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DEDUCTIVE VERSUS EXPRESSIVE POWER: A PRE-GÖDELIAN PREDICAMENT*

There is a distinguished branch of inquiry which goes back to antiquity and captivates the imagination even today. It began with Euclid of Alexandria, and over two millenia later ran into difficulties posed by the foundational work of the great twentieth-century logician, Kurt Gödel—even though, somewhat ironically, Gödel himself ranks as an arch proponent of the branch of inquiry in question. It is called *monomathematics*, a term to be made more precise presently. Suffice it to say at this stage that success in monomathematics requires both *expressive* power (the power to describe structures exactly) and *deductive* power (the power to prove whatever follows logically from one's description).

I. INTRODUCTION

The main question is this. Why was the logical community so slow to realize that monomathematics—the combination of these expressive and deductive aspirations—was impossible?[1] Why was it only some

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[1] The impossibility claim is not that a number theorist, say, would be wrong to maintain that there is a unique structure of natural numbers. A number theorist could well maintain such a thing. But in order to exclude nonstandard structures as possible interpretations of his theory, he would have to formulate that theory in a sufficiently expressive language. But then that language would fail to enjoy a sufficiently powerful logic, that is, a deductive system enabling one to prove any consequence of what the theorist claims. This is but one illustration of the sense in which monomathematics turns out to be impossible.

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time after Gödel's work that it dawned on philosophers, logicians, and mathematicians that the naive ideals were self-defeating and that there were certain inevitable limits to the combination of expressive power in one's mathematical language and deductive power in its underlying logic? Indeed, why was this not apparent even before the 1920s? Why did Gödel have to prove such deep and difficult results to make the lesson sink in, when the lesson could have been learned, in a stark and stripped-down form, years earlier?[2]

My aim is not so much to answer these questions as to show that the puzzlement they express is justified. The intrinsic limitations on monomathematics as a combined expressive and deductive venture are almost immediate, I shall show, upon simple reflection on two central concepts: *proving a result conclusively*, and *describing something exactly*. The interplay between these two concepts leads to what I call the *noncompossibility theorem*, which poses an insuperable difficulty for monomathematics, because it tells us that it is impossible to achieve the combined ideals of exact description and exhaustive deduction.

The noncompossibility theorem shares an interesting feature with other results that have reforged concepts and reshaped philosophical and mathematical inquiry. What come to mind here are the proof in antiquity that $\sqrt{2}$ is not rational, and the subsequent realization of the peculiar richness of the continuum; the liar paradox, and the resulting object-language/metalanguage distinction; Russell's paradox, and the subsequent shift from a logical conception of set to the mathematician's iterative conception; and Cantor's theorem, with its resulting proliferation of higher infinities. As with these other well-known results, the proof of the noncompossibility theorem takes less than half a page. It involves an utterly simple construction, and it uses apparently unproblematic conceptual materials. It takes some time for the result to sink in; and when it does finally sink in, it causes a sinking feeling.

The noncompossibility theorem could have been stated and proved by the end of World War I. For both ingredient concepts—categoricity of theory, and completeness of a system of proof—were well in place by the winter of 1917/18. It is tempting to suggest that, had the theorem been stated in timely fashion, both logical positivism

[2] For an example of a recent exposition that appeals to Gödel's completeness and compactness theorems for first-order logic, and his incompleteness theorem for first-order arithmetic, in order to develop the tension between categoricity and completeness (for the theory of [the ordering of] the natural numbers), see S. Read, "Completeness and Categoricity: Frege, Gödel and Model Theory," *History and Philosophy of Logic*, XVIII, 2 (1997): 79-93. The tension, however, can be brought out more simply and more deeply, without prior knowledge of Gödel's results.

and Hilbert's program might have been affected. Knowledge of the noncompossibility theorem would have preempted, for example, important conceptual confusions or conflations both in Rudolf Carnap's 1927 paper "Eigentliche und uneigentliche Begriffe,"[3] and in David Hilbert's[4] 1928 address to the International Congress of Mathematicians in Bologna. But it is impossible here to pursue any counterfactual speculations about the development of foundational thought, or to elaborate a really detailed historical case for the claim that the two ingredient concepts had been satisfactorily isolated. This is primarily a conceptual study; historical details will be adduced only to heighten the interest of the investigations.

II. MONOMATHEMATICS VERSUS POLYMATHEMATICS

Any mathematician is easily apprised of the difference between two kinds of mathematics. *Monomathematics* is the mathematics of a unique structure. Examples would be: the theory of natural numbers, the theory of the rationals, the theory of the real-number line, and the theory of the usual algebraic operations on the complex plane. In each case, the mathematician has a unique structure in mind (often called the *intended* structure), and is trying to articulate as comprehensively as he can the interesting truths about it. He wants to exhibit them all as true statements concerning the intended structure.

Polymathematics, by contrast, is the mathematics of structures enjoying some definable structural affinity. Examples would be: groups, rings, and topological spaces. In each case, the mathematician has a *variety* of structures in mind, which all have one crucial thing in common. They all satisfy a particular collection of axioms pinning down certain structural features: the axioms for groups, the axioms for rings, the axioms for topological spaces, and so on. In each case here, the mathematician is interested not only in the logical consequences of the (underspecific) set of axioms characteristic of these structures, but also in various embeddings of any one such structure into others within the same class of structures, and in the invariants of such embeddings. The mathematics here is, as it were, more a brand of model theory for the axiom set in question.

III. ON THE UNIQUENESS OF INTENDED STRUCTURES

A Platonic realist, such as Gödel, takes himself to be engaged in some sort of direct intellectual apprehension of the intended abstract

[3] In *Symposion: Philosophische Zeitschrift für Forschung und Aussprache,* I (1927): 355-74.

[4] "Probleme der Grundlegung der Mathematik," *Mathematische Annalen,* CII (1929): 1-9.

structure. The sequence of natural numbers, the dense ordering of the rationals, the hierarchy of the hereditarily finite pure sets, the continuous ordering of the reals, and the complex plane—these are structures for which one has an immediate and intuitive "feel" (or for which one can acquire such a "feel" through appropriate training and problem-solving practice). One thinks one can reason about each of these structures in its own right. This is especially so with the *countable* structures among those just mentioned—the natural numbers, the rationals, and the hereditarily finite pure sets.

The conception of a unique intended structure is taken by some to apply even in the case of the ultimately general mathematical theory, within which (or in terms of which) many mathematicians believe it possible to interpret every other branch of "more particular" mathematical theorizing, namely, set theory. (Here, I mean the theory of pure sets in general, not just the hereditarily finite ones.) Now, the universe of sets—indeed, any model of set theory—does not, as will become clear below, satisfy the hypotheses about models exploited by the noncompossibility theorem. Therefore, the "uniqueness" of any intended model for set theory is not really germane for establishing the pre-Gödelian predicament—which, it is my purpose to show, arises from the noncompossibility theorem. Nevertheless, it is interesting and instructive to examine just how ambitious the expressive ideal of monomathematics can be, insofar as it asserts uniqueness of intended structure even in the case of the most populous abstract universe, namely, that of all (pure) sets.

Here, it is worth recalling Gödel's classic statement of his Platonic realism. In "What Is Cantor's Continuum Problem?"[5] he wrote:

> ...set-theoretical concepts and theorems describe some well-determined reality, in which Cantor's conjecture must be either true or false. Hence its undecidability from the axioms being assumed today can only mean that these axioms do not contain a complete description of *that reality* [emphasis added] (*ibid.*, p. 260).

Note Gödel's choice of the anaphoric phrase 'that reality'. He did not write 'any such reality', despite the punctuation by the intervening period. This textual evidence supports the attribution to Gödel—well

[5] In *Kurt Gödel: Collected Works,* Volume II, Solomon Feferman et alia, eds. (New York: Oxford, 1990), pp. 254-70. The original date of publication of this essay was 1964. The 1964 version was a revised and expanded version of the article by the same title in *American Mathematical Monthly,* LIV (1947): 515-25. Note that the latter postdates Gödel's proof of the consistency of the axiom of choice and the generalized continuum hypothesis (1938).

known as being very careful in his choice of words—of a conception on which the sets form a unique intended structure.

The immediate context of the above quote is a conditional claim by Gödel. The material quoted above is preceded in his text by the words:

> ...if the meanings of the primitive terms of set theory as explained [earlier] are accepted as sound, it follows that the....

One could be convinced of the soundness of the axiomatic conception, however, by having in mind some model or other for the axioms concerned, rather than by having intellectual access to an alleged unique intended model. Why does Gödel here insist on the uniqueness of 'that reality'?

The classical polymathematician will say that any mathematical statement, and a fortiori Georg Cantor's conjecture, will have a determinate truth value in any model (of the appropriate relational type). What truth value this is will, of course, depend on what model is in question. The same conjecture could be true in one model, and false in another. And that, indeed, is precisely what is shown by the combination of Gödel's and Paul Cohen's[6] results establishing the independence of Cantor's *continuum hypothesis* (CH) from the usual axioms of set theory. In the universe of constructible sets given by Gödel, CH is true; whereas in the universe given by Cohen, it is false.

In the essay cited, Gödel went on to give reasons that, in his view, supported the conjecture at the time of writing—a conjecture borne out by Cohen's subsequent result—that Cantor's CH was unsolvable on the basis of the usual axioms of set theory. Since Gödel had already proved the consistency of CH with the usual axioms, these reasons had to indicate the consistency of its negation. Gödel pointed to various "highly implausible consequences" of CH. These were consequences concerning subsets and mappings of the real line, consequences "not known in Cantor's time."[7] The background assumption to which the textual evidence points was clearly that, given one's monomathematical intuitions about the real line itself (a unique structure), the counterintuitive character of such conse-

[6] See Gödel, "The Consistency of the Axiom of Choice and of the Generalized Continuum Hypothesis," *Proceedings of the National Academy of Sciences, U.S.A.*, XXIV (1938): 556-57 (reprinted in *Collected Works,* Volume II, pp. 26-27); and Cohen, "The Independence of the Continuum Hypothesis, I," *Proceedings of the National Academy of Sciences, U.S.A.*, L (1963): 1143-48, and "The Independence of the Continuum Hypothesis, II," *Proceedings of the National Academy of Sciences, U.S.A.*, LI (1964): 105-10.

[7] "What Is Cantor's Continuum Problem?" p. 263.

quences had to be construed as evidence that CH would be false in the unique, intended model of set theory (within which would be a copy of "the" real line).

What further reflections (on matters other than the textual evidence) would justify this attribution, to Gödel, of a monomathematician's expressive conception—a conception of the reals (respectively, pure sets) as forming a unique structure—on the basis of his view of certain consequences of CH as "implausible"? Gödel did not explicitly anticipate the possible objection, at this stage, that a polymathematical attitude to "set-theoretic universes" would be extensible (as needed, for internal consistency) to a similarly polymathematical attitude toward allegedly nonisomorphic and competing "systems of real numbers," or "real lines," thereby blurring one's claimed view of any privileged and unique "system of real numbers" or "real line."

Now, it may be suggested that this apparent failure to anticipate such an objection stems from Gödel's being satisfied that any such implausible consequence φ of CH (modulo the currently accepted axioms Γ) "should be false in any model that answers to our conception of the real line and its subsets, even if there is more than one such model."[8] But such a suggestion would not be well taken. To show that this is so, I shall pursue a dilemma. I shall suppose, first, that our "conception" is captured by the axioms Γ, and then show that this supposition is untenable, because of a result due to Gödel himself. Next, I shall explore the other horn of the dilemma, by supposing that our "conception" is not captured by Γ. I shall argue that, in this case, one confronts philosophical difficulties that are probably insuperable and, in any event, highly likely to have deterred Gödel from taking that horn of the dilemma. The upshot will then be that the suggestion just quoted cannot be sustained as an interpretation of Gödel's platonistic attitude.

So let us suppose, first, that the set Γ of axioms already available to express our conception of the real line and its subsets captures that very conception. Thus, every model of the axioms Γ is supposed to render the "implausible" statement φ false. Thus, those axioms Γ would logically imply $\neg\varphi$. Hence, since those same axioms Γ, taken in conjunction with CH, imply φ—the results "not known in Cantor's time"—the axioms Γ themselves would imply $\neg$CH. But Gödel did not believe that the currently available axioms Γ decided CH negatively.

[8] The suggestion giving rise to this dialectical twist is owed to an anonymous referee, whose words are quoted here.

On the contrary, he had already proved the consistency of CH with those axioms.[9]

So the first horn is untenable; it is illicit to assume that for a model to "answer to our conception" of the real line demands no more than that it satisfy the currently available axioms Γ expressing that conception.

Let us pass, then, to the second horn: let us assume now that the set Γ of currently available axioms does *not* capture our "conception" of the mathematical structure(s) in question. On this new assumption, our "conception" (or, more to the point, Gödel's "conception") of the real line would be neither as specific as *the intended structure itself,* nor as lax as the currently accepted axioms Γ (which, however, the conception would in some sense quasi-theoretically "extend"—that is, all of Γ would be true, according to the conception). Nevertheless (according to the suggestion under consideration), the conception would be exigent enough to admit only of such models as would make the implausible statements φ false. Hence, by contraposition on

$$\Gamma, \mathrm{CH} \vdash \varphi$$

one could conclude to the falsity of CH in all models answering to one's "conception" of set-theoretic reality—yet without having to "have in mind" a unique model answering to that "conception."

The burden of semantic and metaphysical explication now falls on the theorist favoring this second horn of the dilemma. What can such a "conception" be, if it is neither the current axiomatic articulation Γ, nor (capable of determining) a unique intended structure itself? Such a "conception" would be, currently, not fully expressed (since it is supposedly more demanding than Γ) but, at the same time, not be as determinate as, or able intensionally to determine, any unique intended structure. Indeed, if *to be intended* is, in this context, *to be conceived,* then there is no uniquely "intended" structure at all; rather, one has at best a multiply satisfiable "conception" that nevertheless goes beyond current axiomatic expression.

On this conception of "conception," the burden is on the objector to provide an account of their semantic and metaphysical status. Such conceptions would have to reach beyond any current system of axioms and guide all acceptable extensions thereof, and yet fall short of determining any intended structure uniquely or of *being* any unique intended structure. Such 'conceptions' would be intentional, nonlinguistic, and irremediably indeterminate: a tall order for further philosophical development.

[9] "The Consistency of the Axiom of Choice and of the Generalized Continuum Hypothesis," p. 557.

The objector's suggestion, as quoted above, therefore does not stand the test of reflection, once one pursues its broader philosophical consequences. When complaining of the implausibility of certain known (mathematical) consequences φ of CH, Gödel must after all have been thinking of "the" real line and also of "the" universe of sets, insofar as the latter has to contain a copy of "the" real line. Further support for this conclusion derives from the fact (albeit one not known to Gödel) that no axiomatic extension of Zermelo-Fraenkel set theory plus the axiom of choice, whether exploiting large-cardinal axioms or exploiting axioms of determinacy for projective sets (or even for ordinally definable sets), has succeeded in deciding CH (if it is consistent). Some variants of large cardinal axioms—most notably 'there exists a real-valued measurable cardinal' or 'the continuum is real-valued measurable'—do imply the negation of CH; but they do not command much support among set theorists as new axioms.[10]

No matter what further reasonably "evident" or independently motivated axioms have been wrung from our "conception" of set-theoretical reality, they have failed to settle CH—and, a fortiori, failed to settle it as *false*. This is strong circumstantial evidence that the appeal to a "conception" of the kind described two paragraphs back is not vindicated by the activities of the minds presumably best apprised of it. For, despite being less exigent than a unique intended structure, the allegedly multiply satisfiable "conception" has failed to deliver itself of the essential and salient common feature of all models answering to it—to wit, that CH is false. A better way for any Platonist to explain the elusiveness of CH would surely be to appeal to the sheer structural complexities and intricacies of the unique intended model, and the attendant difficulty of encapsulating, in any epistemically evident or accessible claims, what it is about the model that makes CH false.

The case of Gödel turns out, therefore, to be quite instructive for the contrast that is my present theme. He is clearly the quintessential monomathematician, at least in the "expressive" regard, insofar as his conception of a unique structure guides his thinking in the most foundational of all mathematical theories, namely, set theory itself.

I have pursued at some length the exegetical question of whether, on the basis of a certain famous quotation, Gödel can legitimately be regarded as committed to the existence of a unique intended structure (for set theory) that somehow informs our theorizing about sets. However the exegetical chips fall on this score, it is not essential, for

[10] I am indebted here to Harvey Friedman.

my main theme here, to pin this view upon Gödel beyond all reasonable doubt. For I am not indulging in any appeal to his authority; nor am I at all concerned, as it happens, with the full universe of *sets*. The noncompossibility theorem poses a predicament only for our thought about certain kinds of *countable* mathematical structures, such as the structure of the natural numbers. It is much more plausible to claim that Gödel—and indeed, almost every platonistically minded mathematician—regards the *natural numbers* as forming a unique, intended structure. This is the structure which any mathematician takes himself to be talking about when engaged in arithmetical theorizing.[11]

My particular interest in the analogous situation of Gödel and set theory was occasioned, first, by the sheer generality of set theory as a foundation for all of mathematics, and, secondly, by the venerable status that Gödel enjoys as both arch-platonist and foundational iconoclast. It is not essential to my main argument, however, to secure general agreement with my characterization of Gödel's platonism, and least of all as the latter philosophical view concerns sets. All I need is acknowledgment of widespread pretheoretical confidence in "the" structure consisting of the (standard) natural numbers.

III. MONOMATHEMATICAL METHODOLOGY

The monomathematician tries to intuit various simple, obvious, and logically comprehensive truths about his chosen structure, and to express these as axioms in a formal language whose syntax is finitary, precise, and fully understood.

Whatever the epistemic refinements involved in laying down a set of axioms, one general claim about the result in monomathematics is incontestable: the axioms strike one as intuitively evident. They speak directly of the structure, in its own intrinsic terms. For the case of the natural numbers, these will be 0, 1, $+$, $\times$, $<$; for the theory of sets,

[11] Interestingly, the view—represented most notably by Michael Dummett—that our concept of natural number is "indefinitely extensible" is best regarded as a philosophical response to the theoretical incompleteness of formal theories of arithmetic against the background of our conception of the standard natural numbers as constituting a determinate ontology. It is because we take the (unique) intended structure to consist only of the *standard* natural numbers that we are able, either by higher-order reasoning or by the application of reflection principles, to keep on extending our incomplete theories with new principles that are true of the intended structure. Note also that once given the standard natural numbers as the only members of the domain, the atomic diagram of the model is fixed by the noninduction axioms of Peano arithmetic. This reinforces the conviction that our arithmetic thought is directed to a unique intended structure, however incomplete our theorizing about it will have to be.

the relation $\in$ of membership and the operation $\{x|...x...\}$ of set abstraction.[12]

When the monomathematician's aim is to characterize some unique intended structure M, his methodology is as follows. First, he has to choose a language L (in order to talk about the intended structure M) in which all structural nuances of M can be registered (that is, L has to be of appropriate relational type). Then, he has to choose a decidable nonempty set X of axioms in the language L, ensuring that each member of X is intuitively evident and certain as a claim about M. The axioms need to be intuitively evident and certain because of the justificatory weight they are expected to bear. Membership in the set of axioms has to be decidable because we need to be able effectively to tell, when giving proofs, whether each of the premises used in any proof is indeed among the permissible axioms. So, the decidability of the set of axioms is an absolutely general epistemic precondition on the deductive enterprise. Proofs have to convince. To do so, they must be finite, and their starting points must be among the permitted axioms. Moreover, we have to be able effectively to tell when a premise of any given proof is indeed an axiom. Only in that way will we be able to convince ourselves that the conclusion of the proof has thereby been established as a logical consequence of our axioms.

Note, though, that the notion of decidability here is an *informal* one. One need not invoke the mathematical notion of recursiveness, nor the thesis that all decidable sets (of natural numbers) are recursive. Indeed, the notion of decidability will subsequently play a role only in allowing one to calibrate the logical strength of a certain completeness requirement (see (I) below). It will turn out that this completeness requirement can be made very weak and yet still conflict with a categoricity requirement (see (II) below).

To satisfy this requirement of decidability of the set of axioms, it is not enough simply to ensure the finitude of some pathologically defined set of truths and then to propose it as a set of axioms. We need, at the very least, to have the proposer exhibit to us an algorithm for deciding, of any sentence, whether it is in the axiom set proposed. Thus, it will not do simply to choose some large number k and put forward as one's 'axiom set' $\{\varphi \mid \varphi \text{ is true in } N \text{ and length}(\varphi) < k\}$. Although this set is finite, hence (classically) decidable, the proposer

[12] Some of these notions can be defined in terms of their companions. That is, they do not all have to be taken as primitive. For example, the relation $<$ on natural numbers is definable in terms of 1 and $+$; and one can do set theory with just $\in$ primitive, defining set-abstraction terms contextually.

has failed to exhibit an algorithm for deciding membership in it. Moreover, not every one of its members is self-evident. The joint requirement of self-evidence of members, and evident decidability of membership, is epistemically very demanding; but, arguably, satisfied by all well-known and nonpathological axiom sets for important mathematical theories.[13]

$\mathrm{Th}(M)$ will be the set of sentences true in the model M. For each member φ of $\mathrm{Th}(M)$, the monomathematician wants a truth-preserving proof of φ from (some subset of) his set X of axioms about M.

Proofs and refutations are finitary constructions, and they are effectively checkable. A proof is 'of (Δ,φ)', where Δ is a finite set of sentences and φ is a sentence. Δ is the set of premises of the proof and φ is its conclusion. A refutation is 'of Δ', where Δ is a nonempty finite set of sentences. Here again, Δ is the set of premises of the refutation, whose conclusion, of course, is absurdity. Both kinds of construction are sound in the following sense. If there is a proof of (Δ,φ) then every model making all of Δ true makes φ true; and if there is a refutation of Δ, then there is no model making all of Δ true. Proofs and refutations are provided by systems, usually consisting of axioms and rules of inference.[14] We can think of a system, in general (and in a laxer epistemological frame of mind), as simply being an effective enumeration (of proofs or of refutations).

Among the general requirements of the ideal methodology for the monomathematician are those of *completeness* and *categoricity*. Let us understand by *=-literal* an identity statement or the negation of an identity statement. Recall that the monomathematician has a decidable set X of axioms (in some language L) describing an intended structure M. Consider now the following requirement that one should be able to impose on his methods of proof (and of refutation).

(I) *Weak completeness: X-relative refutation in extensions of L.* For any extension L^* of L (via finitely many new extralogical expressions), there is a system of sound refutations such that for any decidable but satisfiable set Y of =-literals of L^*, if $X \cup Y$ has no model, then there is a refutation of some finite subset of $X \cup Y$.

First comment on (I). The intuitive motivation for (I) is as follows. Y is just a simple imagined addendum to the story X about M. If Y

[13] The pathological example here is due to Torkel Franzen.

[14] For a treatment of proofs and refutations as co-inductively definable, see my "Negation, Absurdity and Contrariety," in Dov M. Gabbay and Heinrich Wansing, eds., *What Is Negation?* (Boston: Kluwer, 1999), pp. 199-222.

conflicts with the story told so far (that is, if $X \cup Y$ has no model), then one should be able to tell that it does so—in the sense of "eventually discover," not "effectively decide." And, of course, such discovery can only be sustained by an appropriate refutation.

Second comment on (I). It might be objected that the weak completeness requirement is more general than what might have been envisaged and sought by Hilbert and his followers.[15] For they wanted only of certain special theories that they be *Entscheidungsdefinit*—that is, prove or refute (but not both) every sentence in the language of the theory concerned. Certain selected theories of central interest to mathematicians might turn out to be *Entscheidungsdefinit*, even despite an underlying incompleteness in the system of logic provided for the language.

From this more limited interest in matters of (theory-) completeness, however, it is but a short step of abstraction and generalization to contemplate a conjecture such as (I) above. For the question immediately arises as to how one might expect a variety of theories to be *Entscheidungsdefinit*, if indeed the underlying logic of the language were unable to provide proofs of certain valid arguments. One is far more likely to lay down the more general requirement of logical completeness as a prolegomenon to one's investigations of the *Entscheidungsdefinitheit* of certain select theories formulable in a language equipped with the inferential resources in question. And, once one approaches the matter in this more general spirit, requirement (I) above should appear eminently reasonable, given how weak it appears to be.

Third comment on (I). Just how weak is requirement (I)? A theorem of Harvey Friedman[16] is that on a suitably weak understanding of what is meant by the "models" quantified over, (I) is implied, modulo the system of recursive comprehension arithmetic (known as RCA_0), by 'validity is r.e.', that is, by the *weak* completeness of first-order logic.[17] Conversely, even with X empty and L trivial, (I) implies 'validity is r.e.' over RCA_0. 'Validity is r.e.' is strictly weaker than completeness for logical consequence from sets of sentences in gen-

[15] I am indebted here to Steve Awodey.

[16] Personal communication.

[17] RCA_0 is the system of second-order arithmetic with induction on all Σ^0_1- and comprehension on all Δ^0_1-formulae. A useful introduction to the concepts and results involved in Friedman's program of so-called "reverse mathematics" is S. G. Simpson, "Subsystems of Z_2 and Reverse Mathematics," Appendix to Gaisi Takeuti, *Proof Theory* (Amsterdam: North-Holland, 1986, 2nd edition), pp. 432-46. The main aim of reverse mathematics is to calibrate the logico-mathematical strength of various theorems.

eral; and strictly weaker than compactness. The required understanding of 'model' is that a model is a saturated set of sentences resulting from the atomic diagram of a model in the ordinary sense. This weaker-than-usual sense of 'model' suffices for the logical application to be made of (I) below, and for the philosophical moral to be drawn from the result of that application.

The point of the technical digression in the foregoing comment is to stress that the clash of imperatives to be revealed between our completeness requirement (I) just formulated, and the categoricity requirement (II) to be formulated below, arises even though (I) is, in the sense just clarified, a relatively *weak* completeness requirement. The irresoluble tension arises between two distinct, and prima facie rather modest, methodological aims.

Turning now to the second of these aims, bear in mind that I am still considering the monomathematician's set X of axioms describing an intended structure M. It would be perfectly legitimate to impose the following requirement on the monomathematician; indeed, it would be self-imposed, as a way of explaining his own understanding of his enterprise.

(II) *Categoricity requirement.* Any structure making X true is isomorphic to the intended model M (that is, X is categorical).

Comment on (II). Note that (II) says nothing about the number of possible isomorphisms between any two isomorphic models. There is a more stringent requirement that could be laid down, namely, the requirement that any model isomorphic to the intended model should enjoy exactly one isomorphic mapping onto it. Intended models meeting the more stringent requirement are what Wilfried Sieg[18] calls "accessible domains." As it happens, the models for which the noncompossibility theorem holds are accessible domains; but for that very reason we can leave (II) in its current, weaker, form.

Now, it is important, for an appreciation of the dialectical exposition to follow, that the reader be aware that it is not intended at this stage to bring to bear any substantial results in metalogic which might reflect on either the rationality of the monomathematician's aspiring to attain ideals (I) and (II) or the logical possibility of attaining them. The aim is to make as clear as possible a distinction grounded in what one might call *prefoundational* conceptions of mathematical interests and practice. That is, the distinction between monomathematics and

[18] "Aspects of Mathematical Experience," in Evandro Agazzi and György Darvas, eds., *Philosophy of Mathematics Today* (Boston: Kluwer, 1997), pp. 195-217, here p. 206.

polymathematics is to be drawn by employing notions which are easy to grasp but about which one does not, at this stage, take oneself to know any deep or important metalogical results.

A skeptical historian of logic and mathematics might object that it would be anachronistic to impute ideals (I) and (II) to any thinker during, say, the early 1920s or even the heyday of the Vienna and Berlin Circles. Logicians, mathematicians, and philosophers, the objector might contend, simply did not have the various concepts in sharp enough form to frame any methodological ideals in this way.

But such an objection would be misplaced. The concept of the categoricity of a system of axioms was well established as early as 1902 (E. V. Huntington[19]) but certainly no later than 1910 (O. Veblen and J. W. Young[20]); while the concept of the completeness of a system of logical proof was properly formulated no later than 1918 (Hilbert and Paul Bernays[21]). The more recent studies of Hilbert and Bernays's lecture notes of 1917/1918, and of Bernays's *Habilitationsschrift* of 1918, infirm Warren Goldfarb's[22] contention that first-order logic was really properly understood only by 1928.

During the decade before the publication, in 1928, of Hilbert and Wilhelm Ackermann's *Grundzüge der theoretischen Logik*,[23] in which the definition of completeness of first-order logic was first published, the main figures at Göttingen, and many talented students trained by Hilbert, and many a distinguished visitor invited there by him, would presumably have been conversant with the concept of deductive completeness articulated in 1917/18. So, at least for this relatively privileged group in the decade before publication of the *Grundzüge*, the conceptual materials (completeness and categoricity) were readily at hand. The methodological ideals (I) and (II) would have been intellectually accessible, appealing, and compelling.

Any of these workers, then, could have aspired to meet ideals (I) and (II), only on the charitable assumption that they grasped the constituent concepts. For at least a few months in the year 1929, after Gödel's proof of his completeness theorem for first-order logic, it

[19] "A Complete Set of Postulates for the Theory of Absolute Continuous Magnitude," *Transactions of the American Mathematical Society*, III (1902): 264-79.

[20] *Projective Geometry*, Volume I (Boston: Ginn, 1910).

[21] See Gregory Moore, "The Emergence of First-Order Logic," in William Aspray and Philip Kitcher, eds., *History and Philosophy of Modern Mathematics*, Minnesota Studies in Philosophy of Science, Volume XI (Minneapolis: Minnesota UP, 1988), pp. 95-135; and Sieg, "Hilbert's Programs: 1917-1922," *Bulletin of Symbolic Logic*, v, 1 (1999): 1-44.

[22] "Logic in the Twenties: The Nature of the Quantifier," *Journal of Symbolic Logic*, XLIV, 3 (September 1979): 351-68.

[23] Berlin: Springer, 1928.

might well have seemed that (I) and (II) were attainable. For certainly his completeness theorem, though welcome, came as no surprise; and its proof required some of the main concepts directly involved in (I) and (II)—which concepts certainly sufficed for easy definition of all the other concepts involved in (I) and (II).

The noncompossibility theorem proved below, however, shows that (I) and (II) are not jointly attainable in certain natural and important cases. Could not this impossibility have been evident well before Gödel's completeness proof? After all, only the concepts were needed and not any results, such as compactness or strong completeness. The conclusion forces itself upon one that there was a quite remarkable "blind spot" to the general impossibility established by the noncompossibility theorem.

IV. THE NONCOMPOSSIBILITY THEOREM

The monomathematician could content himself with proving theorems from his axiom set X, thereby learning more about the intended model M. The monomathematician wants to know what it is like to be immersed within this one structure, M; and the obvious way to do that would be to prove theorems about M from (subsets of) the obvious, decidable, (theoretically) complete, and categorical set of axioms X. From the structure M and his intuitions about it, the axioms and theorems should flow. Conversely, from those axioms (and the theorems that follow from them), an intellect not yet apprised of the structure that is their source should be able to work out that it is (isomorphic to) M.

That is all well and good, while thinking quite generally of structure M, and not being apprised of substantial results of metalogic. But even at this "naive" stage, our metalogically uninstructed philosopher of mathematics ought to realize that there is a fundamental difficulty in meeting the two ideals (I) and (II). Consider what is involved in meeting them: given a structure M, the language L of the appropriate relational type is forced upon us; so it is left to us to try to choose a decidable set X of sentences in L, such that: (1) every sentence in X is true in M, (2) for any extension L^* of L, one can choose some "X-relative" sound refutation method such that for any decidable but satisfiable set Y of $=$-literals of L^*, if $X \cup Y$ has no model, then there is a refutation of some finite subset of $X \cup Y$; and (3) any model of X is isomorphic to M, that is, X categorically describes M.

But we have the promised unpalatable result, which shows we are stymied:

Noncompossibility theorem. If M is a countably infinite structure every one of whose individuals is definable, then M cannot be categorically described by any decidable set X of M-truths for which (2) holds.

In the proof about to be given, it matters not whether the mathematician is taken to be speaking and reasoning about the structure M at first order or any higher order. The considerations to be advanced are perfectly general; they hold for all languages, of whatever order (first, second, or higher order), and regardless of the logical vocabulary employed (branching quantifiers, infinite-cardinality quantifiers, and so on).

Proof of the noncompossibility theorem. Let M be a countably infinite structure every one of whose individuals is definable. Thus, for every individual m in M, there is some term t_m in the language that denotes it. Let X be any set of true statements about M.

Introduce a new name a. This gives us the extension L^* of L. Consider $Y =_{df} X \cup \{\neg a = t_m | m \text{ in } M\}$. Suppose for reductio that Y has no model. Then by the desideratum (I) there is a refutation of some finite subset Z, say, of Y. Since Z is a finite set of sentences, each of which is finite, there are only finitely many singular terms involved in members of Z. But M is infinite. Hence some member of M is not denoted by any term occurring in any sentence in Z. Now extend the model M of X by letting the new name a denote such a member of M. We thereby obtain a model for Z—which is impossible, since there is a (sound) refutation of Z.

So by classical reductio ad absurdum, Y must have a model after all. And such a model cannot be isomorphic to the intended model M, because it has to contain a denotation for the name a distinct from the denotation of any term t_m. Quod erat demonstrandum.

There was a considerable window of time during which the reflections embodied in the noncompossibility theorem above should have been transparent. All that is needed for an appreciation of those reflections is a grasp of the concept of categoricity, a grasp of refutations as finitary and sound, and the conviction that refutation modulo axioms is our sole uniform means of access to the falsity of false claims about certain infinite structures.

I remarked above that the noncompossibility theorem is perfectly general, afflicting all languages. Thus, for example, it establishes the noncompossibility for second-order language as much as it does for first-order language. One does not have to know that first-order logic is complete and conclude from that deep fact that first-order arithmetic will have nonstandard models. Nor does one have to know that second-order arithmetic is categorical and conclude from that substantive fact that second-order logic is deductively incomplete. Without knowing either that first-order logic is complete, or that second-order arithmetic is categorical, one can say, on the basis of the noncompossibility theorem: "Look, it doesn't matter whether you try to describe the system N of natural numbers at first order or at second order. Whichever way you set about it, you will not be able to produce a categorical description of N within a language whose deductive system is even weakly complete (in the sense specified in (II))."

The result is so simple and so general that one can be forgiven for wondering why no one discovered it well before Gödel's proof in 1929 that first-order logic is complete. It would not be anachronistic to expect mathematical experts of the day to have been minded of the "flavor" of the method of proof of the noncompossibility theorem. The idea of being able to do with a finite subset of objects of a certain kind what can be done with an infinite set of objects of that kind was not at all new. In 1871, Heinrich Eduard Heine had proved that any real function continuous on a finite closed interval was uniformly continuous. As J. C. Burkill and H. Burkill[24] observe, his argument "contains the seeds of a covering theorem" (*ibid.*, p. 71). In 1894, Emil Borel proved that every covering of a finite closed linear interval by a countable collection of open intervals has a finite subcovering. By 1902, Henri Lebesgue had removed the restriction of countability. By 1905,[25] what is now known as the Heine-Borel theorem "was essentially known": any set in a metric space is compact if and only if every open covering of the set contains a finite subcovering. This theorem is paradigmatically of the flavor required to make the noncompossibility theorem, if not more palatable, then at least digestible.[26]

It would not be anachronistic, either, to expect foundationalists in the early 1920s to be able to engage in the sort of minimally "model-theoretic" thinking involved in the proof of the noncompossibility theorem. All that is required, after all, is contemplation of the *denotation* relation between singular terms of the language and members of the domain of discourse. Alien intruders, avoiding denotation by any singular term of L, obviously destroy categoricity of theories expressed in L. Ever since the early work of geometers such as Veblen and Huntington (not to mention: Hilbert himself), mathematicians were well able to drive a wedge between language and its subject matter. They were flexible enough to appreciate, for example, that a *finite* (hence discrete) system of points modeling the axioms of projective geometry establishes the consistency of those axioms—even

[24] *A Second Course in Mathematical Analysis* (New York: Cambridge, 1970).
[25] As evidenced by Borel's note in *Comptes Rendus*, CXL (1905, I): 298-300.
[26] See Burkill and Burkill, pp. 69-71.

though the primary, intuitive applications of projective geometry are to infinite and continuous two- and three-dimensional spaces.

The reader well-versed in the history of model theory will recognize that the proof of the noncompossibility theorem is very like that of theorem 3 of Anatolii I. Mal'cev[27] (1936), concerning the extendibility of infinite models. Mal'cev proved his result, however, as a corollary to the compactness theorem for first-order logic (which he had generalized to uncountable systems in theorem 1 of the same paper). The noncompossibility theorem culls the most general features of the reasoning involved, in order to bring out the fact—not noted by Mal'cev—that a predicament is thereby generated for any language whatsoever. Robert Vaught[28] has remarked that it is extraordinary that neither Thoralf Skolem nor Gödel observed (before Mal'cev's contribution) that the existence of nonstandard models for Th(N) "is a simple consequence of the compactness theorem [for first-order logic]" (*ibid.*, p. 377). It is even more extraordinary that no logician of their caliber pointed out well before the completeness and compactness theorems that categoricity and even a weak form of logical completeness would be impossible to attain simultaneously.

The predicament established by the noncompossibility theorem is to be distinguished from what is usually called Skolem's paradox. In 1920, Skolem[29] improved a result of Leopold Löwenheim, showing that any model of a theory contained a countable elementary submodel of the theory. It is this version of the Löwenheim-Skolem

[27] "Untersuchungen aus dem Gebiete der mathematischen Logik," *Matematicheskii sbornik*, n. s. I, 3 (1936): 323-36; English translation "Investigations in the Realm of Mathematical Logic," in Mal'cev, *The Metamathematics of Algebraic Systems: Collected Papers, 1936-1967*, Benjamin F. Wells III, ed. and trans. (Amsterdam: North-Holland, 1971), pp. 1-14.

[28] Introductory Note to Gödel's reviews of two of Skolem's papers, in *Kurt Gödel: Collected Works*, Volume I, Feferman et alia, eds. (New York: Oxford, 1986), pp. 376-78. The papers in question by Skolem were "Über die Unmöglichkeit einer Charakterisierung der Zahlenreihe mittels eines endlichen Axiomensystems" (1933), and "Über die Nichtcharakterisierbarkeit der Zahlenreihe mittels endlich oder abzählbar unendlich vieler Aussagen mit ausschliesslich Zahlenvariablen" (1934), both to be found in Skolem, *Selected Works in Logic*, Jens Erik Fenstad, ed. (Oslo-Bergen-Tromsö: Universitetsforlaget, 1970), pp. 345-66.

[29] "Logisch-kombinatorische Untersuchungen über die Erfüllbarkeit und Beweisbarkeit mathematischen Sätze nebst einem Theoreme über dichte Mengen," in Fenstad, ed., pp. 103-36.

theorem that Paul Benacerraf[30] regards as being of greatest importance.

The "Skolem predicament" is that one cannot, at first order, characterize uncountable structures (such as the reals, or the universe of sets) up to isomorphism. Note that the predicament is limited to *first-order* languages. Now, Erik Ellentuck[31] has claimed that

> One of the earliest goals of modern logic was to characterize familiar mathematical structures up to isomorphism...*in a first order language* [emphasis added] (ibid., p. 639).

John Corcoran,[32] however, challenges this view. He adduces the date of Skolem's aforementioned paper and claims "there appears to have been very little interest in first order languages before [1920]" (*ibid.*, p. 192, fn. 7). Thus, Ellentuck (according to Corcoran) must be misdescribing the imputed goal.

The observations below about what will be called the "pre-Gödelian predicament" are not subject to criticism on the same score. For this predicament is independent of any particular choice of, or preference for, first-order languages over other languages.

VI. THE PRE-GÖDELIAN PREDICAMENT

The noncompossibility theorem tells us that whatever axiomatization X about M we were to provide to an inquiring intellect, we could have no guarantee that he would be able both to appreciate (in principle) every consequence of X as following from X, and to "form the proper conception" of M as the source of our inspiration for our axiomatization.

This predicament is *pre-Gödelian.* The adjective adverts to its conceptual, not chronological, provenance. It shows that, even before we try to prove positive results, such as the completeness of first-order logic, we ought to realize that our epistemic and communicative aspirations in mathematics are already thoroughly compromised. We can only come to know truths with certainty by finitary means. But when the structure M about which we are coming to know truths is a countable infinity of definable ele-

[30] See his symposium with Crispin Wright, "Skolem and the Skeptic," *Proceedings of the Aristotelian Society, Supplementary Volume* LIX (1985): 85-115 and 87-137, respectively. For a discussion of this exchange, and an argument for the claim that the Skolem predicament does not afflict the intuitionist, see my and D. C. McCarty's "Skolem's Paradox and Constructivism," *Journal of Philosophical Logic,* XVI, 2 (1987): 165-202.

[31] "Categoricity Regained," *Journal of Symbolic Logic,* XLI, 3 (September 1976): 639-43.

[32] "Categoricity," *History and Philosophy of Logic,* I (1980): 187-207.

ments, the ineluctable consequence is that M cannot be characterized up to isomorphism by any set of axioms relative to which we have a weakly complete refutation procedure. As soon as we employ a logic that is complete in this weak respect, we shall not be able to specify M up to isomorphism. Conversely, any language that allows us to specify M up to isomorphism will fail to have a logic that is complete even in that weak respect. Deductive and expressive power cannot simultaneously be optimized in our thought about any countable infinity of definable elements. This holds not just of first-order thought, but of thought tout court. In particular, such optimization is impossible in the case of our thought about the natural numbers, or about the rational numbers, or about the hereditarily finite pure sets. For in these unique, intended structures, every element is definable.

It is this condition of definability (or distinguishability) which ensures that the noncompossibility theorem does not conflict with Cantor's[33] well-known result (1895) that every countably infinite linear unbounded ordering $(D,<)$ is isomorphic to the ordering of the rationals. Cantor's back-and-forth method is able to generate the sought isomorphism precisely because the members of D are not distinguished and are not being combined by any algebraic operations that would also have to be accommodated in the isomorphism being established. Thus, the mere ordering $(D,<)$ is quite unlike the standard model of the rationals, all of whose elements are distinguished and enter into algebraic operations. One is led to wonder whether Cantor's result had perhaps fostered a widespread but mistaken impression, by the turn of the century, that countable mathematical structures would, in general, be categorically describable.

One would like to think not, and to think that the simple reflections behind the noncompossibility theorem could, in principle, have been accessible to any philosophically minded mathematician (such as Bernays, Hans Hahn, Hilbert, Skolem, Veblen, Weyl, or Young) over a decade before Gödel proved the completeness and compactness theorems for first-order logic.

The stark lesson there for the grasping was: if a mathematician really knows what he is thinking about, then no one could be in a position to deduce any given consequence of his thoughts. And

[33] "Beiträge zur Begründung der Transfiniten Mengenlehre," *Mathematische Annalen*, XLVI (1895): 481-512; here §9: "Der Ordnungstypus μ der Menge R aller rationalen Zahlen, die grösser als 0 und kleiner als 1 sind, in ihrer natürlichen Rangordnung."

anyone who is in a position to deduce any given commitment that a mathematician makes by expressing certain thoughts cannot be sure what the mathematician is thinking about. Or, with apologies to Werner Heisenberg,[34] whose uncertainty principle for quantum mechanics was, perhaps significantly for these investigations, formulated only in 1927: in countably infinite realms, you cannot know both where you are and where you are going.

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[34] "Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik," *Zeitschrift für Physik*, XLIII (May 1927): 172-98.