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NATURAL DEDUCTION AND SEQUENT CALCULUS FOR INTUITIONISTIC RELEVANT LOGIC

NEIL TENNANT

Relevance logic began in an attempt to avoid the so-called fallacies of relevance. These fallacies can be in implicational form or in deductive form. For example, Lewis's first paradox can beset a system in implicational form, in that the system contains as a theorem the formula $(A \& \sim A) \rightarrow B$; or it can beset it in deductive form, in that the system allows one to deduce B from the premisses $A, \sim A$.

Relevance logic in the tradition of Anderson and Belnap has been almost exclusively concerned with characterizing a relevant *conditional*. Thus it has attacked the problem of relevance in its implicational form. Accordingly for a relevant conditional \rightarrow one would not have as a theorem the formula $(A \& \sim A) \rightarrow B$. Other theorems even of minimal logic would also be lacking. Perhaps most important among these is the formula $(A \rightarrow (B \rightarrow A))$. It is also a well-known feature of their system R that it lacks the intuitionistically valid formula $((A \vee B) \& \sim A) \rightarrow B$ (disjunctive syllogism).

But it is not the case that any relevance logic worth the title even has to concern itself with the conditional, and hence with the problem in its implicational form. The problem arises even for a system without the conditional primitive. It would still be an exercise in relevance logic, broadly construed, to formulate a deductive system free of the fallacies of relevance in deductive form even if this were done in a language whose only connectives were, say, $\&$, \vee and \sim . Solving the problem of relevance in this more basic deductive form is arguably a precondition for solving it for the conditional, if we suppose (as is reasonable) that the relevant conditional is to be governed by anything like the rule of conditional proof. To assert the relevance conditional $A \rightarrow B$, one will have to be able *relevantly to prove* B from A ; and characterizing the notion of relevant deduction appealed to here is no more than what I have called the problem of relevance in its deductive form.

So it is the problem in its deductive form to which the present paper is addressed. Moreover, as the title indicates, I am concerned with relevantising *intuitionistic* logic. I do so by applying a method already developed for relevantising *classical* logic in Tennant [1984].

But first, a word on notation. If X is a finite set of sentences and A is a sentence, $X:A$ will be the *sequent* or argument with premisses X and conclusion A . If a system

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is said to contain $X:A$ then this means that A is deducible from X in the system. I shall be considering systems of *natural deduction* and of *sequent proof*. A natural deduction system contains $X:A$ if and only if there is a natural deduction in the system whose conclusion is A and whose undischarged assumptions form the set X . A sequent system contains $X:A$ if and only if there is a proof in the system whose bottom sequent is $X:A$. It is important to understand that what is commonly known as the rule of *cut* is a rule that belongs only to sequent systems. It has no place at all in a system of natural deduction. The rule of cut for a sequent system directly expresses and ensures the unrestricted transitivity of deduction within that system. For example, the rule of cut for intuitionistic logic is the sequent rule

$$\frac{X:A \quad A, Y:Z}{X, Y:Z}, \quad Z \text{ empty or a singleton.}$$

In a system with such a rule of cut any two sequent proofs of the top two sequents can be brought together, with a terminal application of the rule, so as to form a sequent proof of the bottom sequent in the rule.

By contrast, for a natural deduction system (such as for minimal, intuitionistic or classical logic), in which cut is not a rule, the unrestricted transitivity of deduction amounts just to this: if one has a deduction D' of B from Y, A and a deduction D of A from X , then the result of grafting D on top of undischarged occurrences of the assumption A in D' is a deduction, in the system, of B from X, Y :

$$\begin{array}{c} X \\ D \\ Y, (A) \\ D' \\ B \end{array}$$

An important part of the investigations in this paper will concern how to contain the loss of transitivity in a sequent formulation if one gives up the *unrestricted transitivity* ensured by the cut rule; and, correlatively, how to contain the loss of transitivity in a natural deduction formulation if one defines natural deductions so that one no longer has the guarantee in general that the process of accumulating deductions as just described always results in something that counts as a deduction in the new system. This would be the case, for example, if it were a requirement in the new system that all deductions be in *normal form*. A deduction is in normal form if and only if it contains no sentence occurrence standing both as the conclusion of an introduction rule and as the major premiss of the corresponding elimination rule. An occurrence of this kind is said to be *maximal*. In the last figure, for example, some occurrence of A at a point of grafting might be maximal; and the figure accordingly would not count as a well-formed deduction in a system with the normality constraint just described. In the natural deduction system for intuitionistic relevant logic to be given below, we shall be imposing just such a normality constraint. Reasons for doing so will emerge in due course. The normality constraint is just one of the reforms that I shall be pursuing for the sake of relevance. Similar reforms will be undertaken in the sequent system.

The main result of this paper is that the resulting reformed systems agree on their deducibility relations. This is evidence that the motivation behind the reforms is natural. Furthermore I shall prove results that show that the amount of transitivity surrendered in the process is of minor consequence, compared to the relevantist epistemic gains to be had from its local failures.

In order to motivate my approach further, and to contrast it with that of Anderson and Belnap and others, it is worth looking at a simple “proof” of the first Lewis paradox in its deductive form.

The proof purports to show that one can deduce any conclusion B from the inconsistent set of premisses $A, \sim A$. Informally, it runs as follows:

1. Assume A .
2. Then, by \vee -introduction, $A \vee B$ (from 1).
3. Now assume $\sim A$.
4. Then, by disjunctive syllogism, B (from 2, 3 and hence, ultimately, from 1, 3).

In tree form, the argument could be represented as the result of grafting the one-step deduction

$$\frac{A}{A \vee B}$$

onto the one-step deduction

$$\frac{A \vee B \quad \sim A}{B}$$

in order to obtain B “overall” from $A, \sim A$:

$$\frac{\frac{A}{A \vee B} \quad \sim A}{B}$$

Now let us suppose one has no objection to the step of \vee -introduction. Then, in order to reject the overall fallacy of inferring B from $A, \sim A$, there are two options to consider:

1. Reject disjunctive syllogism.
2. Reject transitivity of deduction.

Anderson, Belnap and others opt for (1). They seek to retain unrestricted transitivity of deduction at all costs. Relevance logic has since become bound to the orthodox presupposition that the deducibility relation should be unrestrictedly transitive.

But it is worth examining option (2) more closely in the light of our exercise above with the proof of the first Lewis paradox. Note that each of the two little deductions, before grafting, had this property:

its conclusion is not a logical truth and its undischarged assumptions form a consistent set

But the overall deduction, after the grafting, did not have this property; for the newly accumulated assumptions $A, \sim A$ form an inconsistent set.

Moreover, a simple procedure brings out this inconsistency explicitly. First, take the overall figure after the grafting, and supply the missing detail in the step of disjunctive syllogism:

$$\frac{\frac{A}{A \vee B} \quad \frac{\frac{1 \quad A \quad \sim A}{A}}{B} \quad \frac{1}{B} 1.}{B}$$

where A is the absurdity symbol and the occurrences of the numeral 1 indicate discharge of assumptions by \vee -elimination.

Note that the occurrence of $A \vee B$ in this figure is maximal. Now apply the reduction procedure for \vee in order to get rid of it. The result is

$$\frac{A \quad \sim A}{A}$$

Now shed the terminal application of the absurdity rule. The result is

$$\frac{A \quad \sim A}{A}$$

which is the explicit demonstration of inconsistency promised.

In this simple and degenerate case we have an illustration of what is generally the case: one can normalise and relevantise any deduction so as to obtain one ending either with the original conclusion or with A , and with all undischarged assumptions among the originals. The detailed statement of the general result is given in Theorem 1 below.

The idea that naturally occurs in the light of these considerations is that one could pursue option (2) in such a way that *transitivity of deduction fails only when the grafting does not preserve the property formulated above*. It turns out that there is a very natural system of classical relevance logic with just this feature. It was investigated in Tennant [1984]. In this paper I investigate its intuitionistic analogue.

My main result, Theorem 4 below, is the coextensiveness of a sequent formulation and a natural deduction formulation of a new logic. I call it *intuitionistic relevant logic* (IR). The coextensiveness of the two formulations is what permits me to refer to the system simply as IR, rather than as SIR (for the sequent system) and NIR (for the system of natural deduction).

IR has classical versions in both the sequent formulation and the natural deduction formulation. The classical sequent version SCR is obtained from the sequent formulation of IR by simply allowing more than one formula to appear in sequent succedents. SCR was investigated in Tennant [1984]. The classical natural deduction version NCR would be obtained from the natural deduction formulation of IR by adopting a classical rule of reductio. It remains to be proved that SCR and NCR are coextensive. On the basis of the main result in this paper I conjecture that they are. Henceforth by CR I shall mean SCR.

That the systems IR and CR earn the title of *relevance* logics follows from the obvious failure in both systems of the Lewis paradox $A, \sim A:B$. This result is immediate by inspection of the rules of the systems. It was first noted for the *classical* sequent system CR in Tennant [1984].

But note that in the acronyms CR and IR the letter R is semantically inert! CR differs from the well-known relevance logic R of Anderson and Belnap in important respects. IR differs in the same respects from Došen's intuitionised version of R. These respects are:

1. CR and IR contain disjunctive syllogism ($A \vee B, \sim A:B$); whereas neither Anderson and Belnap's system R, nor Došen's intuitionised version of R, contains disjunctive syllogism.
2. The latter two systems enjoy unrestricted transitivity of deduction. But in CR and IR transitivity does not hold unrestrictedly.

A counterexample to unrestricted transitivity in IR is the following. In IR one can prove both $A:A \vee B$ and $A \vee B, \sim A:B$. By transitivity (cut) one would expect to be able to prove $A, \sim A:B$. But one cannot. (Why this is so will become clear later.) This failure of transitivity, however, is *desirable* for the relevantist, who would not wish to obtain B from the inconsistent set of premisses $A, \sim A$. Moreover, one can show, at least for the language based on $\sim, \vee, \&, \exists$ and \forall (without the conditional), that transitivity in general fails only where it *ought* to fail. The metatheorem for IR that shows this is the following, corresponding to the metatheorem proved for CR in Tennant [1984]:

Suppose the language contains only the logical operators $\sim, \vee, \&, \exists, \forall$. Then every intuitionistic proof of $X:Y$ can be converted into a proof in IR of $X':Y'$ for some subsets X', Y' of X, Y respectively. (Remember Y is empty or a singleton.)

This has the following immediate corollaries, for the language based on $\sim, \vee, \&, \exists$ and \forall , corresponding to similar results for the classical version in Tennant [1984]:

- (i) Every intuitionistically inconsistent set can be proved inconsistent in IR.
- (ii) Every intuitionistic logical truth is a theorem of IR.
- (iii) Every intuitionistic consequence of an intuitionistically consistent set can be deduced from it in IR.

Both the metatheorem above and the three corollaries fail if the conditional is primitive in the language. A counterexample to the metatheorem and to corollary (iii) would be the obvious intuitionistic proof of $\sim A:A \rightarrow B$. There is no proof of this result in IR, nor, obviously, of $\sim A: \quad :A \rightarrow B$, or the empty sequent. A counterexample to corollary (ii) would be $\sim A \rightarrow (A \rightarrow B)$, and a counterexample to corollary (i) would be $\sim(\sim A \rightarrow (A \rightarrow B))$.

It remains to be seen what results in the neighbourhood of this metatheorem and its corollaries could be recovered for the full language in which the conditional is primitive.

The main result mentioned above, however—the coextensiveness of the sequent formulation and the natural deduction formulation of IR—is obtained for the full language *with* a conditional. In fact, it holds for both a weak conditional and a strong conditional. The weak conditional does not require the antecedent to have

been used as an assumption in conditional proof; the strong conditional does. Accordingly, $B:A \rightarrow B$ holds for the weak conditional but not for the strong conditional.

In Tennant [1984] I gave a natural semantics for the *classical* system CR *without* a conditional. I showed that its provable sequents are exactly the *entailments*. An entailment is a substitution instance of a *perfectly valid* sequent. A perfectly valid sequent is one which is (classically) valid but which ceases to be so upon removal of any of its member sentences. It remains to be seen whether and how that semantics might be adapted to the *intuitionistic* system IR *without* a conditional. I conjecture that simply substituting “(intuitionistically)” for “(classically)” would do. Thus I conjecture that in the *intuitionistic* system IR *without a conditional* the provable sequents are exactly those that are substitution instances of intuitionistically valid sequents that cease to be intuitionistically valid upon removal of any of their member sentences. If this conjecture is correct, it would then remain to be seen whether the respective intuitionistic and classical semantics for the relevance systems *without* a conditional could be extended so as to deal with a conditional.

But for the present my concerns are entirely proof-theoretical. I am in part following a suggestion put to me by Horst Luckhardt: that any significant “syntactic” notion of relevance should permit generalisation to the usual inference rules for the conditional. The main result shows that simple changes made to either the sequent formulation or the natural deduction formulation of intuitionistic logic, motivated by the same concern to avoid Lewis’s paradox $A, \sim A:B$ and its close cousin $A, \sim A:\sim B$, but to retain disjunctive syllogism $A \vee B, \sim A:B$, give rise to the same relation of deducibility in the full language containing the conditional as a primitive. It suggests that the system of intuitionistic relevant logic is especially natural.

For reasons explained in Tennant [1979], [1980], if one wishes to avoid proofs of Lewis’s paradox $A, \sim A:B$ and its close cousin $A, \sim A:\sim B$ in the natural deduction system, one must

- (a) ban applications of the absurdity rule $\frac{A \quad \sim A}{\perp}$,
- (b) insist that proofs be in normal form, and
- (c) insist that discharges should not be vacuous in applications of $\sim E$, $\vee E$ and $\exists E$

Reasons for (a), (b) and (c) are the following “proofs” of the first Lewis paradox $A, \sim A:B$ or of the closely related $A, \sim A:\sim B$ that could be constructed by violating (a), (b) and (c) respectively:

$$\frac{A \quad \sim A}{\frac{A}{B}}$$

(application of absurdity rule),

$$\frac{A \quad \frac{\sim B}{A \& B}}{A \quad \sim A}$$

$$\frac{\frac{A}{\sim B} \quad 1}{\sim B}$$

(maximal occurrence of $A \ \& \ B$; proof not in normal form), and

$$\frac{A \quad \sim A}{\frac{A}{\sim B}}$$

(application of $\sim I$ with vacuous discharge of assumption).

Once we have imposed restrictions (a), (b) and (c), then we can recover disjunctive syllogism in the intuitionistic case by liberalizing $\vee E$ (proof by cases) in the way to be stated below.

Similarly, for reasons explained in Tennant [1984], if one wishes to avoid proofs of $A, \sim A : B$ and $A, \sim A : \sim B$ in the sequent system, one must ban cut and ban thinning. If we do not, we will be able to construct the following sequent proofs:

$$\frac{\frac{A:A \quad \frac{A:A \quad A, \sim A: \quad B:B}{A \vee B, \sim A:B}}{A, \sim A:B} \quad (cut),}{\frac{A:A}{A, \sim A:} \quad (thinning).}{A, \sim A:B}}$$

Once we have banned cut and thinning, then we can ensure that we have disjunctive syllogism by choosing carefully the particular form in which we state the sequent rule for introducing \vee on the left. One further result of this choice is that we can prove $A \vee (B \ \& \ \sim B) : A$. Similar careful choice of the form of the sequent rule for introducing $\&$ on the right ensures that we still have $A, B : A \ \& \ B$.

All these ingredients now appear in the statement below of the rules for both the natural deduction version of IR and the sequent version of IR.

The information already provided about what is and is not provable in IR serves to locate IR with respect to other well-known systems as follows:

- Classical logic contains $\sim \sim A : A$; IR does *not*
- Intuitionistic logic contains $A, \sim A : B$; IR does *not*
- Minimal logic contains $A, \sim A : \sim B$; IR does *not*
- Minimal logic does *not* contain $A \vee (B \ \& \ \sim B) : A$; IR *does*

The relevance logic R of Anderson and Belnap, and Došen's intuitionised version of it, and even Anderson and Belnap's logic of first degree entailments *do not* contain $A \vee B, \sim A : B$; IR *does*

For all these other logics, the cut rule

$$\frac{X:A \quad A, Y:Z}{X, Y:Z}$$

is admissible, in the sense that if there are proofs of the top sequents then there are proofs of the bottom sequent. This holds quite generally, for all X, Y , (singleton or empty) Z and A . For IR it is *not* admissible. There are some such X, Y, Z and A such

that there are proofs in IR of the top sequents, but no proof in IR of the bottom sequent.

The picture that therefore emerges is that IR, like minimal logic, is properly contained in intuitionistic logic. And IR overlaps with minimal logic. But it differs from all other known relevance logics by retaining disjunctive syllogism at the expense of (controlled) loss of transitivity.

Natural deduction formulation of IR. In all applications of the introduction rules that follow, the conclusion is said to be *introductory in nature*. In all applications of the corresponding elimination rules, it is required that the major premiss not be introductory in nature. In applications of \vee -Elimination and \exists -Elimination, if C is introductory in nature as a subordinate conclusion, then it is introductory in nature at its occurrence as the main conclusion of the elimination. Henceforth “MPE” will abbreviate “major premiss for an elimination”.

In all applications of rules in which discharge is indicated, the discharge is *obligatory*. That is, there must be an undischarged occurrence of the assumption of the indicated form on which the subordinate conclusion depends. Upon application of the rule all such occurrences must be discharged.

Included within the scope of this last requirement is the introduction rule for the conditional. The requirement makes it the *strong* conditional referred to above. If we relaxed the requirement just for this rule, we would obtain the *weak* conditional. In the interests of uniformity I shall treat of the strong conditional.

Our overall restrictions ensure that every natural deduction is in normal form; that it contains no applications of the absurdity rule (since that rule is absent from the list below, and the discharge requirement on negation introduction prevents it from having applications which could be construed as applications of the absurdity rule); and that it contains no vacuous discharges. (In a deduction in *normal form* no sentence occurrence is both the conclusion of an application of an introduction rule and the major premiss of an application of the corresponding elimination rule.)

I shall adopt the obvious notation $\&I$, $\&E$, etc. for these introduction and elimination rules. Note that, as explained in Tennant [1978], the rules in a system of natural deduction correspond to clauses in an inductive definition of the notion “ P is a proof of A from the set X of undischarged assumptions”. The basis clause is that any occurrence of A is a proof of A from singleton A . In a system of natural deduction there is no need for any rule of cut: what transitivity of proof there is, is a byproduct of the inductive definition of proof just mentioned. Note also that A , the absurdity constant, appears only on its own as a line in a proof; it never occurs as a subformula (nor, therefore, as a formula).

Inconsistency of X is established in a system of natural deduction by giving a deduction of A from X ; while it is established in a sequent system by giving a sequent proof of X : (that is, $X:\emptyset$, where \emptyset is the empty set). Theoremhood of A is established in a system of natural deduction by giving a deduction whose conclusion is A and whose assumptions have all been discharged by the end of the deduction; while in a sequent system it is established by giving a sequent proof of $\vdash A$ (that is, of $\emptyset:A$).

I shall now state the rules of natural deduction for IR in schematic form, and then give some examples of how they are to be understood as clauses in the inductive definition of proof.

<i>Introduction rules</i>	<i>Elimination rules</i>
$\frac{\begin{array}{c} \neg(i) \\ A \\ \vdots \\ A \end{array}}{\sim A} (i)$	$\frac{A \quad \sim A}{A}$
$\frac{A \quad B}{A \& B}$	$\frac{A \& B}{A} \quad \frac{A \& B}{B}$
$\frac{A}{A \vee B} \quad \frac{B}{A \vee B}$	$\frac{\begin{array}{c} \neg(i) \\ A \\ \vdots \\ A \vee B \end{array} \quad \begin{array}{c} \neg(i) \\ B \\ \vdots \\ A \vee B \end{array}}{A/C} (i)$

[In the statement of $\vee E$ the slash notation A/C is to be understood as follows: we allow a subordinate conclusion of either one of the cases to be brought down as main conclusion if the other subordinate conclusion is A . Of course, if both subordinate conclusions are of the same form, the main conclusion has the same form.]

$\frac{\begin{array}{c} \neg(i) \\ A \\ \vdots \\ B \end{array}}{A \rightarrow B} (i)$	$\frac{A \quad A \rightarrow B}{B}$
$\frac{Aa}{\forall xAx}$	$\frac{\forall xAx}{At}$
$\frac{At}{\exists xAx}$	$\frac{\begin{array}{c} \neg(i) \\ Aa \\ \vdots \\ B \end{array}}{\exists xAx} (i)$

In $\forall I$, a does not occur in any assumptions on which Aa depends, and in $\exists E$, a does not occur in $\exists xAx, B$, or in any assumption other than Aa , on which the upper occurrence of B depends.

I shall now illustrate how these rules are to be understood as clauses in the inductive definition of proof, in the context of the normality constraint and the ban on vacuous discharge. I shorten “ D is a deduction of A from X ” to “ $\text{ded}(D, A, X)$ ”.

$\&$ -introduction should be understood as follows: If $\text{ded}(D_1, A_1, X_1)$ and $\text{ded}(D_2, A_2, X_2)$, then

$$\text{ded}\left(\frac{D_1 \quad D_2}{A_1 \& A_2}, A_1 \& A_2, X_1 \cup X_2\right),$$

and the conclusion $A_1 \& A_2$ is introductory in nature.

&-elimination should be understood as follows: If $\text{ded}(D, A_1 \ \& \ A_2, X)$ and the conclusion $A_1 \ \& \ A_2$ is not introductory in nature [normality constraint!], then

$$\text{ded}\left(\frac{D}{A_i}, A_i, X\right)$$

and the conclusion A_i is not introductory in nature.

\sim -introduction is to be understood as follows: if $\text{ded}(D, A, X)$ and A is in X [ban on vacuous discharge!], then

$$\text{ded}\left(\frac{D}{\sim A}, \sim A, X - \{A\}\right)$$

and the conclusion $\sim A$ is introductory in nature.

The reader will now be able to supply similar clauses for the remaining rules.

Note the following simple example of how transitivity fails, given the restrictions we have imposed on the application of rules. We have proofs

$$\frac{A}{A \vee B} \qquad \frac{(i) \frac{A \quad \sim A}{A} \quad B}{B} (i)$$

but it is easy to see by inspection of the rules that there is no proof of B from $A, \sim A$.

Our restrictions on proofs, for intuitionistic relevant logic, that they be in normal form, results in no loss as far as the deducibility relation is concerned. For the normalization theorem (Prawitz [1965]) states that every intuitionistic natural deduction can be converted into one in normal form, of the same conclusion and from (possibly a subset of) the same premisses. So loss of deducibility, if incurred at all, would have to be the result of the other restrictions we have imposed on proofs in intuitionistic relevant logic: the ban on the absurdity rule, and the ban on vacuous discharge of assumptions. How serious is this loss? The answer, given by the following theorems, is that it is not serious at all.

THEOREM 0. *For the language based on $\sim, \vee, \&, \exists$ and \forall (with the conditional missing), every intuitionistic natural deduction of C from X that is in normal form can be converted into a natural deduction in IR of either A or C from some subset of X . Moreover, if the conclusion of the IR deduction is C , then it is not introductory in nature if the conclusion of the original intuitionistic deduction is not.*

PROOF (by induction on the length of intuitionistic natural deductions in normal form). The only case that requires care is that where the intuitionistic deduction ends with $\vee E$. The complication that would arise is that the IR deductions guaranteed by the inductive hypothesis for the two subordinate case-proofs may not have the same conclusion. One may end with A and the other may end with the conclusion B of the original subdeduction. But our rule of $\vee E$ in IR is designed to cater for just such an outcome. The remaining cases, where the intuitionistic deduction ends with some rule other than $\vee E$ (including the absurdity rule) are easy to deal with. The method is essentially that of the proof of the extraction theorem in Tennant [1980].

Combining this result with the normalisation theorem, we have immediately

THEOREM 1. *For the language based on $\sim, \vee, \&, \exists$ and \forall (with the conditional missing), every intuitionistic natural deduction of C from X can be converted into a natural deduction in IR of either A or C from some subset of X .*

Theorem 1 (and, equally well, its counterpart theorem for the sequent calculus, which we shall prove as Theorem 2 below) serves as the metatheorem mentioned above, with the three corollaries discussed there.

We now proceed to develop a structural idea regarding natural deductions in IR that is of use in the proof of Theorem 3 below.

Any deduction ending with an application of an elimination rule possesses a *spine*: that unique sequence of occurrences of MPE's ending with the bottommost one (immediately above the conclusion of the deduction) and choosing as the immediately preceding one (if there is one) the one immediately above the one last chosen. Note that the only elimination rules that permit discharge of assumptions (namely $\vee E$ and $\exists E$) only ever discharge assumptions on which the *minor* premisses of such applications depend. They do not discharge any assumptions on which their major premisses depend.

Thus suppose $MPE_0, MPE_1, \dots, MPE_n$ is a spine with MPE_n as the last one immediately above the conclusion of the deduction. Suppose that each MPE_i rests on the set X_i of undischarged assumptions. Suppose further that X is the set of undischarged assumptions of the whole proof. Then by the last remark, $X_0 \subseteq X_1 \subseteq \dots \subseteq X_n \subseteq X$. Note also that $X_0 = \{MPE_0\}$. Note moreover that if MPE_i is of the form $\sim A$, then the spine ends on MPE_i ; that is, negations can only be terminal (that is, bottommost) members of spines. This is because, obviously, A cannot be an MPE.

Example

$$\begin{array}{c}
 \frac{\forall z(Pz \rightarrow (Q \& \exists xAx)) \dots [MPE_0]}{Pt \quad Pt \rightarrow (Q \& \exists xAx) \dots [MPE_1]} \quad (1) \quad \frac{\forall x(Ax \rightarrow Bx)}{Aa \quad Aa \rightarrow Ba} \\
 \frac{Q \& \exists xAx \dots [MPE_2]}{\exists xAx \dots [MPE_3]} \quad \frac{Ba}{\exists xBx} \\
 \hline
 \exists xBx \quad (1)
 \end{array}
 \quad
 \begin{array}{l}
 X_0 = \{\forall z(Pz \rightarrow (Q \& \exists xAx))\} \\
 X_1 = X_0 \\
 X_2 = X_1 \cup \{Pt\} \\
 X_3 = X_2 \\
 X = X_3 \cup \{\forall x(Ax \rightarrow Bx)\}
 \end{array}$$

Note that this is a natural deduction built up in accordance with the introduction and elimination rules which, as illustrated above, correspond to clauses in the inductive definition of what counts as a deduction in the system. The deduction just given does not contain any step that can be construed as an application of the cut rule. The cut rule has its proper place in the *sequent* version of any logic whose deducibility relation is to be unrestrictedly transitive. To the development of sequent rules I now turn; and, it is important to note, the cut rule is missing from the system of sequent rules that I shall formulate. The sequent rules of IR are the rule of initial sequents in the restricted form $A:A$ (rather than the liberal form $X:A$ for A in X); and rules for introducing logical operators on the right and on the left of sequents. After stating the rules I shall use them to give a (cut free!) sequent proof corresponding to the natural deduction above.

Sequent rules for IR. As in Tennant [1984], X and Y are sets of sentences in the sequent $X:Y$. Furthermore, Y is empty or a singleton. Since we are dealing with sets

and not sequences as antecedents and succedents of sequents, there is no need for such structural rules as permutation and contraction. Indeed, there is only one structural rule in IR, and that is the rule of initial sequents, $A:A$, where A is a sentence. I shall denote the union of X and Y by " X, Y ", and shall shorten " $X, \{A\}$ " to " X, A ".

The remaining rules of IR are for introducing logical operators into bottom sequents, both on the left and on the right of the colon. I shall call these rules &L, &R, etc. They are as follows.

<i>Right</i>	<i>Left</i>
$\frac{X, A:}{X: \sim A}$ where A is not in X	$\frac{X: A}{X, \sim A:}$
$\frac{X: A \quad Y: B}{X, Y: A \ \& \ B}$	$\frac{X, A: Y \quad X, B: Y}{X, A \ \& \ B: Y'}$
$\frac{X: A}{X: A \vee B} \quad \frac{X: B}{X: A \vee B}$	$\frac{X, A: Y \quad Z, B: W}{X, Z, A \vee B: Y, W}$
	where the union of Y and W has at most one member.
$\frac{X, A: B}{X: A \rightarrow B}$ where A is not in X	$\frac{X: A \quad B, Z: W}{X, Z, A \rightarrow B: W}$

The rule of $\rightarrow R$ here is for the *strong* conditional. The weak conditional would be obtained by keeping $\rightarrow L$ as it stands, but relaxing the rule $\rightarrow R$ to

$$\frac{X: B}{X - \{A\}: A \rightarrow B}$$

To complete our list of sequent rules, the ones for the quantifiers are:

$\frac{X: Aa}{X: \forall x Ax}$ where a does not occur in any member of X	$\frac{X, At: Y}{X, \forall x Ax: Y}$
$\frac{X: At}{X: \exists x Ax}$	$\frac{X, Aa: Y}{X, \exists x Ax: Y}$ where a does not occur in any sentence in the bottom sequent.

This completes the statement of sequent rules for IR.

We can illustrate them by giving a sequent proof of the same example for which we gave a natural deduction earlier, when illustrating the notion of *spine*. The

sequent proof would be

$$\begin{array}{r}
 \frac{Aa:Aa \quad Ba:Ba}{Aa, Aa \rightarrow Ba:Ba} \rightarrow L \\
 \frac{Aa, Aa \rightarrow Ba:Ba}{Aa, \forall x(Ax \rightarrow Bx):Ba} \forall L \\
 \frac{Aa, \forall x(Ax \rightarrow Bx):Ba}{Aa, \forall x(Ax \rightarrow Bx):\exists xBx} \exists R \\
 \frac{Aa, \forall x(Ax \rightarrow Bx):\exists xBx}{\exists xAx, \forall x(Ax \rightarrow Bx):\exists xBx} \exists L \\
 \frac{\exists xAx, \forall x(Ax \rightarrow Bx):\exists xBx}{\exists xAx, \forall x(Ax \rightarrow Bx):\exists xBx} \&L \\
 \frac{Pt:Pt \quad Q \& \exists xAx, \forall x(Ax \rightarrow Bx):\exists xBx}{Pt, Pt \rightarrow (Q \& \exists xAx), \forall x(Ax \rightarrow Bx):\exists xBx} \rightarrow L \\
 \frac{Pt, Pt \rightarrow (Q \& \exists xAx), \forall x(Ax \rightarrow Bx):\exists xBx}{Pt, \forall z(Pz \rightarrow (Q \& \exists xAx)), \forall x(Ax \rightarrow Bx):\exists xBx} \forall L
 \end{array}$$

Note that this sequent proof has no applications of the cut rule, which is banned from our system. Our task will be to show quite generally how a sequent proof of the sequent $X:A$ can be obtained from a natural deduction of A from the set X of undischarged assumptions. This task is solved by Theorem 3. As we shall see later from the method of proof of Theorem 3, it is no accident that the ordering of L-rules in the last four steps here should be the same as the ordering of eliminations down the spine of the natural deduction given earlier. A similar observation applies to the top two steps, and the spine of the subdeduction with conclusion Ba in the earlier example.

The sequent calculus just given for IR can be extended to one for intuitionistic logic by adopting the extra rules of thinning on the left and thinning on the right (cf. Dummett [1977, pp. 133–134]). Thinning is also known as dilution:

$$\frac{X:}{X:A} \quad \frac{X:Y}{X, A:Y}$$

THEOREM 2. *For the language based on $\sim, \vee, \&, \exists$ and \forall (without the conditional), if $X:Y$ is provable in the sequent calculus for intuitionistic logic, then for some subsets X', Y' of X, Y respectively, $X':Y'$ is provable in the sequent calculus for IR.*

PROOF. Straightforward by induction on the length of proofs in the sequent calculus for intuitionistic logic, and by inspection of the sequent rules for IR. The method is essentially that of the proof of the dilution elimination theorem in Tennant [1984].

Theorem 2 is the counterpart, for the sequent calculus, to Theorem 1 above for the natural deduction system. In the light of Theorem 4 below, which establishes the coextensiveness of the natural deduction formulation and the sequent formulation of IR, we can regard Theorems 1 and 2 as equivalent versions of the metatheorem stated above, which had the three corollaries discussed there.

It is easy to turn a sequent proof in IR into a natural deduction in IR of the same result—cf. the remark in Prawitz [1965, p. 91] concerning intuitionistic logic. Combining this observation with Theorem 3 below, we shall have our main result:

THEOREM 4. *The natural deduction formulation and the sequent formulation determine the same logic IR.*

The reason why care is needed in the proof of Theorem 3 below is that we have defined sequent proofs in IR in such a way that they are not allowed to contain any

applications of the cut rule. We therefore cannot avail ourselves of the usual method of transforming natural deductions into sequent proofs. This method involves liberal use of the cut rule when forming the desired sequent proofs from sequent proofs already available by inductive hypothesis (cf. Gentzen [1969, Section V, §4, pp. 120–123: transformation of an *NJ*-derivation into an equivalent *LJ*-derivation]. Instead we have to devise a more direct method of transforming natural deductions into sequent proofs. The method in the proof below generalises that in Prawitz [1965, pp. 92–93] for converting natural deductions into sequent proofs. The cases where the natural deduction ends with $\vee E$ or $\exists E$ were not treated explicitly by Prawitz. I owe this extension of his method to Peter Schroeder-Heister. It simplifies an earlier proof of mine using a somewhat different method.

THEOREM 3. *Every natural deduction P in IR of C from X can be converted into a sequent proof in IR of $X:C$*

PROOF (by induction on the number of applications of rules of inference in P). The *basis* is obvious by the rule of initial sequents. Assume the *inductive hypothesis* that the result holds for all natural deductions simpler than P . The *inductive step* falls into cases according to the rule last applied in P .

If P ends with an introduction, then apply the corresponding R-rule to the sequent proofs guaranteed by the inductive hypothesis for the subordinate deductions for the introduction.

If P ends with an elimination, then focus on the topmost MPE of P 's spine. P will have one of the forms displayed on the left below. In each case the displayed formula with the logical operator is the topmost MPE of the spine. Moreover, the set X of undischarged assumptions of P is that formula, plus the union of the displayed X_i . By the inductive hypothesis we assume for each subordinate deduction P_i a corresponding sequent proof S_i . The inductive step in each case is then to form for P the sequent proof displayed on the right by means of an application of the corresponding L-rule:

$\frac{\frac{X_1 \quad P_1}{A \quad \sim A}}{A}$	$\frac{S_1 \quad X_1:A}{X_1, \sim A:C}$
$\frac{\frac{\forall x Ax \quad At, X_1}{P_1} \quad C}{\forall x Ax}$	$\frac{S_1 \quad X_1, At:C}{X_1, \forall x Ax:C}$
$\frac{\frac{X_1 \quad P_1}{A \quad A \rightarrow B} \quad X_2, B \quad P_2 \quad C}{A \rightarrow B}$	$\frac{S_1 \quad X_1:A \quad S_2 \quad X_2, B:C}{X_1, X_2, A \rightarrow B:C}$

$$\begin{array}{c}
 \frac{A_1 \ \& \ A_2}{A_i \ , \ X_1} \quad \frac{S_1}{X_1, A_i : C} \\
 P_1 \quad \frac{X_1, A_i : C}{X_1, A_1 \ \& \ A_2 : C} \\
 C
 \end{array}$$

$$\begin{array}{c}
 (i) \frac{}{Aa, X_1} \quad S \\
 P_1 \quad \frac{X_1, X_2, Aa : C}{X_1, X_2, \exists xAx : C} \\
 \exists xAx \ B \quad (i) \\
 X_2, \frac{B}{B} \\
 P_2 \\
 C
 \end{array}$$

where S corresponds by IH to

$$\begin{array}{c}
 Aa, X_1 \\
 P_1 \\
 X_2, B \\
 P_2 \\
 C
 \end{array}$$

$$\begin{array}{c}
 X_1, \frac{}{A_1} (i) \quad X_2, \frac{}{A_2} (i) \\
 P_1 \quad P_2 \\
 A_1 \vee A_2 \quad A \quad D \quad (i) \\
 \frac{D}{D}, X_3 \\
 P_3 \\
 C
 \end{array}
 \quad
 \begin{array}{c}
 S_1 \quad S_2 \\
 \frac{X_1, A_1 : \quad X_2, X_3, A_2 : C}{X_1, X_2, X_3, A_1 \vee A_2 : C}
 \end{array}$$

where S₂ corresponds by IH to

$$\begin{array}{c}
 X_2, A_2 \\
 P_2 \\
 X_3, D \\
 P_3 \\
 C
 \end{array}$$

and similarly for the cases where P₁ ends with D and P₂ ends with A, where both P₁ and P₂ end with A, and where both P₁ and P₂ end with D. This completes the proof.

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REFERENCES

- A. R. ANDERSON and N. D. BELNAP [1975], *Entailment*, vol. 1, Princeton University Press, Princeton, New Jersey.
- K. DOŠEN [1981], *A reduction of classical propositional logic to the conjunction-negation fragment of an intuitionistic relevant logic*, *Journal of Philosophical Logic*, vol. 10, pp. 394–408.
- M. A. E. DUMMETT [1977] (with the assistance of R. MINIO), *Elements of intuitionism*, Oxford University Press, Oxford.
- I. JOHANSSON [1936], *Der Minimalalkül, ein reduzierter intuitionistischer Formalismus*, *Compositio Mathematica*, vol. 4, pp. 119–136.
- G. GENTZEN [1969], *The collected papers of Gerhard Gentzen* (edited by M. E. Szabo), North Holland, Amsterdam.
- D. PRAWITZ [1965], *Natural deduction: a proof-theoretical study*, Almqvist and Wiksell, Stockholm.
- N. TENNANT [1978], *Natural logic*, Edinburgh University Press, Edinburgh.
- [1979], *Entailment and proofs*, *Proceedings of the Aristotelian Society*, vol. 79, pp. 167–189.
- [1980], *A proof-theoretic approach to entailment*, *Journal of Philosophical Logic*, vol. 9, pp. 185–209.
- [1984], *Perfect validity, entailment and paraconsistency*, *Studia Logica*, vol. 43, pp. 179–198.

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