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On the Necessary Existence of Numbers

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Abstract

We examine the arguments on both sides of the recent debate (Hale and Wright *v.* Field) on the existence, and modal status, of the natural numbers. We formulate precisely, with proper attention to denotational commitments, the analytic conditionals that link talk of numbers with talk of numerosity and with counting. These provide *conceptual controls* on the concept of number. We argue, against Field, that there is a serious disanalogy between the existence of God and the existence of numbers. We give stronger reasons than those advanced by Wright for resisting Field's analogy. We argue that the rules governing the basic numerical notions commit us to the natural numbers as *necessary* existents. We also show that the latest twist in the debate involving 'surdons' leaves both sides in a stalemate.

1 The problem

The question before us is whether numbers are necessary existents, and by what logical criteria we would settle this. We need to enquire which of our statements or linguistic practices, if any, commit us to the existence of numbers, and whether they show such existence to be necessary, or impossible, or merely contingent. We also need to be clear about whether the numerical question is different from the the logical one—that is, whether our inquiry is just like the one into the existence of God.

If there is a God then it is a perfect being. Similarly, for Field, if there are natural numbers then $7 + 5 = 12$. The critics of Anselm's ontological argument maintain that it does not follow from the 'truth' of "God is a perfect being" that God exists. Likewise, for Field, it does not follow from the 'truth' of a mathematical statement such as "$7 + 5 = 12$" that natural numbers exist—or, indeed, that just 5, 7 and 12 exist.

The analogy, though, is broken-backed. Inquiring into the existence and necessity of numbers takes us into quite different logical terrain than inquiring into the existence and necessity of God.

A moment's reflection shows that Field's claim has to be stronger than just presented. He has to maintain that it does not follow from the truth of any mathematically *or conceptually* true statements we may make about numbers that numbers exist. Indeed, if (as will be argued below) the existence of zero is enough, analytically, to yield the existence of every natural number,[1] then Field's task is even harder. He would have to show that it does not follow from the truth of any mathematically or conceptually true statements we may make about numbers that zero exists.

Field thinks—misguidedly—that there is a distinction to be had between truth *simpliciter* of a statement like "7 + 5 = 12" and its truth according to standard number theory. A better view is that this latter kind of truth is bound to be truth *simpliciter*, because of conceptual controls to which we accede on (at least singular) truths of arithmetic. These conceptual controls force statements about numbers into logical connections with statements about ordinary things. The latter statements are true or false *simpliciter*; hence, courtesy of the aforesaid logical connections, the former will be also.

2 The aim of this paper

Field has recently set out an argument for the conceptual contingency of the natural numbers.[2] We shall give an account of his argument, and then criticize it.

We shall formulate precisely, with proper attention to denotational commitments, the analytic conditionals that link talk of numbers with talk of equinumerosity, thereby providing conceptual controls on the concept of number. These conditionals are refinements of 'Hume's Principle'. Then it will be argued, against the nominalist such as Field, that there is a serious disanalogy between the existence of God and the existence of numbers. Stronger reasons than those advanced by Hale and Wright on behalf of the platonic logicist will be given for resisting Field's analogy. These involve a crucial further conceptual control on the concept of number, which is neglected by Wright. The control in question involves counting. Considerations of reflective equilibrium enjoin us to stand by the ontological commitments incurred by our talk about numbers, and, indeed, to acknowledge them as necessary existents.

Field's defective argument leads us to a detailed formal proof of a modal argument, different responses to which correspond to the opposing views on the modal status of numerical existence. The formal analysis reveals that the latest twist in the debate concerning 'surdons' leaves both sides—nominalist and platonic logicist—in a stalemate.

3 The sentential stance

There is a variety of ways one might seek to disclose commitment to the existence of numbers, and cast light on its modal status. We might look for a conclusion in existential form, such as "Numbers exist". But if that conclusion is to be necessary, it will be implied by any set of premisses, unless we have an appropriate relevant logic in which to conduct our argument from our chosen premisses to that conclusion.[3] Suppose now that such a relevant logic is available,[4] so that this sweeping objection can be averted; and let us pose our question again: by what logical criteria would we say which statements or linguistic practices committed us to the existence of numbers, necessary or mere?

There would appear to be at least three distinct levels of diagnosis:

1. statements saying that numbers exist (whether within the language of number theory, or within a many-sorted language for which numbers form one sort);
2. statements presupposing (in the background semantics of the interpreted language) that numbers exist;
3. whole theories (such as mathematical physics) being successful in describing, predicting and explaining phenomena, and this success being best explained by appeal to the existence of numbers.

Level (1) is favoured by the Quinean, for whom "to be is to be the value of a bound variable". At level (1) we have statements such as

$\exists x(x = 0)$
$\exists x$ (x is a natural number)
$\forall x \exists y(x < y)$

At level (2) we have examples such as

$7 + 5 = 12$
The numbers of things not identical to themselves is 0
The number of planets is 9 if and only if there are nine planets

At level (3) the examples are less straightforward. It is at this level that one would advance or reject claims about the conservative extensions, effected by mathematical theories, of physical theories that are, themselves, devoid of any reference to or quantification over mathematical entities. Also to this level would belong claims about reductions in complexity of the deductions behind the predictions and explanations involved in scientific theorizing: claims that the deductions involved in the mathematical-cum-physical theory are shorter, or more easily found, than those involved in the purely physical theory shorn of the mathematics.

Now as far as arguments for the existence (necessary or otherwise) of numbers is concerned, there is no help to be had severally, as it were, from sentences at level (1)—neither from such sentences as assertions with their own peculiar warrants (proofs based on axioms), nor from those very arguments for such sentences as conclusions. Our diffidence concerning sentences at level (1) is indifferent to their extensional or modal logical structure. The extensional examples given above (lacking modal operators) do not, *pace* Quine, themselves convey commitment to numbers—unless taken jointly, as it were. For only the stance or attitude of mind involved in whole-heartedly adopting the relevant form of discourse conveys commitment to the things whereof one speaks.[5] (Adopting the form of discourse is sufficient to secure commitment, given the univocity of the existential quantifier.) And if we were to use modal operators explicitly within the sentences of our theory, this use could only be justified, if at all, after the philosophical argument has been won, regarding the modal character of the entities involved according to the global stance implicit in adopting that form of discourse.

It is with sentences at level (2) that the philosophical argument has to be conducted. For it is here that one begins to understand how talk about numbers meshes with talk about ordinary things. It is here that one appreciates the force of the conceptual controls built into our hybrid talk of numbers and ordinary things.

Finally, the large claims involved at level (3) are not, *pace* Field, strictly relevant to the philosophical problem of the existence of numbers, necessary or otherwise. Their modal/existential status is unaffected by considerations as to whether talk about numbers does or does not (as the case may be) alleviate the deductive burdens involved in physical theorizing.

4 Conceptual controls on numbers

4.1 Counting and Schema (C)

The kind of conceptual control that is relevant here is an analytic equivalence such as

> There are exactly two apples in this basket if and only if the number of apples in this basket = 2.

This is an instance of the general *Schema* (*C*):[6]

> There are exactly n Fs if and only if the number of Fs = $\underline{n}$

where what replaces 'n' is adjectival, and what replaces '$\underline{n}$' is substantival. Once we have laden a statement such as

> the number of Fs = $\underline{n}$

with a truth-value in this way, we are bound to recognize that that truth-value arises from the attachment of a particular number to the given concept F. Once numbers have been acknowledged as entities to which commitment is incurred by using such propositional forms—namely, identity statements—there can be no principled objection to honouring that commitment when our talk proceeds to be about numbers and nothing else, as when we are doing arithmetic proper. If one acknowledges that the number of sides of the Pentagon is 5, and that the number of dwarves in this picture is 7, and that the number of Christ's apostles is 12, then in so thinking one thereby recognizes 5, 7 and 12 as objects. On what grounds, then, could one be entitled to withdraw that recognition when one agrees further that $7 + 5 = 12$?

4.2 Equinumerosity and Hume's Principle (N)

It was remarked above that Field has to maintain that it does not follow from the truth of any mathematically *or conceptually* true statements we may make about numbers that numbers exist. Now among such conceptually true statements we have, according to Hale and Wright,[7] the following biconditional, often referred to as "Hume's Principle", and regarded most famously by Frege as analytic. We shall call it *Schema* (*N*):

> The number of *F*s is identical to the number of *G*s if and only if there are exactly as many *F*s as *G*s.

According to Hale and Wright, there is the following chain of logical implication:

there are exactly as many *F*s as *F*s
$\Downarrow$ by Schema *N*
the number of *F*s is identical to the number of *F*s
$\Downarrow$ by the logic of reference
the number of *F*s exists.

Since the first of these claims is a logical truth (hence necessary), so is the third one.

But is the Humean identity (*N*) really analytic? Not quite; in due course its proper analytic core will be extracted in the form of two conditionals, (*N*1) and (*N*2*) below. The reason why (*N*) is not analytic as it stands at present is that it has not been suitably qualified within the context of a free logic.

Note that the logic of reference involved here is a very weak fragment of second order logic with certain assumptions about the evaluation (as true or false) of statements containing referring terms. Hale and Wright appear to be committed to a Russellian conception of such evaluation. That is, an atomic claim will be true only if all its terms denote. This commitment is entirely reasonable. But it means that we can no longer work in standard logic, based on the background assumption that all singular terms denote just by virtue of being grammatically well-formed. Instead, we have to take seriously the possibility of 'empty', or non-denoting singular terms, even when they are grammatically well-formed. What we need, in short, is a *free* logic.[8] Philosophical discussion about the existence of numbers should be conducted against the background of a logic that is absolutely neutral on the question whether any particular term happens to denote. The whole point is to examine the foundations of our commitment to numbers; and to identify, with the help of our logical techniques, the precise juncture at which explicit existential commitment to numbers is indeed incurred.

5 Why the existence of natural numbers is not Anselmian

The way that Field tries to capture the dependence of the biconditional (N) on the existence of numbers is as follows:

> ...the conceptual truth is not (N) itself but rather that *if numbers exist* then the number of Fs equals the numbers of Gs if and only if there are exactly as many Fs as Gs; ...[9]

And, as Field goes on to say, there is a similar conditioning of the analytic claim according to which God is a perfect being:

> ...what is conceptually true is only that *if God exists then* God is a perfect being.[10]

Field also considers a closer analogy, where the claim about God takes a biconditional form, like the claim about numbers. He offers the following candidate analysis of the nature of God, or of gods (call it (G)):

> The god that created x is identical to the god that created y if and only if x and y are spatio-temporally related.

Once again, says Field, the claim that is true is not really (G), but rather the following conditional of which (G) is the consequent:

> If gods exist then (G).

First we need to clarify precisely where the issue in contention lies. Then we shall show that Field has drawn too crude an analogy between the numbers case and the God case.

5.1 Clarification of existential presuppositions

First, one direction of the biconditional does not have to be conditioned the way that Field maintains. There is nothing wrong with the existentially unconditioned claim

> ($N1$) The number of Fs is identical to the number of Gs only if there are exactly as many Fs as Gs.

This is because, on the Russellian semantics according to which singular predictions are true only if their referring terms denote, the claim "the number of Fs is identical to the number of Gs" can be true only if there is such a thing as the number of Fs and there is such a thing as the number of Gs. This being so, the conditional ($N1$) simply says that the identity of those numbers entails equinumerosity of the Fs and the Gs.

The existential conditioning is needed only for the converse claim

> ($N2$) The number of Fs is identical to the number of Gs if there are exactly as many Fs as Gs

which is not analytically true as it stands. *F* and *G* might have such vast extensions that, although they might be in one-to-one correspondence, they nevertheless enjoy no number as their cardinality. The analytically true claim is rather

($N2^*$) The number of *F*s is identical to the number of *G*s if (there are exactly as many *F*s as *G*s and the number of *F*s exists and/or the number of *G*s exists).

One can see why there is this asymmetry in the need for existential conditioning of the two conditionals that go to make up the biconditional (*N*). With (*N*1), the antecedent "The number of *F*s is identical to the number of *G*s" is phrased in the language of talk about numbers as objects. Its truth presupposes (within the semantics of that language) the existence of numbers as referents for its terms. In (*N*1) the 'number-talker' is spelling out a necessary condition for the identity of numbers numbering the extensions of predicates *F* and *G*. This necessary condition is that there be exactly as many *F*s as there are *G*s. Thus in (*N*1) the 'number-talker' is making a link into the language of the 'mere predicate-extension comparer'. The latter shares with the number-talker the language for comparing predicate extensions. This language contains such expressions as "there are exactly as many ...'s as there are —'s", "there are more ...'s than there are —'s", "there is a ..." and so on. These expressions can be used to make true statements about predicate extensions without those extensions having to enjoy numbers as their cardinalities.

What the number-talker does is extend this common language by adopting the vocabulary for talking about numbers: to wit, the term-forming expression that applies to predicates to form terms designed to refer to natural numbers: "the number of ...'s". In making this linguistic extension, the number-talker needs to forge connections with the common language that is shared with the mere predicate-extension comparer. The number-talker needs to submit to some conceptual controls on the use of the term "the number of ...'s" that is now apt to feature as the dominant operator of terms occurring in existence and identity statements. (*N*1) is such a control. It says, from the vantage point of the speaker who "has the numbers in view", so to speak, what the consequences are, in the shared language, of saying that the number of *F*s is identical to the number of *G*s.

With (*N*2) the situation is different. Here the antecedent "There are exactly as many *F*s as *G*s" is in the language of the predicate-extension comparer. One needs to spell out the conditions under which it will be warranted to make the move from the claim that there are exactly as many *F*s as there are *G*s, to the claim (in the language of the number-talker) that the number of *F*s is identical to the number of *G*s. The mere predicate-extension comparer might, as it were, be innocent about numbers. Or she may, like the nominalist, be 'number-blind', refusing to acknowledge the existence of any abstract objects such as numbers. Therefore, in order to make the consequent palatable to her, the number-talker hedges by saying concessively (via ($N2^*$)) that:

> if the number of *F*s exists and/or the number of *G*s exists then (if there are exactly as many *F*s as *G*s then the number of *F*s is identical to the number of *G*s).

This way the nominalist who wishes to venture no further (in her existential commitments) than the objects involved in her predicational or comparative talk of *F*s and *G*s can be brought to understand what is involved in the number-talker's statement that the number of *F*s is identical to the number of *G*s.

That understanding will be bought only at the cost of a suitably refined meaning of the conditional featuring in the claim just displayed. We have to be able to avoid having the nominalist say "I can accept that! Since there are no such things as numbers anyway, the antecedent of the conditional is false and the whole conditional therefore is true." We need the nominalist to be able to deal with the conditional as having more modal content. We want to be able to say to the nominalist something along the following lines:

> Maybe there are no numbers in this world; but what about other possible worlds in which there are numbers? Since we are giving you an analytic truth in the form of ($N2^*$), you have to consider worlds in which the antecedent might come out true. In such a world, according to ($N2^*$), the consequent will be true also. So ($N2^*$) succeeds in connecting your existentially parsimonious predicate-comparisons to the correct claims about numerical identities in such a world. The non-existence of numbers (if we concede this much to you) in *this* world is neither here nor there. It doesn't make ($N2^*$) vacuously true.

This line would be useless against the nominalist who maintains that not only do numbers not exist in this world, but they also fail to exist in every other possible world: that is, they are conceptually impossible existents. But it is hard to see what possible justification there could be for such a claim once one buys into the metaphysical materials involved in possible worlds semantics.[11] Indeed, on the assumption that arithmetic is consistent, and that our number-talk is an appropriately conceptually controlled extension of our ordinary first-order talk about concrete things, there is no way to the impossibility claim that the nominalist is here advancing.

($N1$) and ($N2^*$) are properly licensed for analytical business. They bring out something important in the content of number-talk. The question, however, is whether they and kindred analytic claims can take us from the content of number-talk to the existence of what it is talk about: for which, see below.

We are now in a position to develop our criticism of Field's analogy between the numerical and the divine.

5.2 Criticism of Field

Our criticism sets out by asking whether Field has really put his finger on a compelling analogy when he likens talk about numbers to talk about God. Is every justification of existential commitment to numbers (as necessary existents) doomed to share the same kind of invalidity as besets the ontological argument for God's existence? We must bear in mind that we need only one successful justification of existential commitment to numbers. Even if Field succeeds in nailing one purported justification with his analogy to the ontological argument, the question arises whether his strategy is guaranteed to apply similarly to all attempted justifications.

It is not guaranteed so to apply. For Field has left out of consideration a very important source of disanalogy between the number case and the God case. This source provides, ultimately, for clinching justification for the existence of numbers, and does so in a way that cannot be matched in the case of analytic principles involving the term 'God'. It therefore escapes Field's strictures.

Here one needs a further development of the all too brief suggestion of Wright[12] that

> ...it is no part of Fregean platonism to regard the existence of the relevant species of abstract objects as entailed just by the way in which their covering sortal is explained. It is not, for instance, the way the concept of direction is explained—*via* stipulation of the equivalences—which entails that there are such things as directions, but those stipulations *together with* the truth of appropriate statements apt to feature on the right-hand sides. What does follow from the explanation is that any one of a relevant class of straight lines has a direction; but the existence of directions is contingent on the existence of members of that class. So the gap between such cases and the uncomfortable precedent of the Ontological Argument seems adequately broad.

Note that what Wright means here by the stipulated equivalences are the simpler-minded 'Fregean' ones that are not conditionalized, in either direction, on the existence of the newly introduced abstract existents in question.

In Hale and Wright (1994), Field's equivalence (G):

> The god that created x is identical to the god that created y if and only if x and y are spatio-temporally related

is also taken as 'Fregean', that is, not conditionalized, in either direction, on the existence of 'the god that created x'. Hale and Wright allow the Fregean detachment to the prime cause of all spatio-temporally related items![13]—but urge, still, that any 'disquieting' conclusion as to God's existence would still require, over and above His being a prime cause of all spatio-temporally items, His possession of distinctive perfections. But this is to sail too close to the wind. One should be wary even of the concession of existence to a 'mere' prime cause of all spatio-temporally related items. The existence of such a thing could have no conceptual guarantee; whereas the existence of numbers does. Let us now take a closer look at the reasons why.

6 Why natural numbers exist necessarily

We saw earlier that there are conceptual controls on the use, by the number-talker, of his number-theoretic vocabulary—in particular, the term-forming expression "the number of ...'s". The main conceptual control that we have looked at so far is the Humean identity (N), refined to ($N1$) and ($N2^*$) above. But it is really our other conceptual control on number-talk that is worth emphasizing—namely, the adequacy condition involving Schema (C). It is Schema (C) that provides analytic access to the (necessary) existence of numbers.[14]

6.1 The centrality of Schema (C)

We have already distinguished Schema (C) from the Humean identity schema (N). Schema (C) adverts to (the outcomes of) counting as a source of further conceptual controls on our talk about natural numbers.[15] Note that the Humean identity schema does nothing to force it to be the case (in free logic) that, if there are no Fs then the number of Fs is 0. It is to address this concern that Schema (C) is put forward:

the number of Fs is $\underline{n}$ if and only if there are exactly n Fs.

Three explanations are in order.

First, the schematic expression '$\underline{n}$' is a place-holder for the numeral for the natural number n. We choose some canonical notation for numerals, like the standard decimal scheme or the binary notation or the (s,0)-notation.

Secondly, the locution "there are n Fs" is to be understood as involving the modifier 'n' adjectivally, not substantivally. That is, it is a locution available to the nominalist who confines herself to the unextended language of ordinary predication and quantification and predicate-extension comparison. In connection with Schema (C), all that is necessary on the part of the nominalist is that she understand the inductive scheme of definition that allows one to generate, for each *n,* the appropriate sentence of first-order logic that expresses the claim that there are exactly n Fs. This sentence will not involve quantification over or reference to numbers as objects, unless F itself is a predicate intended to have numbers in its extension. But suppose F is not such a predicate. Suppose F is a harmless predicate like "...is an apple". Then for example the claim that there are exactly two Fs is captured by the sentence

$$\exists x \exists y(\neg x = y \wedge Fx \wedge Fy \wedge \forall z(Fz \supset (z = x \vee z = y)))$$

The full technical details of the definitional scheme for "there are n Fs" need not detain us here.[16]

Thirdly, a point already implicit in the foregoing, but crucial for an appreciation of the credentials of arithmetic as a store of analytic truth: the predicates that can be substituted for F are those that allow one to distinguish and re-identify objects falling in their extensions. One need only be able to grasp that it is meaningful to say such things as "This F is the same as that F", or "This F is distinct from that F", in order to be entitled to put such F into the slots of the Schema (C). Thus Schema (C) governs all our thinking about objects in general. It covers physical objects and abstract objects, contingent existents and necessary existents, and in particular the numbers themselves.[17]

Note, however, that Schema (C) provides for the equivalence of two different ways of making statements about finitude. Since there are infinitely many natural numbers, neither side of the schema will be true (for any n) when F is '...is a natural number'. But Schema (C) still governs thought about natural numbers insofar as, for example, we have the true instance

> the number of prime numbers between 10 and 20 is 5 if and only if there are exactly 5 prime numbers between 10 and 20.

The derivability of all instances of Schema (C) is as important an adequacy condition on a logicist theory of number as is the derivability of all instances of Tarski's adequacy schema for a semantic theory of truth.[18] Moreover, Schema (C) is all that has to be met by a constructivist theory of number, which is committed to the existence of all natural numbers but not to the existence of any infinite numbers. That Schema (C) is crucial has more recently been emphasized by Dummett as well:[19]

> ...what is constitutive of the number 3 is not its position in any progression whatever, or even in some particular progression, nor yet the result of adding 3 to another number, or of multiplying it by 3, but something more fundamental than any of these: the fact that, if certain objects are counted 'One, two, three', or, equally, 'Nought, one, two', then there are 3 of them. The point is so simple that it needs a sophisticated intellect to overlook it; and it shows Frege to have been right, as against Dedekind, to have made the use of the natural numbers as finite cardinals intrinsic to their characterisation. ...this represents, not a trifling detail, but a fundamental principle, of his philosophy of arithmetic.

Dummett even ventures the view[20] that, had Dedekind's own theory been available to Frege at the time of the *Grundlagen*, then

> Frege's deepest objection to it would have been that it attempted to characterise the totality of natural numbers purely in terms of its internal structure, and relegated their application as finite cardinals to an appendix to the theory.

The purpose of Schema (C) is precisely to heed such an objection, and to make the application of natural numbers as finite cardinals central to our logico-mathematical theory of natural numbers. Schema (C) is an adequacy condition on any theory of numbers and counting, a condition which connects use of the term-forming operator "the number of ...'s" with our use of quantifiers, negation and identity in our numerically non-committal, ordinary discourse about concrete things. Thus it is analogous to Tarski's famous adequacy condition on truth:

⌜S⌝ is true if and only if S

which connects use of the truth predicate and designations of sentences with the use of those sentences themselves. Arguably, (C) is a philosophically deeper condition than the Humean identity (N), at least for finite numbers; since for these (C) implies (N). Significant also is the fact that (C) is completely expressed (albeit schematically) at first order:

$\#xFx = \underline{n}$ if and only if there are exactly n Fs

once granted that the variable-binding term-forming operator $\#x\Phi x$ is a first order expression.[21] By contrast, (N) involves second order quantification on at least one side, insofar as it deals with the equinumerosity of concepts F and G.

6.2 Field's analogy re-visited

Moreover, (C) has no analogue in the case of talk about God.[22] That is, talk about God cannot be made to dovetail via similar equivalences with ordinary talk about concrete things and events in the natural world, without giving extraordinary hostage to non-conceptual fortune. Such equivalences as would be put forward by even the most anodyne theologies would have to be non-analytic. The attempts so far made to link the existence of God to the natural, everyday world (whether or not via 'equivalences' of the kind sought for a proper analogy with the case of talk about numbers) involve appeal to initial acts of cosmic creation, to miracles within an otherwise law-governed natural order, or to contingent 'cosmological' assumptions such as the assumption that there will always be life in the universe.[23] It is a highly non-analytic task to make God make contact with the world as spoken of in 'God'-free language.

Take, for example, the principle (G) above:

The god that created x is identical to the god that created y if and only if x and y are spatio-temporally related.

The existence of just one spatio-temporal thing would be enough to permit us to traverse the biconditional (G) 'right-to-left' and conclude to the existence of a creator god. There is no question but that there exists at least one spatio-temporal thing, and indeed that there exist distinct spatio-temporally related things. A great many of these, including artefacts, have been 'created' by natural processes, of which we can give a god-free account. But perhaps the proponent of (G) suspects that the ultimate wherewithal—fundamental particles, or raw energy—for such mundane events of creation within space and time must itself have had a divine origin. Not even big bang cosmology, however, commits us to there being a moment or event of creation of the physical universe; so, *a fortiori*, does not even pose the need to postulate a god responsible for any such 'event' of creation.[24]

Furthermore, there is not much of a theory to be had about God Himself in the language for talking exclusively about God. God does not seem to have much interesting internal structure. He becomes interesting only when featuring as a posit in various hypothetical interactions with human destiny. And, insofar as this positing might be brought into the form of an equivalence (such as Field's principle (G) above), its claim to analyticity is undermined by the lack of other conceptual, rather than empirical, controls on the positing.

6.3 Core conceptual truths about natural numbers

With the natural numbers, however, matters stand differently. We have the theory of arithmetic, dealing exclusively with the structure of the system of natural numbers, quite apart from their applications in counting. The theory is formulated in a language for talking only about numbers. Indeed, as we know from Gödel's first incompleteness theorem, the whole truth about the additive and multiplicative structure of the natural numbers is so complicated that it cannot be encompassed by finitary, formal means.

But this does not prevent the central core of conceptual truths about the natural numbers from yielding to purely conceptual analysis. Consider the following simple principles for the language containing the constant 0, the successor function sign $s(\,)$, and the term-forming operator $\#x\Phi(x)$ (meaning "the number of Φs"):

Existence of zero
If there are no Fs then $\#xFx = 0$

Ratchet Principle
If $\#xFx$ exists and there is exactly one more G than there are Fs, then $\#xGx$ exists

Principle of Succession
If $t = \#xFx$ and there is exactly one more G than there are Fs, then $\#xGx = s(t)$

From these three principles one can derive all the Peano-Dedekind axioms about natural numbers, including the induction axiom schema. (It should be stressed that the derivations can be carried out in a free logic, where great care is taken not to smuggle in unjustified existential assumptions.[25])

The Ratchet Principle is really toothless ontologically. All it expresses is the thought that if one has gone so far as to acknowledge the existence of any one natural number, then there is no reason to refuse to recognize the 'next' number. That seems reasonable: not even the nominalist opponent wishes to visit on the Platonist a prematurely truncated initial segment of the natural number series, denying the Platonist all (and only) the numbers after some allegedly 'final' one! The Ratchet Principle can be expressed by the rhetorical question 'Wherever you are, why stop there?' There is no reason not to regard the Ratchet Principle and the Principle of Succession as analytic. To the objection that they express 'conditional existence' claims, the reply would be: their antecedents involve existential commitments; and all the principles are doing is drawing out the analytic consequences of entering into such commitment. With the numbers, it's all or nothing, as it were. One cannot have any one of them without having the next one. For part of what being a natural number is, is to occupy the place that it does within the whole series of natural numbers. How could 17 possibly exist without 18 to follow it? Even Field seems to grant this much. Notice how his own existential conditionalization of the claim (*N*2):

> The number of *F*s is identical to the number of *G*s if there are exactly as many *F*s as *G*s

used not the antecedent "If the number of *F*s exists and the number of *G*s exists..." but rather "If natural numbers exist...".

Given the Principle of Succession (hence the Ratchet Principle), one sees that the full brunt of existence for the natural numbers is carried by the principle concerning the existence of zero. So Kronecker was wrong! Kronecker's God need not have given us the integers, before Man did all the rest. He needed to give us only the number 0. We have the analytic Ratchet Principle and Principle of Succession, fleshing out our conception of the natural number series. Man can then do all the rest.

We need therefore to enquire into the analyticity of Existence of Zero.[26] 'How many *F*s are there?' a nominalist may inquire. 'None', might be the reply from the expert on *F*s. The nominalist will understand: it is a contraction of 'Not one', or of 'It is not the case that there is at least one *F*'. There is no talk of numbers yet. Suppose the expert on *F*s proceeds on his way and encounters another inquirer, who chooses to pose the same query as follows: 'What is the number of *F*s?' This second inquirer has chosen to ask the same question, but in the language of numbers. Note that he has not asked 'What, were numbers to exist, would be the number of *F*s?' He wants to know what the number of *F*s is—that is, how many *F*s there are. When he is told 'Zero' (or 'Nought') he is completely satisfied. His query has been answered in an entirely apt way. He has asked for a number, and has been told it. That number tells him exactly how many *F*s there were.

6.4 On incurring commitment to the natural numbers

From within the framework of number-talk, as Carnap would have said, there is no question but that numbers exist: especially the number 0, *primus inter pares*, which is the number of things not identical to themselves. The external question 'Are there numbers?' is answered affirmatively by default. Now Carnap did not envisage the tiresome possibility that a nominalist of Field's persuasion may inhabit that framework in a completely different spirit, explicitly conditionalizing all questions and all claims expressible within that framework on the existence of the very entities that the framework presupposes anyway. It is a precondition of participation in such discourse that one accept that one is thereby committing oneself to the belief that the natural numbers exist. If one does not do so, one is not paying one's dues; one is not dealing in common coin. One is seeking to have all the benefits of participation in the discourse, without any of its attendant commitments.

Note that it is not being claimed that there would be breakdown in mathematical communication between the nominalist and other participants in the discourse. It is not being claimed that the nominalist, while denying in philosophical mood that numbers exist, would fail to secure uptake with assertions such as "Every prime number is smaller than some other prime number". All that is being claimed is that he does not have a properly developed philosophical understanding of what he is up to. There is a dissonance between his practice, on the one hand, and, on the other hand, the verbal formulae by means of which he seeks to maintain his ontological neutrality, or worse, his ontological hostility, towards numbers.

Once the language of arithmetic has been adopted, one has taken on a substantival view of numbers in addition to the adjectival one. The adjectival view is embodied in such locutions as "There are *n* *F*s". The substantival view is embodied in such locutions as "The number of *F*s is $\underline{n}$". By employing the latter, one incurs commitments. These are, first, to the number zero, as the number of non-self-identical things; and, thereafter, to each natural number *n* in turn, as the number of numbers preceding *n*. In any world in which one uses a rich enough first-order language—with the identity predicate, the existential quantifier, negation and the numerical term-forming operator #—one has (on reflection) to acknowledge the existence of zero. For in any such world there are no things that are not self-identical; whence 0 is the number of such things; whence 0 exists. Moreover, any world is such that the conceptual apparatus sustained by such a language *could* be deployed, even if it is not. Thus, even in a world without thinking subjects, if a thinking subject were to start thinking about it (by means of the minimal linguistic resources just described) then the existence of 0 would have to be acknowledged.

Suppose *p* is a proposition which, like the proposition that 0 exists, is logically independent of anyone's being minded of any proposition *q* (entertaining *q*, wondering whether *q* was true, adopting *q* as an assumption, judging that *q* is true...). Thus *p* is unlike the proposition

I am wondering whether I am wondering about anything

whose truth requires me to be wondering whether *Q*, where *Q* is the proposition that I am wondering about something. Once given that

If anyone were to consider whether *p*, they would have to acknowledge that *p*

we can infer that *p*. If anyone were to consider whether 0 exists, they would have to acknowledge that 0 exists. Hence 0 exists.

Why can one maintain the premiss? One can only consider the question whether 0 exists by framing the thought $\exists x(x = 0)$ in a language one of whose sentences can be thus regimented. Moreover, one has to be thinking of 0 as a number, that is, thinking of "0" as a term which, if it denotes anything at all, denotes a number. One cannot be thinking of "0" as a term which might denote some person, or physical object.

Now what about the objection that even if we can be satisfied that numbers (being necessary existents) cannot be physical objects (since these are contingent existents), we nevertheless face the generalized (or abstract) 'Caesar problem' insofar as we still do not know whether 2 is the Zermelo ordinal $\varnothing$ or the von Neumann ordinal $\{\varnothing, \{\varnothing\}\}$?[27] My response to this is that 2 is none of these abstract 'candidates' among which we cannot in principle, supposedly, choose one 'as' (or *to be*) 2. 2 is *sui generis* as a logical object. This is because we can embark on number-talk, governed by its sense-conferring rules, without having any inkling of sets. Our everyday language for talking about concrete things can be extended to enable us to talk about the number of *F*s, for any predicate *F* of our language. Conceptual controls can be placed on the new numerical expressions so that the erstwhile talker about concrete objects can become a talker about concrete objects and about the numbers of concrete objects (and: of numbers!) with such-and-such properties. The structuralist (or practising mathematician) may, if he wishes, 'identify' the number 2 with a particular abstract object occupying the second place of some progression within his set-theoretic hierarchy; he will not end up making any (disguised) false statements 'about' 2 in its relation to other *numbers* by so doing. (The Caesar problem is a metaphysical one, not a mathematical one.)

The question being considered is whether 0, *qua number*, exists. Our answer is that it does. 0 is a very special number; it is the number of any empty concept—in particular, the number of things that are not self-identical. Expressed more formally, 0 is $\#x(\neg x = x)$.

6.5 The meaning of '0': getting something for nothing

When we first learn that 0 is the number of any empty concept (of which $\neg x = x$ is a paradigm), we are grasping, for the first time, the meaning of the term "0". We learn that, if we have a *reductio* of *Fa,* for arbitrary *a* (that is: if there are no *F*s), then 0 is the number of *F*s:

$$\dfrac{\begin{array}{c}\underbrace{\overline{Fa}^{(i)}\qquad\overline{\exists!a}^{(i)}}\\ \vdots\\ \bot\end{array}}{0 = \#xF(x)}{\scriptstyle(i)}$$

We shall call this the rule of 0-introduction. It is 'meaning conferring' for the symbol "0". Anyone considering whether 0 exists, then, is committed to the simple conceptual truth

$$(S_0)\quad 0 = \#x(\neg x = x)$$

from which it follows that 0 exists. It therefore follows that 1 exists, for 1 is the successor of 0, namely the number of things identical to 0. It therefore follows that 2 exists, for 2 is the successor of 1, namely the number of things identical either to 0 or to 1...and so on.

A predictable objection from the nominalist must now be anticipated and defused. The objection is: if one is to accept this argument for the existence of 0, namely the number of non-self-identical things, why should one not accept an analogous argument for the existence of Pegasus, the winged horse?

The reply to this objection is that there is too great a disanalogy between the role of the rule of 0-introduction and the role of whatever rule would be proposed as 'meaning conferring' for the term "Pegasus". Note that our rule of 0-introduction makes 0 the number of Fs where F is *any* empty concept. But is there any introduction rule with conclusion "Pegasus = ...", where the right hand side is filled out with a similarly schematic and general term? It would appear not. The name "Pegasus", if analytically equivalent to any term formed by means of variable binding operators, will be equivalent to some particular descriptive term, in which the primitive predicates are not schematic. The best the opposition could accordingly do here would be to give a rule for some 'analytically true' conclusion such as "Pegasus is the winged horse":

$$\dfrac{W(\text{Pegasus}) \qquad H(\text{Pegasus}) \qquad \begin{array}{c} \underbrace{\overline{W(a)}^{\,(i)} \quad \overline{H(a)}^{\,(i)} \quad \overline{\exists ! a}^{\,(i)}} \\ \vdots \\ a = \text{Pegasus} \end{array}}{\text{Pegasus} = \iota x(W(x) \wedge H(x))}\,(i)$$

and it is clear from this rule that the conditions that have to be met in order to justify the assertion "Pegasus $= \iota x(W(x) \wedge H(x))$" involve, among other things, proof that one is in a position to assert the atomic predications "W(Pegasus)" and "H(Pegasus)". But then Pegasus must have been shown to exist!—for how else would one be able to assert these atomic predications? And if the opposition objects to the rule just proposed on their behalf, let them provide a better rule, if they can, that is free of the defect just identified. But they will not succeed. So, by contrast with any non-denoting term of fiction, mythology or bad science, 0 is in decidedly good standing.

Having failed, then, to confute these claims on behalf of 0 by getting Pegasus in as a stablemate, our opponent will no doubt still object that we cannot help ourselves to the simple 'conceptual truth' (S_0) unconditionally. Rather, we are entitled only to the conditional claim

($S_{\exists 0}$) if 0 exists then $0 = \#x(\neg x = x)$.

This, the opponent will say, is sufficient commitment on the part of the would-be user of the language of arithmetic, even when that language involves the operator

$\#x(...x...)$. $(S_{\exists 0})$ is enough of an undertaking. To assert $(S_{\exists 0})$ shows enough respect for the rules of the language for substantival talk of numbers, even equipped as it is with its conceptual controls on counting. To ask him to assert the consequent (S_0) unconditionally would be to ask too much.

Our opponent is asking for special dispensation, therefore, to condition all of our arithmetical talk on the assumption that 0 exists. In particular, he wants the rule of 0-introduction to take this conditionalized form, where $\exists!0$ means $\exists x(x = 0)$:

$$\dfrac{\exists!0 \qquad \begin{array}{c} \underbrace{\overline{Fa}^{\,(i)} \quad \overline{\exists!a}^{\,(i)}} \\ \vdots \\ \bot \end{array}}{0 = \#xF(x)}\,(i)$$

It does not matter if at times he does not make that assumption $\exists!0$ explicit in his arithmetical theorizing, or in his applications of theorems of arithmetic to derive consequences of various empirical beliefs. That 0 exists is nevertheless to be regarded as an assumption conditioning all that he says. He does not assert that assumption; he merely exploits what follows from it, that is, what would be true if the assumption were true.

But this is completely disingenuous; and leads to a peculiar sort of incoherence. For, consider how this nominalist opponent has to regard ordinary arithmetical theorems *P*. He has to regard any such theorem *P* not as assertable outright, but rather as deducible, by means of this existentially conditioned rule of 0-introduction, from the premiss that 0 exists; or else he has to regard conditionals of the form "if $\exists!0$ then *P*" as assertable outright on the basis of those same rules. Now, however, let us have the nominalist express his nominalism in the same breath. Let us have him say "0 does not exist". This existential denial is one of his beliefs. From this belief $\neg\exists!0$ it follows logically, for any arithmetical statement *P* (indeed: for any statement *P*) that if $\exists!0$ then *P*. So he will be at a loss to delineate just which *P* are to count, by his lights, as genuinely assertable arithmetical claims. If he eschews the formulation according to which it is a conditional "if $\exists!0$ then *P*" that is asserted, and reverts instead to the (relevant) deducibility of *P* from the premiss $\exists!0$, then he stands open to the challenge to explain why he chooses that (by his lights, false!) premiss with which to start his mathematical reasoning. He will also face the embarrassing circumstance of having every application of arithmetical truths in reasoning about the empirical world produce empirical conclusions depending not only on whatever empirical assumptions are involved, but also on the assumption that 0 exists![28]

The only way out would be for the nominalist to refrain from denying the existence of 0, and just to refuse to be drawn into asserting it.

Against this tactical disavowal of full-blooded commitment to the objects about which we claim to be able to speak, one can advance an objection from the constructivist theory of meaning. We do not master the term "0" by being allowed the sly philosophical misgiving that 0 might not exist (or worse: that 0 necessarily does not exist). For if we were to condition the introduction rule in the way indicated on the further existence assumption $\exists!0$ then we would not be introducing 0 for the first time, enjoying just one salient occurrence in a conclusion that has the most general schematic form of identity statement possible, namely $0 = \#xF(x)$.

The constructivist has a high-level theory of the canonical inferential contexts that ultimately confer meanings on our logico-mathematical expressions.[29] These canonical contexts are the introduction rules (for warranted assertions) and the elimination rules (for warranted denials).[30] These sense-conferring inferential patterns involve just one occurrence at a time of the expression on which they confer sense. They allow one to *attain to* an understanding of the expression without having to presuppose that understanding within the 'subordinate materials' of the canonical pattern of inference.

The rule of 0-introduction, for example, does not tell the learner only what thing 0 would be provided only (perhaps *per impossible*, if the extreme nominalist were to have his way) that 0 were to exist. Rather, the rule of 0-introduction tells the learner what 0 really *is.* And once he starts speaking the language in which the term "0" is a primitive expression, governed by that rule, he should appreciate that what 0 is it is *of necessity*—the number of any empty concept, particularly of any concept (such as non-self-identity) that is necessarily empty.

It is a demand of reflective equilibrium that the sly ontological disclaimers just mentioned in connection with the 'existentially conditioned' rule of would-be '0-introduction' not be allowed to play a rôle in the account of our grasp of 0, $s(\,)$ and $\#x(...x...)$. The extra conditional assumption(s) demanded by the disclaimer require us already to grasp that to whose grasp the rule is supposed to allow us to attain. Thus they are a gross deviation from what is otherwise a smooth theoretical pattern, a pattern which is true to the communicative commitments of the competent language-user. Rather than have the nominalist's ontological scruples botch the theory, the theory should be allowed to disabuse the nominalist of those scruples.

The Dummettian anti-realist, or intuitionist, maintains that a proper theory of meaning, treating of the conditions for correct assertion and the inferential commitments undertaken by asserters, can issue in an injunction to reform certain logical practices revealed by the theory as devoid of justification. Such a theory of meaning, we now see, can also issue in an injunction that we stand by the manifest ontological commitments arising from our ways of speaking and thinking, insofar as these are ways that can be imparted and acquired in a principled and orderly fashion.

The conception of analyticity at work here is one that is shorn of the usual dogma that an analytic statement can involve no ontological commitments. It is analytically true that $0 = \#x(\neg x = x)$, and this in turn implies that 0 exists. To grasp the meaning of "0" is to know what number it denotes, not merely to know that it would denote such-and-such a number provided only that that number existed.

This line of argument, followed through, leads one also to the necessary existence of the natural numbers, at least if necessity is construed as truth in every possible world about which predicative and quantificational thought is possible, or within which such thought is possible about it. For, once such thought is possible, so is its extension (if it is not yet extensive enough) to thought about numbers. One can introduce the expressions 0, $s(\,)$ and $\#x(...x...)$ by way of conservative extension of any pre-existing language containing identity, negation and predication. The necessary possibility of doing so guarantees the necessary existence of the natural numbers.

7 Dispensability for physical theorizing proves nothing

Field believes that it is conceptually contingent whether numbers exist. Their existence cannot be settled on purely conceptual grounds, such as those above involving necessary equivalences within our language. Rather, he claims, "other grounds are required". The most important "other grounds" that he envisages would be an argument that established that our talk about numbers is "extremely useful"; more strongly, that such talk is theoretically indispensable. As Field describes his own programme in *Science without Numbers*, one undermines the indispensability claim by showing how each and every physical-and-mathematical theory is but a conservative extension of the synthetic physical theory that eschews the mathematics concerned. Now Field's argument, if successful, is a purely conceptual one itself;[31] and should therefore, by his own lights, establish, if not the conceptual impossibility of the numbers themselves, then at least the conceptual impossibility of establishing their conceptual necessity. So we would appear to have

(a) if numbers are dispensable in science, then it is conceptually impossible to establish the conceptual necessity of numbers.

But it should be an a priorist article of faith that

(b) if numbers are conceptually necessary, then it must be conceptually possible to establish their conceptual necessity!

From (a) and (b) it would appear to follow that there is an inconsistency in maintaining both

(1) numbers are dispensable in science, and
(2) the existence of numbers is conceptually necessary:

$$\dfrac{\begin{array}{c}\text{(1) numbers are dispensable in science}\\ \vdots\text{(a)}\\ \neg\Diamond(\vdash\Box(\text{numbers exist}))\end{array}\qquad\begin{array}{c}\text{(2) }\Box(\text{numbers exist})\\ \vdots\text{(b)}\\ \Diamond(\vdash\Box(\text{numbers exist}))\end{array}}{\bot}$$

Field, accepting (a) and (b), holds (1); accordingly, he rejects (2). It would appear, likewise, that anyone holding (2) would have to reject (1). But this is not so. It is open to one who holds (2) to concede (1) also, by refusing to accept the argument (a). Indeed, (a) is maintained only on the shakiest of grounds. The proponent of (a) would seem to believe that the only way to establish the conceptual necessity of numbers is to show that numbers are indispensable in science. But not only is this not the *only* way, it is not even *a* way!

There is no point in becoming embroiled in speculative argument as to what, exactly, Field takes himself to be accomplishing, and what, exactly, he would actually be accomplishing were his dispensability arguments to go through.[32] Instead, let us try here a different tack on behalf of the necessity theorist. If the dispensability claim is correct, so what? Why should necessary existents have to be appealed to when explaining natural phenomena? Until this simple question receives a satisfactory answer, the nominalist goes begging for an argument (a), and there is no easy route to the alleged inconsistency between (1) and (2). Field is inferring from the thought that it is not necessary for us to use the numbers, to the thought that it is not necessary for the numbers to exist. Indeed, he goes further—to the thought that, although they might have existed, the numbers nevertheless do not actually exist. Field's conclusion is that the existence of natural numbers is contingent. Nothing could be clearer than that his inference to this conclusion (from the alleged dispensability of natural numbers for physical theorizing) is fallacious.

8 The ambivalent modal status of surdons

8.1 Some notions defined

Hale and Wright[33] seek to raise problems for the contingency theorist about numbers by pointing out that the contingency contemplated is "absolutely insular", in that it is both *barren* and *brute.* A *brute* contingency q by definition *violates* this principle:

($P_1(q)$) q will be contingent on something—there will be circumstances on which q's obtaining depends (and whose citation can thus contribute to its explanation).

A *barren* contingency by definition *violates* this principle:

($P_2(q)$) q will have things contingent on it—there will be circumstances whose obtaining depends on q (would be otherwise if q did not obtain).

Thus an *absolutely insular contingency* is a proposition q such that

$$(\Diamond q \wedge \Diamond\neg q) \wedge (\neg P_1(q) \wedge \neg P_2(q)).$$

Let us use the following abbreviations:

$C(q) =_{df} (\Diamond q \wedge \Diamond\neg q)$, meaning "$q$ is contingent";
$AI(q) =_{df} (\neg P_1(q) \wedge \neg P_2(q))$, meaning "$q$ is absolutely insular".

Hale and Wright contend that

> The conception that mathematical objects exist or not as a matter of strong contingency...clashes not merely with the presumption that contingencies should be contingent on something but, equally, with the converse principle that bona fide contingencies are presumptively *things on which other bona fide contingencies are contingent.* ...The suggestion is, then, that [principles ($P_1(q)$) and ($P_2(q)$)] constrain the ascription of strong contingency [to q].

The suggestion, then, is that there are no absolutely insular conceptual contingencies. Let us abbreviate this claim as

$$I =_{df} \neg\exists q(C(q) \wedge AI(q)).$$

Note that by a 'contingency' one is to understand a claim or proposition that is contingent, not an entity or class of entities whose existence is contingent. Thus the existential quantification in the formula is over propositions. Hale and Wright offer no compelling argument, however, for principle (I), apart from the observation that

> ...the notion that there can be absolutely insular contingencies...is at odds with the idea that the realm of contingency forms a single integrated system—a tree-like structure in which the connecting links are dependencies and in which every node is linked to others.[34]

Field's reply to this is to introduce the concept of a *surdon.*[35] A thing x is a surdon if and only if

(Ax) the existence and state of x are in no way dependent on the existence and state of anything else; and

(Bx) the existence and state of nothing else [is] in any way dependent on the existence and state of x.

Let us use the following abbreviations:

$Sx =_{df} (Ax \wedge Bx)$, meaning "x is a surdon";
$\Sigma =_{df} \exists x Sx$, meaning "Surdons exist".

8.2 Repairing Field's modal argument

Field writes

> This certainly seems to be a conceptually consistent concept; but [*Ax*] and [*Bx*] guarantee insularity, so principle [(*I*)] immediately guarantees the existence of surdons—indeed, the conceptual necessity of their existence. ...even (Hale and Wright) should balk at the idea that establishing the existence of mathematical entities is as easy as this!

Hale and Wright's response[36] to Field's brief argument here is to accept it, but then render it toothless by running a parallel argument from the "presumed conceptual consistency of the thought that surdons do not exist" to the conclusion that "the existence of surdons, so far from being a conceptual necessity, is actually *impossible!*" They ask one to "reflect that the aura of conceptual consistency given off by the supposition that surdons exist is perfectly matched by the supposition that they do not."

We shall seek to make completely clear just what the logical structure of these opposed positions is; for both sides present their arguments rather swiftly, and very informally. A more detailed logical analysis will be valuable in diagnosing exactly how the two sides have reached a stand-off.

Field's argument had the following form:

There are no absolutely insular conceptually contingent claims
'Surdon' is a conceptually consistent concept
[*Ax*] and [*Bx*] guarantee (absolute) insularity
ergo, Necessarily surdons exist

This is as far as Field spells out the argument. In our logical notation, the first two premisses of Field's argument would read

$\neg\exists q(C(q) \wedge AI(q))$ (or, more briefly, I)
$\Diamond\exists xSx$ (or, more briefly, $\Diamond\Sigma$)

and the conclusion would read

$\Box\exists xSx$ (or, more briefly, $\Box\Sigma$).

But how should we render the third premiss of Field's argument?

"[*Ax*] and [*Bx*] guarantee absolute insularity"

—of what, precisely? It cannot be of surdons themselves—for that would be a category mistake, since it is propositions, not entities, of which absolute insularity is correctly predicated. It must be the claim, then, that surdons exist. So the third premiss, less elliptically, should read

"[*Ax*] and [*Bx*] guarantee absolute insularity of the claim that surdons exist"

But the claim that surdons exist is the claim Σ. So the third premiss reads

[*Ax*] and [*Bx*] guarantee that $AI(\Sigma)$
i.e. $(Ax \wedge Bx) \supset AI(\Sigma)$
i.e. $Sx \supset AI(\Sigma)$

The variable *x*, however, is still free. How should we turn this into a closed sentence apt for truth-bearing? There are really only two ways, both of them equivalent, which can be read in English as "the existence of surdons is absolutely insular":

$\forall x(Sx \supset AI(\Sigma))$; or
$(\exists xSx) \supset AI(\Sigma)$, i.e. $\Sigma \supset AI(\Sigma)$

We shall opt for the latter regimentation, and abbreviate it further as *J*. Our question now is whether Field's argument, sympathetically regimented as follows, is valid:

$$\neg\exists q(C(q) \wedge AI(q))$$
$$\Diamond\Sigma$$
$$\underline{\Sigma \supset AI(\Sigma)}$$
$$\Box\Sigma$$

But unfortunately this argument is not valid, even when extra logical structure is supplied by expanding all the abbreviations we have introduced. For the argument to be valid, it needs to have its first and third premisses strengthened to their necessitations. Since the first premiss is a metaphysical principle of great generality, and the second premiss is a definitional truth, neither side to the debate should object to such strengthening. So we are dealing now with the necessitated principles

$\Box\neg\exists q(C(q) \wedge AI(q))$ (or, more briefly, $\Box I$)
"There can be no absolutely insular contingencies"
$\Box(\Sigma \supset AI(\Sigma))$ (or, more briefly, $\Box J$)
"The existence of surdons would have to be absolutely insular"

These two modally strengthened principles are inconsistent with the claim that Σ is contingent. That is, the following set of sentences is inconsistent, on the basis of the logical structure therein revealed and our definition of C:

$\Box\neg\exists q(C(q)\wedge AI(q))$
$\Box(\Sigma\supset AI(\Sigma))$
$\Diamond\Sigma$
$\Diamond\neg\Sigma$

The proof of this inconsistency in the modal logic S5 is as follows:[37]

$$\dfrac{\Diamond\Sigma \qquad \dfrac{\dfrac{\dfrac{\dfrac{\Diamond\Sigma \quad \Diamond\neg\Sigma}{C(\Sigma)} \qquad \dfrac{\overset{(1)-}{\Sigma} \quad \dfrac{\Box(\Sigma\supset AI(\Sigma))}{\Sigma\supset AI(\Sigma)}}{AI(\Sigma)}}{C(\Sigma)\wedge AI(\Sigma)}}{\exists q(C(q)\wedge AI(q))} \qquad \dfrac{\Box\neg\exists q(C(q)\wedge AI(q))}{\neg\exists q(C(q)\wedge AI(q))}}{\bot}}{\bot}\,(1)$$

Note that two premisses of this *reductio* have $\Box$ dominant, and two have $\Diamond$ dominant. Let Δ be the set consisting of the two premisses that have $\Box$ dominant.

8.3 Two ways of looking at the reductio

Depending now on which of the $\Diamond$ premisses of the *reductio* one then chooses to discharge for negation introduction, one will obtain a proof either of the valid argument closest in spirit to the invalid one actually given by Field:[38]

$$\dfrac{\begin{array}{l}\Delta\\ \Diamond\Sigma\end{array}}{\Box\Sigma}$$

or of the parallel argument of Hale and Wright:

$$\dfrac{\begin{array}{l}\Delta\\ \Diamond\neg\Sigma\end{array}}{\neg\Diamond\Sigma}$$

Our *reductio* establishes

$\Delta,\ \Diamond\Sigma,\ \Diamond\neg\Sigma \vdash \bot$

Which of the two possibility claims is to be given up? By taking two more steps Field's argument establishes, from Δ, that

if $\Diamond\Sigma$ then $\Box\Sigma$

(in a nutshell: if surdons could exist then they must exist). Hale and Wright's argument likewise establishes, from Δ, but *via* a different couple of steps, that

if $\Diamond\neg\Sigma$ then $\neg\Diamond\Sigma$ (hence $\Box\neg\Sigma$)

(in a nutshell: if surdons might not exist then they cannot exist). Each side, as pot, calls the kettle black. The mutual complaint is that the other side is forced into an unseemly rush from a possibility to a necessity.

That there can be no absolutely insular contingencies ($\Box I$) and that the existence of surdons would have to be absolutely insular ($\Box J$) ensure that surdons exist necessarily if possibly (Field), and that they cannot exist if they might not (Hale and Wright).

Even if the platonist thinks numbers are surdons, this is no embarrassment; indeed, it is as it should be. Field sees the proven passage to

$$\Box\exists x(x \text{ is a surdon})$$

from the anti-insularity principle, the definitional truth about surdons, and the possibility claim $\Diamond\exists x(x \text{ is a surdon})$, as embarrassingly swift. But his very proof tells us that establishing the necessary existence of surdons is no harder than establishing their possible existence!—so why should he think he can help himself so easily to the premiss

$$\Diamond\exists x(x \text{ is a surdon})?$$

Moreover, since Field clearly thinks that surdons are possible but not necessary, the inconsistency proof really leaves him with no other option but to conclude further to the falsity of the anti-insularity principle:

$$\Box J,\ \Diamond\Sigma,\ \Diamond\neg\Sigma \vdash \neg\Box I$$

Field could then assert the remaining (undischarged) premisses $\Box J$, $\Diamond\Sigma$ and $\Diamond\neg\Sigma$; but, if he does so, he is obliged to assert the conclusion $\neg\Box I$ which follows from them. This position for Field, then, elaborating on our inconsistency proof, would be:

$\Box J$: the existence of surdons would have to be absolutely insular
$\Diamond\Sigma$, $\Diamond\neg\Sigma$ } the existence of surdons is contingent
ergo, $\neg\Box I$: there could be absolutely insular contingencies.

Hale and Wright, on the other hand, would have to opt for the following divergent elaboration of our inconsistency proof:

$$\Box I,\ \Box J,\ \Diamond\Sigma \vdash \neg\Diamond\neg\Sigma, \text{ i.e. } \Box\Sigma$$

because they are happy with the anti-insularity principle $\Box I$, and believe in the actual (hence possible, hence necessary) existence of surdons—e.g., the natural numbers. This position for Hale and Wright, then, would be:

$\Box I$: there could not be any absolutely insular contingencies
$\Box J$: the existence of surdons would have to be absolutely insular
$\Diamond\Sigma$: it is possible that surdons exist
ergo, $\Box\Sigma$: it is necessary that surdons exist.

This argument is a highly plausible reconstruction of their position. Hale and Wright, on this analysis, agree with Field on two premisses, namely

$\Box J$: the existence of surdons would have to be absolutely insular
$\Diamond\Sigma$: it is possible that surdons exist.

One would not expect any disagreement here, since the concept of surdon has been so defined as to secure the truth of both these claims. If we suppress these common premisses, the two opposing positions can be summarized as follows:

Field:

$\Diamond\neg\Sigma$: it is possible that no surdons exist
ergo, $\neg\Box I$: there could be absolutely insular contingencies

Hale and Wright:

$\Box I$: there could not be any absolutely insular contingencies
ergo, $\Box\Sigma$: it is necessary that surdons exist

One man's *modus ponens*, then, turns out to be two men's *modus tollens.*

From a constructivist point of view, Hale and Wright's acceptance of

$\Diamond\Sigma$: it is possible that surdons exist

as a premiss would need much more honest toil than Field is willing to contemplate; so much, in fact, that the conclusion

$\Box\Sigma$: it is necessary that surdons exist

would be justified as soon as the hard day's work was done—provided only that one accepted the general principle that Field rejects, namely that there could not be any absolutely insular contingencies. It would be purely conceptual work, of the kind that delivers diamonds only in boxes.

As always in metaphysical debates, we have reached a logical stand-off. Our *reductio* gives a perfect example of apory: a few individually plausible claims that turn out (perhaps surprisingly at first) to be jointly inconsistent. What one chooses consistently to assert then determines what one has consistently to deny. The real work then remains to be done: to find some independent justification for what one chooses consistently to assert.

Unfortunately, this will not be found in the debate between Field, on the one hand, and Hale and Wright, on the other. The conclusion to be drawn is that Hale and Wright should have availed themselves of the modal line of thought developed above, that ran from considerations of reflective equilibrium and the necessary possibility of numerical thought, to the necessary existence of numbers. The theory of meaning set out above for a language for pure and applied arithmetical thought provides the best arguments available for both the existence and the necessity of numbers.

Notes

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1. It is a dogma of logical empiricism, and of the analytical philosophy of logic and language that has followed it, that no analytic statements could carry any existential commitments. But they can and do. For a more fully developed challenge to the dogma, in the light of a survey of the history of the analytic/synthetic distinction, see Chapter 9 of my book *The Taming of The True*, Oxford University Press, 1997.

2. 'The Conceptual Contingency of Mathematical Objects', *Mind*, 102, 1993, pp. 285–299.

3. I owe this point to Timothy Smiley.

4. A good candidate is intuitionistic relevant logic. See my book *Autologic*, Edinburgh University Press, 1992, and my paper 'Intuitionistic Mathematics Does Not Need *Ex Falso Quodlibet*', *Topoi*, 1994, pp. 127–133.

5. That this is so is apparent from the refusal, on the part of a nominalist such as Field, to take himself as committed to the existence of numbers simply because of his (quasi-assertoric) use of statements of number theory. Such a nominalist is not really holding these statements true. Rather, he seeks to explain his use of them solely in terms of their ulterior usefulness in enabling him to make genuinely true (hence ontologically committing) statements about physical objects in space and time. Such a nominalist is not whole-heartedly adopting numerical discourse.

6. The derivability of all instances of Schema (C) was first imposed as an adequacy condition on a logicist theory of number in my book *Anti-Realism and Logic*, Clarendon Press, Oxford, 1987, at p. 234. There, however, it was called Schema (N). In this paper we shall reserve the label (N) for 'Hume's Principle', which is different (see below).

7. Wright C., 'Why Numbers Can Believably Be: A Reply to Hartry Field', *Revue Internationale de Philosophie*, 42, 1988, pp. 425–473; Hale, R. and C. Wright, 'Nominalism and the Contingency of Abstract Objects', *Journal of Philosophy*, 89, 1992, pp. 111–135; and Hale, R., 'Dummett's Critique of Wright's Attempt to Resuscitate Frege', *Philosophia Mathematica*, 3, 1994, pp. 122–147.

8. That we need a free logic is most obvious when one uses variable-binding term-forming operators as primitive in the language. Descriptive terms such as "the F", regimented as ιxFx, and set-terms such as "the set of all Fs", regimented as $\{x|Fx\}$, spring immediately to mind here. If we allow that such terms are grammatically well-formed provided Fx is, then we have to allow in general for failure of denotation on the part of singular terms. This is because of such descriptive terms as $\iota x(Px \wedge \neg Px)$ and the notorious set-term $\{x|\neg x \in x\}$. A free logic for handling precisely these sorts of terms, and respecting the Russellian analysis of the truth conditions of atomic predications, can be found in *Natural Logic*, Edinburgh University Press, ch. 7.

9. *loc. cit.*, p. 287. See also Wright's acceptance of this at p. 80 of 'Field and Fregean Platonism', in A.D. Irvine, ed., *Physicalism in Mathematics*, Kluwer, 1990, pp. 73–93, and at p. 450 of the overlapping paper, 'Why Numbers Can Believably Be', *loc. cit.*

10. *loc. cit.*, p. 287; Field's emphasis.

11. The only other route is to treat the modal operators for necessity and possibility as primitive in the object language, without offering any semantical treatment of them in the metalanguage—which is what Field does, even though he is not arguing for the view that the natural numbers *could not* exist.

12. at pp. 84–85 of 'Field and Fregean Platonism'.

13. *loc. cit.*, p. 179.

14. Immediately after the passage quoted from Wright in the text above, he says that the gap narrows when we move back from the case of directions to the case of natural numbers. It will emerge that this is an over-hasty and misguided concession to the nominalist of Field's persuasion. The concession is made because Wright overlooks the further conceptual controls that are available in the case of numbers.

15. Alex Oliver raised the point that in *Grundlagen* §75 Frege says he can prove that 'if there are no *F*s then the number of *F*s is 0', and asked how Frege's proof depends on anything more than the equivalence principle (N). The answer is that Frege needs the existence of zero. Frege's second-order logic is not a free logic, so he gets 0 for free!

16. See, for example, Field, *Science without Numbers*, at p. 21.

17. *Cf.* Dummett, *Frege: Philosophy of Mathematics*, Duckworth 1991, pp. 43–44.

18. See footnote 6 above.

19. *Frege: Philosophy of Mathematics*, p. 53. Note, however, that Dummett does not state the general adequacy condition on a theory of number, nor indicate how it may be met.

20. *op. cit.*, p. 61.

21. Note that we regard the descriptive term-forming operator and the set term-forming operator as first-order expressions when they are taken as primitives in the language. There is accordingly no reason why we should not regard the numerical term-forming operator as a first-order expression when it is taken as a primitive.

22. *This* is how Hale and Wright ought to have countered Field's analogy between the Fregean equivalence principle for numbers, and the Ontological Argument.

23. This assumption is made at the outset of the most recent cosmological argument for God's existence (*and* for our future bodily resurrection in a Judeo-Christian heaven!). See Frank J. Tipler, *The Physics of Immortality*, Doubleday, 1994. Even when we are blinded by the science of an evangelical physicist, we cannot lose sight of the fact that the theological claims involved are at best dubious interpretations of certain cosmological hypotheses: hypotheses that are part of a conjectural physical theory that is advanced as the result of an attempted inference to the best explanation of all our observations. Such a theory, if true, is only contingently true. There is no succour in any of this for one who wishes to regard Field's principle (G) as analytic—as Field himself would no doubt agree.

24. For a convincing case for atheistic quietism over the Big Bang, see A. Grünbaum, 'Origin versus Creation in Physical Cosmology', in L. Krüger and B. Falkenburg, eds., *Physik, Philosophie, und die Einheit der Wissenschaften*, Spektrum Akademischer Verlag, 1995, pp. 221–254.

25. The derivations are in *Anti-Realism and Logic, op. cit.*, ch. 25. It should be stressed that they are all contained within intuitionistic relevant logic. Moreover, one can derive every instance of Schema (C). Note that the Ratchet Principle follows from the Principle of Succession.

26. It is rather amusing that the notation for zero was introduced only after notation for non-zero natural numbers (by the ancient Hindus). Once again, the order of discovery does not match the order of logical justification.

27. Stewart Shapiro raised this objection.

28. Note that for the theorist who maintains the necessary existence of 0, this would be no embarrassment. His opponent, however, in order to avoid the embarrassment, would have to ensure conservative extension results of the following form:

if Σ is our synthetic (mathematics-free) scientific theory, and M is our mathematical theory, then for all synthetic sentences ϕ, $\Sigma, M, \exists!0 \vdash \phi \Rightarrow \Sigma \vdash \phi$.

29. See the treatment in *Anti-Realism and Logic*, Chapters 10 and 13.

30. *Cf. The Tanning of The True, op. cit.*, ch. 12, for a rule-based constructive falsifiability semantics.

31. The success of any mathematical-cum-physical theory is of course an empirical matter. But the question whether it conservatively extends the purely physical theory is not.

32. Whether they go through, indeed, is a fraught question, and one which must lie beyond the scope of this paper. Would it be enough for Field's project, for example, simply to establish a conservative extension result for each physical theory that we ever actually develop?—or would he need a more general result, to the effect that mathematics conservatively extends every possible physical theory (whether or not it be true of our world)? Stewart Shapiro has shown (in 'Conservativeness and Incompleteness', *Journal of Philosophy*, 80, 1983, pp. 521–531) that Field's 'platonistic' system is not deductively conservative over nominalist physics. The latter is rich enough to permit formulation of surrogates for the natural numbers, and therefore also an independent Gödel-sentence. The latter is in the nominalistic language; but is provable in the platonistic theory, since the nominalistic theory can be modelled in the set theory that the platonist adjoins.

33. in 'Nominalism and the Contingency of Abstract Objects', p. 133.

34. *loc. cit.*, p. 134.

35. 'The Conceptual Contingency of Abstract Objects', pp. 296–7.

36. 'A Reductio Ad Surdum? Field on the Contingency of Mathematical Objects', §IV.

37. Note that we use the S5 rule of $\Diamond$-elimination at the final step, with major premiss $\Diamond\Sigma$. The subordinate proof for that step has, among its undischarged assumptions other than Σ, ones with $\Diamond$ dominant as well as ones with $\Box$ dominant. This step would not be correct in S4, which requires all such undischarged assumptions to have $\Box$ dominant.

38. Henceforth, we shall write as though the valid version given here is to be credited to Field.

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