established directly within the proof theory of classical and intuitionistic logic.)

It is a standard intuitionistic objection to the classical mathematician's use of the law of excluded middle (or any of its equivalents) that it involves "an unjustified generalization to infinite domains of a procedure which is legitimate for finite domains only".[10] In mathematical domains, given the effective undecidability of certain sentences, intuitionistic bivalence fails. Have we, in Wittgenstein's words, fallen prey to "misunderstandings concerning the use of words, caused, among other things, by certain analogies between the forms of expression in different regions of language"? (*Phil. Inv.*, para 20) Elsewhere (para 125) we hear what in this context might well be a

chiding voice of intuitionism: "...we lay down rules, a technique for a game, and...then when we follow the rules, things do not turn out as we had assumed...we are therefore as it were entangled in our own rules. This entanglement in our rules is what we want to understand... It throws light on our concept of *meaning* something. For in those cases things turn out otherwise than we had meant, foreseen." The illicit extrapolation noted above is perhaps another sublimation of the logic of our language. "...the philosophical problems...are solved...by looking into the workings of our language, and that in such a way as to make us recognize those workings: in despite of an urge to misunderstand them." (para 109)

We have looked into the workings of the logical operators: or, rather, looked into two ways of looking at them. We have recognized in those workings two conceptions of truth, in agreement in those domains in which, arguably, we learn our language: sounding, as it were, a perfect octave in this middle range. The only discordant notes are struck when we extend the range of our interpretations to the infinite and undecidable. There the happy conflation ends; and substantive problems of meaning and metaphysics begin.

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NOTES

* This paper grew out of a suggestion provoked by a paper of Hintikka at the 1976 Colloquium on Language and Reality in Biel, Switzerland. Versions have been read at Edinburgh, Dundee, Oxford and the 1977 Wittgenstein Colloquium in Kirchberg.

[1] in H.E. Rose and J.C. Shepperdson (eds.): *Logic Colloquium '73*, North Holland, Amsterdam, 1975.

[2] For a discussion of Dummett's argument see D. Prawitz: 'Meaning and Proofs: On the conflict between classical and intuitionistic logic', *Theoria* 43 (1977), 2–40.

[3] J. Hintikka: *Logic, Language Games and Information*, Clarendon, Oxford, 1972.

[4] "...what a proposition is is in one sense determined by the rules of sentence formations..., and in another sense by the use of the sign in the language game. And the use of the words 'true' and 'false' may be among the constituent parts of this game..." (*Phil. Inv.*, para 136.)

[5] with minor differences: Hintikka talks of 'Myself' and 'Nature' instead of T and F, and has players invent names for individuals rather than co-ordinate them with variables. The innovations introduced here are intended to serve a later purpose; see below.

[6] Without, however, making it like the games proposed by Lorenzen and Lorenz in their theory of dialogical games.

[7] Cf. Hintikka's appendix to 'Language Games for Quantifiers', in *op.cit.*

[8] 'Toward a Foundation of General Proof Theory', in L. Henkin, P. Suppes, A. Joja and G. Moisil (eds.): *Logic, Methodology and Philosophy of Science IV*, North Holland, Amsterdam, 1973.

[9] 'On the idea of a general proof theory', *Synthese* 27 (1974), 63–77.

[10] A. Fraenkel and Y. Bar-Hillel: *Foundations of Set Theory*, North Holland, Amsterdam, 1973, p. 234.