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Delicate Proof Theory

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1. Introduction

Prior would have loved Prolog proof-finders. The only data type in Prolog is the *term*, and indeed the term *in Polish notation* (with unnecessary parentheses supplied, unfortunately!). Prolog-based proof-finders would move the little old lady to exclaim 'Why, it's Polish all the way down!' For the proofs themselves are represented as Polish terms, their functors being names of rules of inference. At certain argument places there are terms for formulae; and these in turn are in Polish. And the Prolog computation that finds a proof could itself be presented as one big Polish term, representing a meta-proof in the Horn fragment of first-order logic. If this ever became a standard way of doing computational logic, I think Prior ought, timelessly, to be deeply pleased.

Sharing those Polish proclivities, I have been investigating how best to represent in Prolog decision algorithms (or proof-finders) for constructive and relevant systems of propositional logic. These computational investigations have required one to study the nuances of systemic constitution that influence how readily one may find a proof if there is one. I call this study *delicate proof theory*. And my purpose in this chapter is to impart just a little of its flavour. Unfortunately, I let some impurities intrude; I resort to infix notation with easily surveyable examples, since everyone is so much more at ease reading them that way.

Delicate proof theory is concerned with the fine structure of proofs. It complements gross proof theory. Gross proof theory is typically concerned with such matters as the coextensiveness of deducibility relations of different systems of proof. It shows, that is, that they are systems of

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proof for one and the same logic. A well-known gross result is Gentzen's *Hauptsatz*, showing that one can always get rid of applications of the Cut rule in any sequent proof. The other rules match the output of those rules plus Cut.

Delicate proof theory, by contrast, answers to the intuition that there is more to the field of logic than meets the eye from the vantage-point of a turnstile. Its major concern is with *how* we effect a deductive transition from premisses to conclusion. It looks at ways of ploughing furrows. Even with a fixed repertoire of axioms and rules of inference for a given logic, this can be a fascinating question. How many 'distinct' lines of reasoning are there? Can we explicate them by appeal to proofs in some preferred form? How can we always find the 'shortest' proof? Or the most 'direct' one? Or one using a weakest possible subset of the given premisses? How is the complexity of the decision problem affected by insisting on having proofs in some preferred form?

I do not address, let alone give answers to, such questions in this chapter. Despite my title, this chapter can be thought of as hovering at the gross level and setting the stage for future investigations at the delicate level. All inquiry needs to be conducted within a framework that imparts some orientation and sense of purpose. Arthur Prior was a prime example of a philosopher-logician who pursued his logical concerns with constant regard for their philosophical pay-off. I intend these investigations in the same spirit, because I do believe that proof theory has yet to enjoy its philosophical heyday.

I also believe that heyday is not far off. The computational philosophy of mind is a broadening paradigm within philosophy and cognitive science. It is a paradigm against which prominent heretics, such as Searle and Putnam, have to mark their distinctive departures. It stresses, as the essence of ratiocination, the *effective processing of finitary representations*. Applying rules of inference, and transforming the proofs thus constructed, are prime examples of such processing. But to date we have paid very little attention to the fine structure of that processing and the corresponding fine structure of its products. If we do pay attention to it, I believe there are great gains to be made, especially in computational logic.

This discipline has been driven in its early history by the expressive exigencies of available programming languages. It has been limited also by the computational expertise of workers whose training in logic has been very basic and orthodox. Logic as taught in a mathematics or computer science department is usually not very informed by philosophical subtleties to do with either the nature of reasoning, or the

explicative promise of computational logic as potentially contributing an important piece of the jigsaw of human cognitive competence. Rather, the dominant paradigm in computational logic—resolution theorem-proving—is so machine-friendly that deductive problems are decided without even producing *proofs*! And if a 'proof' *were* to be produced within this paradigm, it would bear absolutely no relation—homomorphic, permutational, or otherwise—to the abstract objects that mathematicians, philosophers, or logicians would call proofs.

Delicate proof theory offers a possible foundation for a radical reorientation in computational logic. The aim would be to study transformations on proofs, and various kinds of 'normal form' of proofs, so as to be able to implement in our mechanical proof-finders various syntactically subtle constraints on proof-search. The search will be for *real proofs*—nice, readable, suasive natural deductions in which every step is made clear; from which one can read off the premisses and conclusion; from which one can understand the 'line of reasoning' pursued; from which one can see how the information has been unpacked from the premisses, and rearranged so as to yield the necessarily following conclusion.

The present author has indeed produced such a proof-finder for minimal logic, programmed in Prolog. It incorporates a host of constraints from delicate proof-theoretic investigations, and the resulting performance is remarkable. Those detailed investigations have been published separately at monograph length (see Tennant 1992).

My more limited aim in this chapter is to set the wider stage, within which one will be able better to appreciate how to adapt the computational methods to other naturally related logical systems. This chapter is therefore concerned with various logical systems that are important for philosophy, mathematics, the explication of natural reasoning, etc. These include the well-known systems M, I, and C of minimal, intuitionistic, and classical logic; the system **R** of relevance logic due to Anderson and Belnap (see Anderson and Belnap 1975), and Došen's intuitionistic version D thereof (see Došen 1981); and relevant systems CR and IR, which I have derived in a delicate way from C and I respectively (see Tennant 1984, 1987*a*,*b*). I would argue that these systems CR and IR are better codifications than R or D of what we intuitively take to be 'relevant reasoning'. CR and IR are two important systems of relevant logic that satisfy a principle of *restricted* transitivity of deducibility. Similar in this respect are the systems T of *truth-table logic* and K of *Kalmar logic* (see Tennant 1989).

From an orthodox perspective, each of CR and IR results by relevantizing the respective parent systems C and I. CR is a system of *classical* relevant logic because it still retains, for example, $\sim\sim A : A$, the law of double negation elimination. Both CR and IR are *relevant* systems because in neither of them can one prove the Lewis paradoxes $A, \sim A : B$ and $A : B \vee \sim B$. (Nor, for that matter, can one prove $A, \sim A : \sim B$.) From the constructive and relevant perspective that I advocate, however, IR is the correct system of reasoning. CR can be understood as resulting from IR by adopting the invalid rule of double negation elimination (or the rule of classical reductio). C then results if one commits the fallacy of relevance, allowing a contradiction to imply any proposition. And I results from IR by making the same mistake.

Kalmar logic is a very restricted logic, which aims only to trace the deducibilities involved in evaluating sentences as true or false under assignments of truth-values to their atoms. Truth-table logic results from Kalmar logic by allowing the derivation of an absurdity to reflect equally badly on any premiss involved in the reductio. More details about these two unusual systems will emerge below.

I aim to bring out the naturalness of the relationships among these systems, as far as this is possible. I do so by trying to develop a uniform method for 'co-ordinatizing' these logical systems within an abstract system space. All the features and structure involved in locating these systems with respect to one another will provide the raw material for further delicate investigations within any chosen system. I presuppose only a basic familiarity with the methods of natural deduction, as set out in Tennant (1978).

2. *What are the Desirable Features of a System of Proof?*

A system of proof should do more than simply generate the right consequence relation. It should formalize in a natural way the structure of reasoning by which thinkers could, in principle, establish those consequences. (I say 'in principle' in order to cover the necessary extrapolations to arguments so long that we could not feasibly cope with them.) In no particular order of importance, the following features of systems of proof deserve consideration. After presenting the list, I shall elucidate each item a little further. The presence of an item on the list does not mean that I endorse it; at this stage it is simply worth considering.

1. Unrestricted transivity of deduction.
2. Separability of rules for logical operators.
3. Reduction procedures for logical operators.

4. Harmony between introduction and elimination rules.
5. Normalizability of proofs.
6. Proofhood (hence deducibility) closed under uniform substitution.
7. Relevance.
8. Preservation of preferred species of truth.
9. Adequacy as metalogic for truth theory.
10. Deduction theorem.
11. Duhemianism on denial.
12. Nice 'mereology' of proof.
13. Containing what one wants of classical logic under a simple translation.
14. Interdeducibility of two sentences should be sufficient for their interreplaceability, *salve veritate*, in all statements of deducibility.

I now deal with each of these items in turn.

2.1. *Unrestricted Transitivity of Deduction*

Proofs should 'accumulate'. In mathematics, one expects that if one proves various lemmata from various sets of axioms respectively, and then proves a theorem from some axioms and those lemmata, then the theorem should follow by means of proof from all the accumulated axioms:

If P_1 is deducible from X_1,
...,
P_n is deducible from X_n,
and Q is deducible from $X_0 \cup \{P_1, ..., P_n\}$,
then Q is deducible from $X_0 \cup X_1 \cup ... \cup X_n$.

Thus we expect a mapping $\mapsto$ on proofs as follows:

$$\begin{array}{c} X_1 \; ... \; X_n \\ \Pi_1 \; ... \; \Pi_n \\ P_1 \; ... \; P_n \\ \\ X_0, P_1, ..., P_n \\ \Pi_0 \\ Q \end{array} \quad \mapsto \quad \begin{array}{c} X_0 \cup X_1 \cup ... \cup X_n. \\ \Pi \\ Q \end{array}$$

The way to understand this display is as follows. Suppose, for each i from 1 to n, that you have used the set X_i of assumptions in the proof Π_i whose conclusion is P_i. (To help fix ideas: members of the X_i may be axioms of some mathematical theory; and the P_i may be mathematical lemmata.) Suppose further that, in a separate deductive enterprise, you gathered up all the lemmata $P_1, ..., P_n$ and availed yourself further of the set X_0 of assumptions (axioms); and that from this combined set of assumptions you give the proof Π_0 with conclusion Q. (To continue the example: Q is some sought theorem of mathematical interest.)

The axioms are the starting-points. The lemmata are deductive 'half-way houses': that is, they are *conclusions* of the proofs $\Pi_1, ..., \Pi_n$ shown at the top of the display above; but they are *assumptions* of the proof Π_0 shown at the bottom of the display.

What the displayed transformation shows is that one should be able to 'accumulate' all these proofs $\Pi_0, \Pi_1, ..., \Pi_n$ in order to obtain a proof Π of the overall conclusion Q from the set $X_0 \cup X_1 \cup ... \cup X_n$ of combined assumptions. The union $X_0 \cup X_1 \cup ... \cup X_n$ is the set of 'ultimate starting-points' for the deduction of Q.

It is clear that this assumption of *unrestricted* transitivity is what many a logician unreflectingly takes to lie behind all our mathematical practice. Indeed, the method of accumulation commonly assumed is that the proof Π can be obtained simply by grafting copies of Π_i wherever needed over assumption occurrences of P_i in the proof Π_0. This tacit assumption behind our practice is suggested by the way I have positioned the proofs $\Pi_1, ..., \Pi_n$ above the respective indicated assumptions $P_1, ..., P_n$ in the proof Π_0 below. The simple thought is that within the proof Π yielded by the transformation, the status of the P_i as half-way houses need not be obscured. They can still be 'seen' within Π, 'half-way down', as it were. *But this is no more than a suggestion.* I do not think the simple thought is correct. I want very much to leave open the possibility that the mapping $\mapsto$ applies to the given proofs $\Pi_0, \Pi_1, ..., \Pi_n$ in a non-straightforward way to produce the proof Π: that is, not necessarily by 'modular patching', or grafting as just described, but by reshaping or transforming the given proofs $\Pi_0, \Pi_1, ..., \Pi_n$ and possibly radically rearranging their syntactic materials in pursuit of Π. In this process the occurrences of the P_i 'half-way down' might even be obliterated; that is, Π might not contain any trace of the deductive strategy that involved picking the P_i as useful lemmata. All that *would be* required is that the mapping $\mapsto$ be *effective*. We should be able to find Π mechanically from $\Pi_0, \Pi_1, ..., \Pi_n$, without need for further ingenuity or creative deductive insight.

The simple thought—that one can simply treat the P_i as grafting-points in the creation of the longer proof Π—is permissible only when the proof system allows proofs such as Π to contain formulae occurrences such as the P_i. These occurrences would in all likelihood stand as conclusions of applications of introduction rules (the terminal steps of the proofs Π_i) and as major premisses of applications, within Π_0, of the corresponding elimination rules. As such they would be what I call *I/E formula occurrences* (within the proof Π).

Now a proof is usually said to be *normal* just in case it contains no I/E formula occurrences. It follows that, if one requires of proofs in a given system S that they be in normal form, the simple thought about transitivity-by-accumulation will not do for S. One will require the more subtle provision according to which the mapping $\mapsto$ applies to the given proofs $\Pi_0, \Pi_1, ..., \Pi_n$ in a non-straightforward way in order to yield a result *Π in normal form.*

2.2. Separability of Rules for Logical Operators

Rules should deal with 'one occurrence of an operator at a time'. The best-known examples are the system of natural deduction and the system of sequents. Separability makes study of language fragments easier. It serves a molecular, as opposed to an holistic, theory of meaning. It helps with heuristics for finding proofs, and it also helps one to compare different logical systems. (How do their respective rules for introducing a conditional, for example, differ?)

2.3. Reduction Procedures for Logical Operators

These are the operator-by-operator expression of Prawitz's 'inversion principle' (see Prawitz 1965). Roughly, by means of an elimination applied to the dominant operator in a sentence, one should not be able to deduce anything that one did not have to be able to establish in order to be entitled to introduce it in the first place. That is, one may only 'unpack' what one had to be able to 'pack in'. The reduction procedures may need fine-tuning when we work with a system of proofs that have to be in normal form anyway, and for which only restricted transitivity holds. Reduction procedures may exist without all proofs being normalizable.

2.4. Harmony between Introduction and Elimination Rules

The introduction rule should be the *weakest* one that enjoys a reduction procedure with the elimination rule; and the elimination rule should be the *strongest* one that enjoys a reduction procedure with the introduction rule. The existence of a reduction procedure is necessary but not sufficient for harmony. The requirement of harmony is important, I believe, in order to give an anti-realist account of how introduction and elimination rules respectively *fix* and *respect* the meanings of the logical operators. It also allows one to account for the possibility of logical operators having found their way into a growing language by virtue of selective forces governing the process of communication. I cannot repeat that account here; the interested reader will find it in Tennant (1987*a*).

2.5. *Normalizability of Proofs*

One should be able effectively to convert any proof Π into one in normal form, say Σ. Standardly, the normal proof Σ will have the same conclusion as Π, but the undischarged assumptions of Σ might form a *proper* subset of the set of undischarged assumptions of Π. In systems in which proofs have to be in normal form anyway, normalizability is trivial. Normalizability requires reduction procedures, which in turn require each operator to have its own introduction and elimination rules (separability). Proof in normal form can serve as criteria of identity for arguments: two arguments are identical if and only if they have the same normal form.

There is a variety of normal form theorems in the general mould 'If there is a proof of Q from X then there is a proof of Q from (some subset X' of) X *with feature* F'. Such a theorem could be called an F-normal form theorem. Where F is an easily detectable feature of a proof under construction, an F-normal form theorem is a great aid in computational logic. For it guarantees that, in searching for a proof of Q from X, one need only consider proofs with the feature F. When one has several different normal form theorems, one can exploit them together only if the features F involved are *compossible*. Think of each F-normal form theorem as saying 'When looking for a proof, look only within *this* spotlit region'. What we then require is that the spotlights provided by the various F-normal form theorems all overlap in some non-empty region.

As a special case of normal form theorems, we have what may be called *filtering theorems*. These are of the form 'If $X : C$ has a proof, then $F(X,C)$', where again F is a detectable feature of the premiss set X and conclusion C.

Here is an example (reasonably easy to prove) of an F-normal form theorem:

> If there is a proof in minimal logic of $X : C$, then there is a proof in which all applications of $\supset$-elimination have conclusions that are either atomic or disjunctive.

And here is an example of a filtering theorem:

> If there is a proof of $X : C$, then if C is atomic C is an accessible positive subformula of some member of X.

(For explanation of 'accessible positive', see Tennant 1992.)

My own experience in devising proof-search algorithms for subclassical propositional logics has been that *enormous* speed-up is afforded by the right combination of F-normal form theorems and filtering theorems.

What is so nice about such theorems is that they allow one to do what I would call *intrinsic* or *endogenous* proof-search. That is, proof-search proceeds only with attention paid to the patent syntactic structure of the formulae in the sequent $X : C$ to hand. One exploits only intrinsic, effectively determinable properties of X and C. One does *not* examine extrinsic, relational properties of $X : C$, such as counter-exemplifiability in some specially concocted matrix that happens to be characteristic for some axiomatization of the calculus. Instead one constrains one's search just by paying attention to what is staring one in the face, as it were. I would conjecture that this is how the cortex does it; but the metaphor of 'staring in the face' might have to be altered!—'staring in the homunculus' face' might be more appropriate . . .

Moreover, these nice theorems affording intrinsic constraints on proof-search can be obtained only by pursuing delicate proof theory. Hence my concern to set out here some of the basic structural ideas to which one can have recourse in such investigations.

I would venture the following methodological conjecture: *Having a metalogical insight in the form of an F-normal theorem or F-filtering theorem (for decidable F) yields much greater benefits for computational logic than those to be obtained by refining one's implementation of an algorithm once the latter has been found, or by 'descending' to a lower-level programming language in order to have the algorithm executed more quickly.*

2.6. *Proofhood (hence Deducibility) is Closed under Uniform Substitution*

This is implicit in our frequent practice of proving results schematically, or in thinking of primitive extra-logical symbols as 'place-holders'. It is explicit in axiomatic systems that have a rule of substitution as a primitive rule of inference. It fails in systems that require non-vacuous and thorough discharge of assumptions:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{(1)-}{B}\quad B\supset C}{C}\qquad \dfrac{\overset{(2)-}{A}\quad A\supset(C\supset D)}{C\supset D}}{D}\,{\scriptstyle(1)}}{B\supset D}\,{\scriptstyle(2)}}{A\supset(B\supset D)} \quad \underset{[A \text{ for } B]}{\mapsto} \quad \dfrac{\dfrac{\dfrac{\dfrac{\overset{(1)-}{A}\quad A\supset C}{C}\qquad \dfrac{\overset{(1)-}{A}\quad A\supset(C\supset D)}{C\supset D}}{D}\,{\scriptstyle(1)}}{A\supset D}\supset\text{I would be vacuous}}{??}$$

This display is intended to illustrate the process of trying to substitute A uniformly for B in the proof on the left. The would-be proof resulting from this substitution is shown on the right. Because of what the substitution has wrought higher up, it now transpires that the final step of the 'proof' on the right would be incorrect. This is because there are, by that stage, no more undischarged assumption occurrences of A; yet at least one such occurrence is needed in order to prevent that step of $\supset$I from being vacuous.

2.7. *Relevance (via Restricted Transitivity of Deduction)*

Some logicians insist (and I agree) that we must avoid the Lewis paradoxes $A, \sim A : B$ and $B : A \vee \sim A$ (and also $A, \sim A : \sim B$). Proofs should not trade illicitly on inconsistency of premisses or logical truth of conclusion. Hence we must ban the absurdity rule, or *ex falso quodlibet*.

To avoid getting its effect indirectly, we must therefore also disallow 'vacuous' discharge of assumptions. The standard derivation of $A, \sim A : B$ shows that we have to give up *either* disjunctive syllogism $A \vee B, \sim A : B$ (as do Anderson and Belnap) *or* unrestricted transitivity of deduction.

We can *restrict* transitivity of deduction by requiring proofs to be in *normal form*. To regain disjunctive syllogism, we can relax $\vee$-elimination to allow cases to close off with contradiction.

The resulting system(s) satisfy an *epistemically gainful condition of restricted transivity*.

The epistemically gainful condition of restricted transitivity:

> If P_1 is deducible from X_1,
> ...,
> P_n is deducible from X_n,
> and Q is deducible from $X_0 \cup \{P_1, ..., P_n\}$,
> then *either* Q *or* $\bot$ is deducible from *some subset of* $X_0 \cup X_1 \cup ... \cup X_n$ (which may even be the empty set).

Thus we now expect only a mapping $\mapsto$ on proofs as follows:

$$\begin{array}{c} X_1 \;...\; X_n \\ \Pi_1 \;...\; \Pi_n \\ P_1 \;...\; P_n \\ X_0, P_1, ..., P_n \\ \Pi_0 \\ Q \end{array} \quad\mapsto\quad \begin{array}{c} X\,(\subseteq X_0 \cup X_1 \cup ... \cup X_n) \\ \Pi \\ \bot/Q \end{array}$$

What is important here is that we allow Π to deliver as conclusion not necessarily the conclusion Q, but rather the conclusion of absurdity ($\bot$) in its stead.

I cannot stress too strongly how significant a relaxation this is of the normally unreflective demand that deducibility be unrestrictedly transitive. The restricted principle is epistemically gainful for the following reason. First, one may learn that Q follows from a *proper* subset of what would otherwise have been the set of overall assumptions of Π. This may even, in the extreme case, be the empty set—that is, one would discover that Q is, after all, a logical truth, so one would not want it to have to depend on any set of assumptions (that are not logically true). Secondly, if one does *not* obtain Q as the conclusion of Π, one learns something stronger instead: namely, that the accumulated premisses are *inconsistent*!—so who would *want* a proof of the deep result Q 'from' them anyway?

It remains only to point out that the restricted, epistemically gainful condition *is all that one needs for the cumulative deductive development of mathematics*. (This is because we naturally assume that mathematics is consistent.)

Moreover, since the condition ensures that all *inconsistencies* remain provable, it also guarantees that *we have enough logic for the hypothetico-deductive method in science*—which, at root, involves only the tracking down of inconsistencies. For further development of the case for restricted transitivity, see Tennant (1994).

2.8. *Preservation of Preferred Species of Truth*

One requires truth to be preserved from the premisses of any proof to its conclusion, under any interpretation of extra-logical primitives (*soundness* of proof).

> What counts as truth?
> How is it *preserved*?

Diverging answers are given to these important questions by opponents in the realist–anti-realist debate. The anti-realist would argue that classical truth is just plain truth with some illegitimate classical trappings—namely, the principle of bivalence and the concomitant law of excluded middle, law of double negation elimination, etc. The pared-down, licit notion of truth for the anti-realist is such that (some version of) intuitionistic logic manages to preserve truth from premisses to conclusions *when it ought to be preserved*: that is, when warrants for the premisses can be transformed effectively into warrants for the conclusion. Because of the constructive character of warrants, a strictly classical proof of a conclusion from warrantable premisses will not necessarily produce a warrantable conclusion. The anti-realist's complaint, then, is: since *having a warrant* is what it is to be *true*, strictly classical proofs do not necessarily produce true conclusions from true premisses!

Again, limitations of space prevent me from developing this theme further here; the reader should consult Tennant (1987*a*).

2.9. *Adequacy as Metalogic for Truth Theory*

To do truth theory in a metalanguage for an object language, we should not need a metalogic stronger than the logic of the object language. Consider Tarski's well-known condition of adequacy on truth theory:

> For every sentence S of the object language, there should be proofs, in the metatheory, of its metalinguistic translation $\underline{S}$ from the truth-predication $T(S)$ and vice versa.

This condition concerns the *theory* of truth given in the metalanguage. But is it not a requirement of philosophical stability that the *metalogic* that one uses to develop a theory of *truth* for an object language should not itself require, for its own justification, appeal to a notion of truth that would turn out, in the context of the object language, to be richer than the one *being defined* for the object language?

It is difficult to give precise formal expression to this thought. I would venture something along the following lines:

> Condition of adequacy on metalogic for the condition of adequacy on truth theory:
>
> The condition of adequacy on a truth theory should be met by a truth theory whose logic is that of the object language, and whose truth-theoretic rules result by replacing every truth-bearer placeholder A in a logical rule by $T(A)$.

Intuitionistic relevant logic (IR) meets this condition, but truth-table logic (T) does not. The reason why T does not is very simple: as we shall

see below, all its elimination rules provide only for absurdity ($\bot$) as a conclusion. Thus one cannot, for example, prove $A \,\&\, B$ from $T(A \,\&\, B)$ in T-based truth theory. One can, however, prove $\bot$ from ${\sim}(A \,\&\, B)$, $T(A \,\&\, B)$; hence also prove ${\sim}{\sim}(A \,\&\, B)$ from $T(A \,\&\, B)$! One can only come that tantalizingly close to the desired result $T(A \,\&\, B) : A \,\&\, B$. (There is nothing special in this about &; similar remarks apply to both $\vee$ and $\supset$.)

To see by contrast how IR meets the conditions, we have the following inferential truth theory for a language L, where the logic of L is IR, and the basic principles of the truth theory are 'truth-predicated' on the rules of IR, which also form the metalogic for the theory.

Some preliminary explanation of notation:

1. The box $\square$ next to a discharge stroke indicates that one must have used an assumption of the form indicated; and one discharges all available occurrences of it. In such a case I speak of *non-vacuous* and *thorough* discharge.
2. The symbol $\varnothing$ next to a discharge stroke indicates that one does not need to have used an assumption of the form indicated; but, if one has, one discharges it at all its available occurrences. In such a case I speak of *possibly vacuous but thorough discharge*.
3. The notation $\bot/C$ in the statement of the rule of $\vee$-elimination is to be understood as follows: if either one of the two case proofs has absurdity ($\bot$) as its conclusion, then one may bring the conclusion of the other case proof down as the main conclusion. In such a case I speak of the rule of $\vee$-elimination being *in $\bot/C$-form*.

Note how the rules immediately below, for an IR-based truth theory, are simply the rules for IR (see further below) with occurrences of the truth-predicate T grafted on at appropriate places:

$$\dfrac{\begin{matrix}\square\,\overline{TA}^{\,(i)}\\ \vdots\\ \bot\end{matrix}}{T{\sim}A}\,(i) \qquad\qquad \dfrac{TA \quad T{\sim}A}{\bot}$$

$$\dfrac{TA \quad TB}{TA \,\&\, TB} \qquad\qquad \dfrac{TA \,\&\, B}{TA} \quad \dfrac{TA \,\&\, B}{TB}$$

$$\frac{TA}{TA \vee B} \qquad \frac{TB}{TA \vee B} \qquad\qquad \frac{\begin{array}{cc} & \overset{\square\!-\!-\,\text{(i)}}{TA} \\ & \vdots \\ TA \vee B & \bot/C \end{array} \quad \begin{array}{c} \overset{\square\!-\!-\,\text{(i)}}{TB} \\ \vdots \\ \bot/C \end{array}}{\bot/C}\,\text{(i)}$$

$$\frac{\begin{array}{c} \overset{\square\!-\!-\,\text{(i)}}{TA} \\ \vdots \\ \bot \end{array}}{TA \supset B}\,\text{(i)} \qquad \frac{\begin{array}{c} \overset{\varnothing\!-\!-\,\text{(i)}}{TA} \\ \vdots \\ TB \end{array}}{A \supset B}\,\text{(i)} \qquad\qquad \frac{TA \supset B \quad TA.}{TB}$$

2.10. *Deduction Theorem*

In discussions of the deduction theorems one has to be very clear about exactly what is meant by the deducibility sign $\Rightarrow$. There is a weak reading and a strict reading. The *weak* reading of $X \Rightarrow Q$ is that there is a proof of Q whose set of undischarged assumptions is *some subset of* X. The *strong* reading of $X \Rightarrow Q$ is that there is a proof of Q whose set of undischarged assumptions is *exactly* X. I shall adopt the strong reading. I follow the convention also that when I write X, P on the left of $\Rightarrow$, P is understood not to be a member of X.

The deduction theorem links the implication connective $\supset$ with deducibility $\Rightarrow$:

(1) If $X, P \Rightarrow Q$ then $X \Rightarrow (P \supset Q)$.
(2) If $X \Rightarrow (P \supset Q)$ then $X \cup \{P\} \Rightarrow Q$.

(1) is unproblematic; but what about the closely related

(1′) If $X \Rightarrow Q$ then $X - \{P\} \Rightarrow (P \supset Q)$?

Note that (1′) allows $Q \Rightarrow (P \supset Q)$; whence $\varnothing \Rightarrow Q \supset (P \supset Q)$.

And (2) can cause two kinds of problem:

(2*a*) One might want $Q \Rightarrow (P \supset Q)$ without $Q, P \Rightarrow Q$.
(2*b*) One might want $\sim P \Rightarrow (P \supset Q)$ without $\sim P, P \Rightarrow Q$.

We could replace (2) with

(2′) If $X \Rightarrow (P \supset Q)$ then for some subset Y of $X \cup \{P\}$ either $Y \Rightarrow Q$ or $Y \Rightarrow \bot$.

This would allow one to have $\sim P \Rightarrow (P \supset Q)$, and therefore $\sim P, P \Rightarrow \bot$, without incurring the Lewis paradox $\sim P, P \Rightarrow Q$.

The precise form of deduction theorem that holds for a system turns essentially on the form of $\supset$-introduction rule (rule of conditional proof) in that system, and the system's transitivity property. In IR, for example, the rule of $\supset$-introduction has two parts:

$$\frac{\begin{array}{c} \overset{\square\!-\!\text{(i)}}{A} \\ \vdots \\ \bot \end{array}}{A \supset B}\,\text{(i)} \qquad \frac{\begin{array}{c} \overset{\varnothing\!-\!\text{(i)}}{A} \\ \vdots \\ B \end{array}}{A \supset B}\,\text{(i)}$$

I shall call this the *split* form of conditional proof. Note that it appears to be guided by the thought that $A \supset B$ is somehow equivalent to $(\sim A) \vee B$. The irony, however, is that $A \supset B$ does not imply $(\sim A) \vee B$ in IR. This failure is to be expected, given the same failure in I, of which IR is a subsystem. The converse, though, holds in IR as it does in I. (The reader should see below, in the discussion of discharge requirements, what is meant by the symbols gracing the discharge strokes here. At this point the following explanation should be given: The half-rule on the left says that if one has really reduced A to absurdity, one can infer $A \supset B$ and thereby discharge all the assumption occurrences of A that have helped precipitate the absurdity. The half-rule on the right says that if one has proved B, one may infer $A \supset B$, regardless of whether A was actually used as an assumption to prove B; but, *if* A was so used, one discharges all its occurrences.)

With this rule of $\supset$-introduction, the deduction theorem in form (1) and (1′) above hold for IR. But (2) fails, since we have $A \Rightarrow \sim A \supset B$ in IR, but not $A, \sim A \Rightarrow B$. We do, however, have $A, \sim A \Rightarrow \bot$; and indeed (2′) holds quite generally for IR.

2.11. *Duhemianism on Denial*

Once one has proved a set of sentences inconsistent, one should be able to conclude the negation of *any* one of them from the remaining ones. This is one of two slight changes that produce an important extension of the system of Kalmar logic, which I call truth-table logic. (I introduced the terms 'Kalmar logic' and 'truth-table logic', and the systems to which they refer, in Tennant 1989.) The other slight change is to allow applications of elimination rules to have within their scopes applications of those rules with the same major premiss (see below).

Kalmar logic is just the logic of true 'evaluative' statements of deducibility. Suppose one has a consistent set X of formulae, each of which is an atom or the negation of an atom. Let Q be any formula involving exactly those atoms. If an atom is in X, treat it as true. If its negation is in X, treat it as false. Under the resulting evaluation, if Q is true, then $X : Q$ is valid in Kalmar logic; and if Q is false, then $X, Q : \bot$ is valid in Kalmar logic. These are the only valid sequents of Kalmar logic.

To secure this last result, it is important not to let the rule of negation-introduction in Kalmar logic discharge arbitrary assumptions. For example, Kalmar logic provides a proof of $\bot$ from the assumptions $\sim A$, $A \,\&\, B$. In Kalmar logic one may extend this by a step of negation-introduction to obtain $\sim A : \sim(A \,\&\, B)$. But one cannot use negation-introduction to obtain $A \,\&\, B : \sim\sim$A. Kalmar logic may therefore be said to be *non-Duhemian*. Its rule of negation-introduction allows one to discharge only the assumption of highest degree that has participated in the reductio. If we relax this restriction—that is, if we 'Duhemianize' Kalmar logic—then we get precisely truth-table logic.

It is also important to prohibit applications of an elimination rule within the scope of another with the same major premiss, in order to make Kalmar logic prove only the sequents intended. Thus in Kalmar logic one has both $A \,\&\, \sim A, A : \bot$ and $A \,\&\, \sim A, \sim A : \bot$. One does not, however, have $A \,\&\, \sim A : \bot$. This is because of the scope restriction just described on &-eliminations.

2.12. Nice 'Mereology' of Proof

Think of a proof as tree-structured. One progresses logically through the proof, following the lines of reasoning, roughly by descending within it towards the conclusion at the bottom. (Special allowance has to be made for the way one goes from a disjunction to the case assumptions it yields for proof by cases; and from existential statements to their arbitrary instantiations for proof by existential elimination.) An initial fragment of a proof is any subtree subtended by any formula occurrence within it. Ideally, we would wish in general to know what assumptions such an occurrence depends on, and how it follows from them. That is, *any initial fragment of a proof should be a proof*. Likewise, we may want to forget about how that formula occurrence has been derived, and simply take it as given, and be able to see how the proof goes on from there. That is, *the residue after pruning away any initial fragment of a proof should be a proof*. These features would enable one to read off more logical information from a proof: for, in addition to the overall deducibility-statement established by the proof, one would be able to

come to know all the deducibilities that hold by virtue of all its initial and residual proof-fragments.

2.13. Containing what One Wants of Classical Logic under a Simple Translation

This enables one to give relative consistency proofs for various classical theories. Various translations have been devised in the literature, of which the simplest and best known is the double negation translation.

'What one wants' of classical logic needs explication:

> The translation t *concentrates deducibility in S to deducibility in S′* iff for every X, Y if $X \Rightarrow Y$ in S then for some subsets X', Y' of X, Y respectively, $fX' \Rightarrow fY'$ in S'.

Double negation concentrates deducibility in classical logic to deducibility in truth-table logic. (For a proof of this result, see Tennant 1989.)

2.14. Interdeducibility of Two Sentences should be Sufficient for their Interreplaceability, Salve Veritate, *in All Statements of Deducibility*

This deceptively reasonable condition is easily violated, even though it is commonly met (in C, I, M, R).

For example, if $\supset$I requires non-vacuous and thorough discharge, and proofs have to be in normal form, then $(A \,\&\, B) \supset A$ is interdeducible with $A \supset (B \supset A)$, yet the former is a theorem and the latter not. This example also shows how proofs can fail to accumulate even when the accumulated premisses form a consistent set.

Another example: In CR and IR A is interdeducible with $A \vee (B \,\&\, \sim B)$. But while $A \vee (B \,\&\, \sim B)$ is deducible from $(B \,\&\, \sim B)$, A is not.

Having decided what features are desirable in a system of proof, we have now to consider how to choose a system.

3. Options for Choices in Setting Up Systems of Proof

I spoke above of the rule of $\vee$-elimination being in $\bot/C$-form. One could in the same way speak of the usual rule of $\vee$-elimination being in plain C-form. (This is the form of the rule that requires both case proofs to have the same conclusion C.) Furthermore, if one were to restrict applications of $\vee$-elimination to cases where both case-proofs ended with the conclusion $\bot$ (absurdity), we would be able to speak of the rule being in $\bot$-form.

The same holds quite generally for the elimination rules for & and ⊃. (Note that the rule of ~-elimination is already in ⊥-form; it does not have any more general *C*-form. The rule of ~-elimination is also in both serial and parallel form, in a sense of these terms that will be explained shortly.) As we have seen for ∨, and as we shall now see for & and ⊃, elimination rules can be in either *C*-form or ⊥-form. Choosing the latter form constrains the range of applicability of the rule, and therefore also the deducibility relation that it helps to generate.

In order to see how the possibility of ⊥-form arises for the rules of &-elimination and ⊃-elimination, one has to cast them first in a form that has a structural similarity to the form of ∨-elimination, in so far as it involves discharge of assumptions within subproofs. Usually the rules of &-elimination and ⊃-elimination are stated in what I call *serial form*:

$$\frac{A \,\&\, B}{A} \qquad \frac{A \,\&\, B}{B} \qquad\qquad \frac{A \supset B \quad A}{B}$$

There is, however, an alternative form that one can use in a system of natural deduction. The idea comes from the Gentzen sequent calculus; but we do not go so far as to make the rule apply to *whole sequents* above and below the inference stroke (a choice that means the nodes of a proof tree become cumbersomely labelled). Rather, we simply take over the idea from the sequent calculus that the eliminative use of a premiss should be made, as far as is possible, in 'one fell swoop'. One should not have to invoke *repeated* occurrences of *A* & *B*, say, *with each of these occurrences requiring a fresh copy of its subproof above it*, whenever one needs *A* or needs *B*. This drawback of the serial form of elimination rules for & and ⊃ is overcome by the following *parallel forms* of these rules:

$$\frac{A \,\&\, B \qquad \begin{array}{c} {}^{(i)}\overline{A}\,,\,\overline{B}^{(i)} \\ \vdots \\ C \end{array}}{C}\,(i) \qquad\qquad \frac{A \supset B \quad A \qquad \begin{array}{c} \overline{B}^{(i)} \\ \vdots \\ C \end{array}}{C}\,(i)$$

These parallel forms are also *C*-forms, since they provide for a general conclusion *C*. There can also be parallel forms of these rules in ⊥-form:

$$\frac{A \,\&\, B \qquad \begin{array}{c} {}^{(i)}\overline{A}\,,\,\overline{B}^{(i)} \\ \vdots \\ \bot \end{array}}{\bot}\,(i) \qquad\qquad \frac{A \supset B \quad A \qquad \begin{array}{c} \overline{B}^{(i)} \\ \vdots \\ \bot \end{array}}{\bot}\,(i)$$

These are indeed the forms of &-elimination and ⊃-elimination required for truth-table logic.

From our discussion of forms of E-rules, we see that we have the following options concerning E-rules:

Form of E-rules

⊥-form or C-form?

Pure strategy 1:
Have all E-rules in ⊥-form

Pure strategy 2:
Have all E-rules in *C*-form

Hybrid strategy:
Have some E-rules in ⊥-form, some in *C*-form;
have ∨E in ⊥/*C*-form

Serial form or parallel form?

Serial strategy:
Have some E-rules in serial form (&E, ⊃E in standard systems)

Parallel strategy:
Have all E-rules in parallel form (&E, ⊃E in truth-table logic)

These choices are not always exclusive. Some systems (such as classical logic C) can be formulated with E-rules in either form. This is because the extra power available in the other rules (most notably the classical rules of negation) allow one to manage even with the restriction to ⊥-form. But other systems, by contrast (such as truth-table logic T) depend constitutively on choosing just one form of E-rule—in the case of T, the ⊥-form.

I note here for subsequent consideration that choosing to have E-rules in parallel form will combine with differing discharge requirements to affect the deducibility relation. We cannot consider this in greater detail here, as those discharge requirements have not yet been described.

So much for the form of E-rules. What about the form of proofs? By far the most important strategic and structural choice is whether to insist that proofs be in normal form:

Form of proofs

Normality strategy:	
Proofs must be in normal form	
Abnormality strategy:	
Proofs need not be in normal form	

I have already spoken of vacuous versus non-vacuous discharge rules, and of thorough discharge. Discharge of assumption occurrences occurs when a discharge rule is applied within a proof. A proof establishes its conclusion on the basis of the assumptions which enjoy at least one undischarged occurrence by the stage at which the conclusion is reached.

If it is *permitted* that *no* assumption occurrences be available for discharge, we have *vacuous* discharge. If it is *permitted* that *not all* available assumption occurrences be discharged by the application of a discharge rule, we have *selective* discharge. (Of course, a selective discharge rule could allow one at times to discharge *all* available occurrences. In general it permits the discharge of *any number* of the available assumption occurrences.) If it is *required* that *all* available assumption occurrences be discharged, we have *thorough* discharge.

Finally, a notion we have not yet mentioned explicitly: *contraction of assumptions*. If it is permitted that more than one available assumption occurrence (in its particular subproof) be discharged, then we have *contractive* discharge. The well-known *contraction-free* systems of logic have only *non-contractive* discharge: that is, they do not allow more than one available occurrence of an assumption to be discharged.

We therefore have a variety of options concerning discharge rules. It will be evident to the reader what these are, and how varied might be the combinations of discharge requirements, as they affect individual discharge rules, that would help to determine a system of proof. Moreover, the choice of a parallel as opposed to a serial form of &-elimination and/or $\supset$-elimination can contribute even more complexity to the overall picture. We are now in a position to appreciate this point, which was foreshadowed earlier. With the serial form of &-elimination, one might have $A \& B$ immediately above one occurrence of B, and have $B \& C$ immediately above another occurrence of it. Further down in the proof one might apply $\supset$-introduction, discharging $A \& B$; and still further down, apply $\supset$-introduction again, but this time discharging $B \& C$. With a *parallel* form for &-elimination, however, all occurrences of B might be discharged with but the one use of major premiss $A \& B$. The other conjunction $B \& C$

would thereby be rendered otiose. If, therefore, we were also requiring *non-vacuous and thorough discharge*, the second application of $\supset$-introduction might go begging for an assumption $B \& C$ to discharge. One might be able to get round such a problem by relocating applications of &-elimination and $\supset$-introduction so as to get all the original formulae back into the deductive picture. All I wish to do by way of this (possibly tractable) problematic example is point out that one has to proceed delicately on such matters, paying a lot of attention to the fine structure of proofs under different regimes of rule-selection and constraints.

I shall now list the main options concerning discharge requirements. Two of them will have characteristic symbols that I shall use later to label indicated assumptions in the statement of a discharge rule. Each such symbol represents the discharge requirement on applications of the rule in question. (As it happens, only two of the options are taken in the limited range of examples I give of systems below.)

Discharge requirements

Vacuous discharge of assumptions prohibited	
Selective and contractive discharge	
Selective because non-contractive discharge	
Thorough, hence contractive discharge	$\square$
Vacuous discharge of assumptions allowed	
Selective and contractive discharge	
Selective because non-contractive discharge	
Thorough, hence contractive discharge	$\varnothing$

Prohibiting vacuous discharge is a necessary step if one wishes to avoid irrelevant reasoning. Otherwise, for example, one could reach $\perp$ from assumptions A, $\sim A$; and then claim to be applying the rule of classical reductio (vacuously, of course!) to obtain the irrelevant conclusion $\sim B$. Or—almost as bad—one could claim to be employing the rule of $\sim$-introduction (vacuously, of course!) to obtain the irrelevant conclusion $\sim B$.

This raises the next important choice facing one in the design of a logical system:

Relevance

Reformist strategy: All proofs should establish their conclusions relevantly from their premisses; in particular, avoid the Lewis paradoxes
Quietist strategy: Permit proofs that do not establish their conclusions relevantly from their premisses; in particular, permit the Lewis paradoxes

It is obvious that anyone choosing the reformist strategy must ban the absurdity rule *ex falso quodlibet*:

$$\frac{\bot}{A}$$

But what is often not appreciated is that the reformist strategy requires one also to *avoid I/E formulae*—that is, it requires proofs to be *in normal form*. Otherwise, one could deduce B from A, $\sim A$ by exploiting a conjunctive I/E formula occurrence that makes for a technically non-vacuous discharge, but of an assumption that normalization would show to be spuriously relevant:

$$\cfrac{\cfrac{\cfrac{A \quad \overline{\sim B}^{(1)}}{A \,\&\, \sim B} \;\text{.......... conjunctive I/E formula}}{A} \quad \sim A}{\cfrac{\bot}{B}\,(1)} \;\text{.......... } \sim B \text{ only spuriously relevant to } \bot.$$

The last step in this proof is classical reductio. If the proof were normalized, however, the true character of this last step would become apparent: it is really a step of *ex falso quodlibet*. This rule, when added to the standard introduction and elimination rules that make up minimal logic M, yields intuitionistic logic I. The normalized version of the proof just given is

$$\cfrac{\cfrac{A \quad \sim A}{\bot}}{B}$$

Adding the rule of classical reductio to I yields the system C of classical logic. Indeed, there are at least four well-known ways of 'going classical' from proper subsystems of classical logic. When the proper subsystem in question is intuitionistic logic, each of these extra four

strictly classical rules allows one to derive any of the remaining three. The four rules are: classical reductio, double negation elimination, the law of excluded middle, and the rule of dilemma. From left to right these are, respectively:

$$\cfrac{\begin{matrix}\overline{\sim A}^{(i)}\\ \vdots \\ \bot\end{matrix}}{A}\,(i) \qquad \frac{\sim\sim A}{A} \qquad \frac{}{A \vee \sim A} \qquad \cfrac{\begin{matrix}\overline{A}^{(i)}\\ \vdots \\ B\end{matrix} \quad \begin{matrix}\overline{\sim A}^{(i)}\\ \vdots \\ B\end{matrix}}{B}\,(i)$$

So the next option we have to consider is whether to 'go classical' or non-constructive:

Classicism versus constructivism

Constructivist strategy: Do not allow any one of (1) classical reductio (2) double negation elimination (3) law of excluded middle (4) dilemma
Non-constructivist strategy: Adopt any of (1)–(4)

Note that the interderivability of (1)–(4) fails if the absurdity rule (*ex falso quodlibet*) is not available. We need this rule in order to derive each of (1) and (2) from either (3) or (4). In this regard (1) and (2) represent *prima facie* stronger ways of going classical than do (3) and (4). Even in the absence of the absurdity rule, however, interderivability of all four rules can be restored provided that we state the rule of dilemma in $\bot/B$-form.

With our interest still focused on the question of relevance, we have to consider two further related choices: whether we have *restricted* as opposed to *unrestricted* transitivity of deduction; and whether we have *disjunctive syllogism*. The need to decide these two crucial issues arises from the famous proof that Lewis gave of his first paradox. In essence it runs like this:

Proof of the first Lewis paradox

Consider the unobjectionable proof

$$\frac{A}{A \vee B}$$

Suppose now that one had a proof of disjunctive syllogism:

$$\begin{array}{c} A \vee B,\ \sim A \\ \vdots \\ B \end{array}$$

By grafting the first proof on to the top occurrence(s) of $A \vee B$ in the bottom proof, one would obtain a proof

$$\begin{array}{c} A,\ \sim A \\ \vdots \\ B \end{array}$$

The relevantist, not wishing to accept that there is a proof of the first Lewis paradox, is confronted with two stark alternatives:

Two relevantist alternatives

The Anderson–Belnap–Meyer *et al.* alternative:
Retain unrestricted transitivity of deduction
Reject disjunctive syllogism

The other alternative:
Restrict transitivity of deduction
Retain disjunctive syllogism

In the second alternative, it suffices to restrict transitivity of deduction only by the little that is required in order to avoid the Lewis paradox. It is remarkable how little indeed needs to be cut off Cut. One can still have the epistemically gainful condition of transitivity! One nice way to understand it is as a modified cut rule, which requires in the conclusion only some subsequent of what one normally has. Thus the usual form of the Cut Rule is:

$$\frac{X : Y, A \quad A, Z : W}{X, Z : Y, W}$$

The modified cut rule would state only that:

If there is a proof of $X : Y, A$ and a proof of $A, Z : W$ *then* for some subsets X', Y', Z', W' of X, Y, Z, W respectively there is a proof of $X', Z' : Y', W'$.

In the Lewis proof, all that accumulation of the two proofs would yield is the proof that $A, \sim A$ is inconsistent. That is, in accordance with the modified cut rule there is a proof of $A, \sim A : \varnothing$. (Remember that the empty set $\varnothing$ is a subset of any set, and so of $\{B\}$.)

But how, in the second alternative, does one ensure that disjunctive syllogism is provable in the absence of the absurdity rule? After all, the usual proof of disjunctive syllogism (not available in minimal logic) is:

$$\dfrac{A \vee B \quad \dfrac{\dfrac{\overline{A}^{(1)} \quad \sim A}{\bot}}{B} \quad \overline{B}^{(1)}}{B}\,(1)$$

Note the application of the absurdity rule in the first case proof. How can one salvage disjunctive syllogism if this move is no longer available to us? The answer lies in the $\bot/C$-form of the rule of $\vee$-elimination discussed earlier. If we adopt this more liberal form of $\vee$-elimination, we can simply 'close off' any case leading to absurdity, and conclude to what follows in the *other* case:

$$\dfrac{A \vee B \quad \dfrac{\overline{A}^{(1)} \quad \sim A}{\bot} \quad \overline{B}^{(1)}}{B}\,(1)$$

So we can have disjunctive syllogism after all. And just to fix other adjustments clearly, note what happens when one grafts proofs in pursuit of the first Lewis paradox:

$$\dfrac{\dfrac{\overline{A}^{(1)}}{A \vee B} \quad \dfrac{A \quad \sim A}{\bot} \quad \overline{B}^{(1)}}{B}\,(1)$$

The grafting produces a disjunctive I/E formula. So the last figure is not a proof, in so far as we require proofs to be in normal form. We can, however, apply the mapping $\mapsto$ that we discussed earlier in connection with restricted transitivity of deduction. This mapping essentially calls upon us to *normalize* the quasi-proof figure above. Applying the reduction procedure for the $\perp/C$-form of $\vee$-elimination, the resulting proof is simply

$$\frac{A \quad \sim A}{\perp}$$

This is just as we said it could be.

I have posed a series of choices among clearly spelled-out alternatives when we seek to set up a system of proof. To summarize, the main choices concern:

Serial form v. parallel form of E-rules
$\perp$-form v. C-form of E-rules
Normal v. abnormal form of proofs
Discharge requirements (vacuous, selective, thorough, contractive)
Relevance
Constructivism
Restricted v. unrestricted transitivity of deduction
Disjunctive syllogism
Form of conditional proof.

The choices are of course not independent of each other, as discussion above has already served to illustrate. Different constellations of choices issue in the following main systems. I am giving as examples only a very few of the possible systems that could be generated by making different choices. Because the statement of each system's rules of inference would take up considerable space, it is worth setting them out in a more compressed table for ease of reference. A compendium of their contrasting features will be given subsequently. I confine myself to full statements of rules for the systems T, IR, CR (which satisfy only restricted transitivity), and for M, I, C (the orthodox ones that satisfy unrestricted transitivity). Anderson and Belnap's system R of relevance logic turns out to be anomalous; one cannot locate it naturally within the system of 'coordinates' that I have developed.

We now state the various rules of inference in highly schematic form. Not only do we have place-holders for sentences; we also have place-holders for discharge conventions, and for the form of conclusion allowed by certain rules. The material in square brackets in the name of a rule is part of that name. When this material is schematic (using the variables δ or γ or both), it can be instantiated from among the possible values (for δ and γ) indicated below the rule schema, so as to obtain an exact form of the rule.

Rule Schemata

~-Introduction [δ]	~-Elimination
$\delta\overline{\ A\ }$ (i) . . . $\dfrac{\perp}{\sim A}$ (i) δ may be □ or $\varnothing$	$\dfrac{A \quad \sim A}{\perp}$

&-Introduction	&-Elimination [δ;γ]
$\dfrac{A \quad B}{A\&B}$	δ — (i) A , δ — (i) B . . . $\dfrac{A\&B \quad \gamma}{\gamma}$ (i) γ may be $\perp$ or C δ may be □ or $\varnothing$ or κ

(κ is like □ but in addition allows *exactly* one of A, B to be discharged)

	&-Elimination [serial]
	$\dfrac{A\&B}{B} \quad \dfrac{A\&B}{B}$

∨-Introduction

$$\frac{A}{A\vee B} \qquad \frac{B}{A\vee B}$$

∨-Elimination [δ;γ]

$$\dfrac{A\vee B \qquad \begin{matrix}\delta\,\overline{A}^{\,(i)}\\ \vdots\\ \gamma\end{matrix} \qquad \begin{matrix}\delta\,\overline{B}^{\,(i)}\\ \vdots\\ \gamma\end{matrix}}{\gamma}\,(i)$$

δ may be □ or ∅
γ may be ⊥, or C or ⊥/C

⊃-Introduction [simple]

$$\frac{B}{A\supset B}$$

⊃-Elimination [γ]

$$\dfrac{A\supset B \qquad A \qquad \begin{matrix}\square\,\overline{B}^{\,(i)}\\ \vdots\\ \gamma\end{matrix}}{\gamma}\,(i)$$

γ may be ⊥ or C

⊃-Introduction [δ; γ]

$$\dfrac{\begin{matrix}\delta\,\overline{A}^{\,(i)}\\ \vdots\\ \gamma\end{matrix}}{A\supset B}\,(i)$$

γ may be ⊥ or B
δ may be □ or ∅

⊃-Elimination [serial]

$$\frac{A\supset B \quad A}{B}$$

Absurdity rule

$$\frac{\bot}{A}$$

Classical reductio [δ]

$$\dfrac{\begin{matrix}\delta\,\overline{\sim A}^{\,(i)}\\ \vdots\\ \bot\end{matrix}}{\sim A}\,(i)$$

δ may be □ or ∅

Law of excluded middle

$$\overline{A\vee\sim A}$$

Double negation elimination

$$\frac{\sim\sim A}{A}$$

Dilemma [δ;γ]

$$\dfrac{\begin{matrix}\delta\,\overline{A}^{\,(i)}\\ \vdots\\ \gamma\end{matrix} \qquad \begin{matrix}\delta\,\overline{\sim A}^{\,(i)}\\ \vdots\\ \gamma\end{matrix}}{\gamma}\,(i)$$

δ may be □ or ∅
γ may be B or ⊥/B

All the logical systems discussed here have in common the rules of ∼-elimination, &-introduction, and ∨-introduction. The table below shows how the remaining rules for each system are framed. The asterisk against the ticked entry for K concerning ∼-introduction reminds us that the assumption discharged by any application of ∼-introduction must be the one of highest degree among the undischarged assumptions of the subordinate proof (whence K is non-Duhemian).

Choice of rules

	C	I	M	K	T	IR	CR
Proofs to be in normal form				√	√	√	√
No E-rule in scope of another with same MPE				√			
~-Introduction (□)				√*	√	√	√
~-Introduction (∅)	√	√	√				
&-Elimination (κ;⊥)				√			
&-Elimination (□;⊥)					√		
&-Elimination (□;*C*)						√	√
&-Elimination (serial)	√	√	√				
∨-Elimination (∅,*C*)	√	√	√				
∨-Elimination (□;⊥)				√	√		
∨-Elimination (□;⊥/*C*)						√	√
⊃-Introduction (simple)				√	√		
⊃-Introduction (□;⊥)				√	√	√	√
⊃-Introduction (∅;*B*)	√	√	√			√	√
⊃-Elimination (⊥)				√	√		
⊃-Elimination (*C*)						√	√
⊃-Elimination (serial)	√	√	√				
Absurdity rule	√	√					
Law of excluded middle	√						√
Double negation elimination	√						√
Classical reductio (□)							√
Classical reductio (∅)	√						
Dilemma (□;⊥/*B*)							√
Dilemma (∅;*B*)	√						√

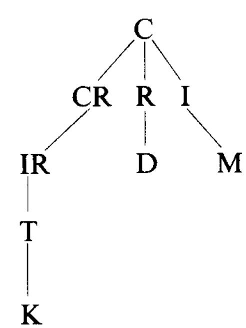
4. *Consequences of Choices in Setting Up Systems of Proof*

Given these choices of rules for the various systems, we now see that the features we have been discussing are distributed across those systems as follows. An affirmative entry means that the system has *some* formulation in which that feature is realized, even if it has not been so in the canonical versions given above:

Features

	C	I	M	K	T	IR	CR
E-rules							
Serial form	√	√	√	×	×	√	√
Parallel form	√	√	√	√	√	√	√
⊥-form	√	√	√	√	√	×	√
C-form	√	√	√	×	×	√	√
∨E in ⊥/*C*-form	√	√	×	×	×	√	√
Proofs							
All systems *can* be formulated with a normality requirement							
Abnormal	√	√	√	×	×	×	×
Discharge							
Vacuous	√	√	√	×	×	×	×
Selective	√	√	√	×	×	×	×
Thorough	√	√	√	√	√	√	√
Contractive	√	√	√	√	√	√	√
Relevance	×	×	×	√	√	√	√
Constructivism	×	√	√	√	√	√	×
Transitivity							
Restricted	×	×	×	√	√	√	√
Unrestricted	√	√	√	×	×	×	×
Disjunctive syllogism	√	√	×	×	×	√	√
Conditional proof							
Unitary	√	√	√	×	×	×	×
Split	√	√	√	√	√	√	√

The choices of rules, with the various restrictions chosen on discharge of assumptions, normality of proofs, etc., yield our important logical systems in the following containment relations:



By way of summarizing the most important consequence of choices of rules with regard to the desiderata set out above, we provide here also a checklist of the most salient entries:

Remaining desiderata

	C	I	M	K	T	IR	CR
Separability	×						×
Reduction procedures							
Harmony							
Normalizability	?						
Substitution							×
Truth-preservation	?						?
Adequacy as metalogic for truth theory				×	×	√	
Deduction theorem						almost	
Duhemianism on denial				×			
Nice 'mereology'	√	√	√				
Containing what one wants of C under a simple translation					√	√	
Interdeducibility implies deductive interreplaceability					×	×	×

We have not yet given a rule formulation for Anderson and Belnap's system R. Indeed, none is readily available. The system R is usually given instead by the following list of axioms and rules of inference.

(1) $A \rightarrow A$
(2) $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$
(3) $(A \rightarrow B) \rightarrow ((C \rightarrow A) \rightarrow (C \rightarrow B))$
(4) $(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$
(5) $(A \,\&\, B) \rightarrow A$
(6) $(A \,\&\, B) \rightarrow B$
(7) $((A \rightarrow B) \,\&\, (A \rightarrow C)) \rightarrow (A \rightarrow (B \,\&\, C))$
(8) $A \rightarrow (A \vee B)$
(9) $B \rightarrow (A \vee B)$
(10) $((A \rightarrow C) \,\&\, (B \rightarrow C)) \rightarrow ((A \vee B) \rightarrow C)$
(11) $(A \,\&\, (B \vee C)) \rightarrow ((A \,\&\, B) \vee C)$
(12) $(A \rightarrow {\sim}A) \rightarrow {\sim}A$
(13) $(A \rightarrow {\sim}B) \rightarrow (B \rightarrow {\sim}A)$
(14) ${\sim}{\sim}A \rightarrow A$

(15) $(A \rightarrow B) \rightarrow (((A \rightarrow B) \rightarrow C) \rightarrow C)$
(16) $(((A \rightarrow A) \rightarrow A) \,\&\, ((B \rightarrow B) \rightarrow B)) \rightarrow (((A \,\&\, B) \rightarrow (A \,\&\, B)) \rightarrow (A \,\&\, B))$
(17) $A \rightarrow ((A \rightarrow A) \rightarrow A)$
(special case of $A \rightarrow ((A \rightarrow B) \rightarrow A)$: Assertion).

It is hard to locate R as the result of a constellation of choices of the kind that generated the other systems above. All the axioms except (11) can be derived in a decidable Gentzen system known as LR. The distributivity axiom (11) has to be postulated separately; and indeed makes the resulting system R undecidable. The problem with R is that the discharge conventions that have to be adopted in its natural deduction formulation are exceptionally complex, because of their selectivity, especially with regard to ⊃I (see Prawitz 1965). It is precisely this complexity involved in 'keeping track of how assumptions have been used' which makes the decision problem for LR of such awesome computational complexity. An unpublished result of Urquhart's is that the decision problem for LR is at best ESPACE hard, and at worst space hard in a function primitive recursive in the generalized Ackermann exponential.

5. *Applications to Computational Logic*

These theoretical considerations, to my mind, make R and LR hopelessly ill-advised choices for computational investigations. That is why, in pursuit of better results on the computational front, I have chosen an avenue of investigation that will lead from M to the systems IR and CR, whose deducibility relations are *not* unrestrictedly transitive, but which satisfy the epistemically gainful condition of transivity. The guiding thought is that heeding requirements of relevance should *speed up* the search for proofs, not catastrophically retard it.

Whether this conjecture about the complexity pay-off of my preferred analysis of relevance is borne out will only become clear once computational investigations along present lines have been taken further. So far I have confined these investigations to minimal logic M. This is because it is a well-known system and suitably neighbourly to IR. The methods developed for M promise to extrapolate smoothly to IR. In particular, IR affords a much stronger 'relevance theorem' than is available for M. That is, one can prove that any IR-provable sequent $X : C$ involves a much tighter relationship of variable-sharing than is the case with M. This means that one can filter much more powerfully before undertaking proof-search in earnest.

For M, I have written a proof-finder in Prolog, which is based on just a few of the normal form and filtering theorems delivered so far. It has managed to prove, in less than a second each, the thirty-three most difficult problems culled by John Slaney, of the ANU's Automated Reasoning Project, from 50,000 problems that he generated. The problems involve as many as 170 occurrences of propositional variables and logical operators each.

This power of computational proof-search has come from abandoning axiomatic approaches, and learning to appreciate the inferential fine-structure available in systems of natural deduction. The yeoman service of Prior and colleagues like Carew Meredith (see Prior 1963*b*) in their laborious axiomatic derivations of not-so-hard theorems of subclassical logics can now be emulated in mere milliseconds. I cannot help feeling that if Prior were alive today he would by now have abandoned axiomatic approaches and taken up the inferential one, based on separable operators, harmoniously balanced singular rules (treating one occurrence of a logical operator at a time), reduction procedures, and normalizability. The resort to harmony was, after all, implicit in Belnap's response to Prior's problem of the run-about inference ticket (see Prior 1960*a* and Belnap 1962).

What natural deduction and delicate proof theory promise now is a whole book of run-about inference coupons. By exploiting the affinities among systems along the lines revealed above, we can develop a variety of proof-search methods that can transfer reasonably invariantly from one logical system to another. Prior's zeal for the purity of Polish formulae deserves a successor: a zeal for the purity of Polish proofs, which is after all what natural deductions really are.

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Logic is . . . about the real world.

A. N. Prior, 'A Statement of Temporal Realism'