TELLER, P. [1979]: 'Quantum Mechanics and the Nature of Continuous Physical Quantities', *The Journal of Philosophy*, **76**, pp. 345–61.

VAN FRAASSEN, B. C. [1982]: 'The Charybdis of Realism: Epistemological Implications of Bell's Theorem', *Synthese*, **52**, pp. 25–38.

MINIMAL LOGIC IS ADEQUATE FOR POPPERIAN SCIENCE*

Assume that Popper's 'no counterexample' interpretation of scientific laws is correct ([1972], pp. 68–69):

> The theories of natural science, and especially what we call natural laws, have the logical form of strictly universal statements; thus they can be expressed in the form of negations of strictly existential statements or, as we may say, in the form of *non-existence statements* (or 'there-is-not' statements). . . . In this formulation we see that natural laws . . . insist on the non-existence of certain things or states of affairs, proscribing or prohibiting, as it were, these things or states of affairs: they rule them out. And it is precisely because they do this that they are *falsifiable*. If we accept as true one singular statement which, as it were, infringes the prohibition by asserting the existence of a thing (or the occurrence of an event) ruled out by the law, then the law is refuted.

Assume also that Popper's analysis of scientific method, as set out in that book, is correct. Assume, that is, that the experimental refutation of scientific hypotheses has the following deductive structure, which I shall call *Schema P*:

Hypotheses, Boundary conditions

|

Predictions, Observational reports

|

Contradiction

Logic is needed for the downward passages indicated within this schema. As Popper himself puts it elsewhere ([1970], at p. 18):

> . . . in the empirical sciences (logic) is almost exclusively used critically—for the re-transmission of falsity. . . . in the empirical sciences logic is mainly used for criticism; that is, for refutation.

* I am grateful to an anonymous referee for the *BJPS* for comments on an earlier draft.

So far so good. But then he immediately goes on to say:

> Now, what I wish to assert is this. If we want to use logic in a critical context, then we should use a very strong logic, the strongest logic, so to speak, which is at our disposal; for we want our criticism to be *severe*. In order that the criticism should be severe we *must use the full apparatus; we must use all the guns we have. Every shot is important.... Thus we should (in the empirical sciences) use the full or classical or two-valued logic. If we do not use it but retreat into the use of some weaker logic—say, the intuitionist logic, ... —then, I assert, we are not critical enough* ... (my last emphasis).

It can be shown that this last claim is simply wrong. Theorem *I* below guarantees that *intuitionistic logic is adequate for all instances of Schema P, on the assumption that the no-counterexample interpretation is correct.* Theorem *M* extends this to show that even *minimal* logic is thus adequate.

Some preliminaries: Minimal logic is determined by the introduction and elimination rules for the logical operators. If one adds the absurdity rule, one obtains intuitionistic logic. If one then adds one of the well known classical negation rules (such as double negation elimination) one obtains classical logic. A set of sentences is (classically, intuitionistically, minimally) *inconsistent* if and only if the absurdity sign is deducible from it (in classical, intuitionistic, or minimal logic respectively). The single turnstile will represent deducibility throughout, subscripted by *C*, *I* or *M* as appropriate.

THEOREM *I* Every classically inconsistent set of first order sentences in $\sim$, $\vee$, &, $\supset$ and $\exists$ is *intuitionistically* inconsistent.

Sketch of proof

Straightforward by induction on the length of those natural deductions that can be built up using the introduction and elimination rules for the logical operators concerned; plus (for intuitionistic logic) the absurdity rule; plus (for classical logic) a classical rule of negation such as double negation elimination. To be precise, we prove by induction on Π the statement

> For every classical proof Π from assumptions Δ: if the conclusion of Π is a sentence ϕ then we can find an intuitionistic proof of $\sim\sim\phi$ from Δ; and if the conclusion of Π is the absurdity sign $\wedge$ then we can find an intuitionistic proof of $\wedge$ from Δ.

In the inductive proof (which is left to the reader as an exercise) the basis is obvious, and the inductive step falls into cases according to the last rule applied in Π. Easy intuitionistic manoeuvres produce the desired intuitionistic proof corresponding to Π when one assumes as given (by inductive hypothesis) the intuitionistic proofs corresponding to the immediate subproofs of Π. Note that when the last rule applied in Π is the classical rule of double negation elimination, the desired intuitionistic proof is obtained by virtue of the fact that $\sim\sim\phi$ follows intuitionistically from $\sim\sim\sim\sim\phi$.

What happens if we now add the universal quantifier as a primitive in the language? The following theorem gives an answer.

THEOREM *II* Suppose Δ is a classically inconsistent set of first order sentences in the full language in which ∀ is also primitive. Then the result of inserting double negations immediately after every universal quantifier prefix in members of Δ is *intuitionistically* inconsistent.

Proof

As above. But now when we consider in the inductive step the case where Π ends with an application of universal introduction, we find we have to rest content with an intuitionistic proof in which universal introduction is applied to the doubly negated conclusion of the proof given by the inductive hypothesis.

Remarks on Theorems I and II The proofs of these theorems are constructive. Their history starts with Glivenko (for the propositional case). Gentzen and Gödel proved results in the neighbourhood of those just stated, using slightly different syntactic transformations on formulae. Theorem *I* is Problem 9.11.13 in J. L. Bell and M. Machover [1977], page 444; it is an easy corollary of their Theorem 9.11.12. For a non-constructive proof of these results using Henkin's method see my [1978], pages 125–30.

In the constructive proofs of Theorem *I* and *II* indicated above, the case in the inductive step dealing with deductions ending with ⊃-introduction calls for the *absurdity rule* in order to obtain the required intuitionistic proof from those whose existence is guaranteed by the inductive hypothesis. Similarly in the proof by Henkin's method the absurdity rule enters in the case dealing with ⊃ in the inductive step of the proof that every consistent complete set with witnesses has a natural model. But the absurdity rule is not available in minimal logic. The next two theorems (labelled *M* and *MM*) show that this difficulty is confined to the connective ⊃. We drop ⊃ from the language, and strengthen *I* and *II* by substituting 'minimally' for 'intuitionistically':

THEOREM *M* Every classically inconsistent set of first order sentences in ∼, ∨, &, and ∃ is *minimally* inconsistent.

THEOREM *MM* Suppose Δ is a classically inconsistent set of first order sentences in ∼, ∨, &, ∃ and ∀. Then the result of inserting double negations immediately after every universal quantifier prefix in members of Δ is *minimally* inconsistent.

Proofs By inspection of the constructive proofs of Theorems *I* and *II* indicated above, or by inspection of the proof by Henkin's method in my [1978]. (These results for minimal logic appear to be new—a bonus from the approach based on the rules of natural deduction, as opposed to Hilbert-style axiomatizations of logical systems.)

Theorem *I* shows conclusively that intuitionistic logic is adequate for Popperian science on the 'no-counterexample' interpretation. Theorem *M* strengthens this by showing that the same holds for minimal logic, provided that we avoid using $\supset$ in the regimentation of sentences of our scientific theories. These considerations in favour of intuitionistic and minimal logic are not necessarily impugned should the Popperian give up the no-counterexample interpretation and thereby introduce the universal quantifier as a primitive into the object language. For Theorems *II* and *MM* extend Theorems *I* and *M* respectively to take care of the universal quantifier by means of double negation insertions.

Now there are examples where this double negation insertion is not *needed* in order to secure intuitionistic inconsistency. An obvious one is

$$\text{(Q)}\qquad \dfrac{\dfrac{\mathrm{Ft} \qquad \dfrac{\forall \mathrm{x}(\mathrm{Fx} \supset \mathrm{Gx})}{\mathrm{Ft} \supset \mathrm{Gt}}}{\mathrm{Gt}} \qquad {\sim}\mathrm{Gt}}{\Lambda}$$

In some cases, however, one does need to resort to the double negation insertions cautioned by the last result, as can happen when the universal quantifier occurs within the scope of a negation. For example,

$$\forall x \sim\, \sim Fx,\ \sim \forall x Fx \vdash_C \Lambda \quad \text{but} \quad \forall x \sim\, \sim Fx,\ \sim \forall x Fx \nvdash_I \Lambda$$

In order to secure *intuitionistic* inconsistency in this case we have to follow the advice of Theorem *II*; and indeed by Theorem *MM* we obtain even more:

$$\forall x \sim\, \sim Fx,\ \sim \forall x(\sim\, \sim)Fx \vdash_M \Lambda$$

But it is by no means a foregone conclusion that occurrences of the universal quantifier in scientific contexts *P* will ever be of this kind. Further syntactic investigation of the sentences involved in contexts *P* could well reveal (*Q*) to be a simple example of what is *generally* the case in contexts *P*. That is, the prospect remains of defining a syntactic property **P** generally true of contexts *P* such that if Δ is clasically inconsistent and satisfies **P**, then Δ is intuitionistically (or even minimally) inconsistent.

If this proves to be the case, then even by giving up the no-counterexample interpretation the Popperian will have no methodological grounds for preferring classical logic to intuitionistic or even minimal logic. What I have shown, however, is that he definitely has no such grounds if he adopts the no-counterexample interpretation.

Finally, a word on our metalogic. All the theorems above have constructive proofs, which will be afforded by an intuitionistic metalogic. But a critic may object that Theorems *M* and *MM*, which I am offering as showing the adequacy of *minimal* logic for Popperian purposes, may not be provable in a *minimal* metalogic. This misgiving can be allayed. For, provided we avoid the use of ⊃ in the statement of the metatheorems, they can be proved in *minimal relevant* logic. This stability property is demonstrated in Tennant [forthcoming]. But it is worth noting that one need not strive officiously after such stability; for Popper himself would allow the use of the full classical logic in criticism of his own position.

NEIL TENNANT
University of Stirling

REFERENCES

BELL, J. L. and MACHOVER, M. [1977]: *A Course in Mathematical Logic*. North Holland.
POPPER, K. [1970]: 'A realist view of logic, physics and history', in A. D. Breck and W. Yourgrau (*eds*.), *Physics, Logic and History*, Plenum Press, pp. 1–30.
POPPER, K. [1972]: *The Logic of Scientific Discovery*, revised edition. Hutchinson.
TENNANT, N. W. [1978]: *Natural Logic*. Edinburgh.
TENNANT, N. W. [forthcoming]: 'Perfect validity, entailment and paraconsistency', *Studia Logica*. Abstract in *Journal of Symbolic Logic*, **49** (1984).