Existence and Identity in Free Logic: A Problem for Inferentialism?

Abstract

Peter Milne (2007) poses two challenges to the inferential theorist of meaning. This study responds to both. First, it argues that the method of natural deduction idealizes the essential details of correct informal deductive reasoning. Secondly, it explains how rules of inference in free logic can determine unique senses for the existential quantifier and the identity predicate. The final part of the investigation brings out an underlying order in a basic family of free logics.

1. Background

1.1 Free v. unfree logic

Standard, unfree logic has among its rules of natural deduction the following (see, for example, the system in the classic monograph Prawitz 1965).

Universal Introduction (∀-I)

$$\frac{\begin{array}{c}\vdots\\ \varphi\end{array}}{\forall x\varphi^{a}_{x}}$$

where a does not occur in any assumption on which φ depends

Universal Elimination (∀-E)

$$\frac{\forall x\varphi}{\varphi^{x}_{t}}$$

Existential Introduction (∃-I)

$$\frac{\varphi^{x}_{t}}{\exists x\varphi}$$

Existential Elimination (∃-E)

$$\frac{\exists x\varphi \qquad \begin{array}{c}\overline{\varphi^{x}_{a}}^{\,(i)}\\ \vdots\\ \psi\end{array}}{\psi}\,(i)$$

where a does not occur in $\exists x\varphi$, or in ψ, or in any assumption, other than φ^{x}_{a}, on which the upper occurrence of ψ depends

Reflexivity of identity

$$\frac{}{t=t}$$

Substitutivity of identity

$$\frac{t=u \quad \varphi(t)}{\varphi(u)}$$

Let t be any term, not necessarily closed. Then $\exists!t$ is short for $\exists x\ x{=}t$, where x is alphabetically the first variable not free in t.

In *unfree* logic we have, for every closed term t,

$$\vdash \exists!t.$$

We also have

$$\vdash \exists x\ x{=}x.$$

So unfree logic is committed to a denotation for every closed term, and to the existence of at least one thing.

A logic is *free* when it is no longer based on the assumption that all singular terms denote. A free logic is *universally* free when in addition it is no longer based on the assumption that the universe is non-empty.

In Tennant 1978, rules of natural deduction were provided for a system of universally free logic that will here be called $\mathrm{F}\mathcal{U}_{NL}$. The salient rules for $\mathrm{F}\mathcal{U}_{NL}$ can be found in §7. (Rules for the connectives are omitted.)

1.2 Inferentialism

An inferentialist theory of meaning holds that the meaning of a logical operator can be captured by suitably formulated rules of inference (in, say, a system of natural deduction).

The problem to be addressed is: *Can one be an inferentialist about the logical operators in free logic?*

Prima facie, the answer should be affirmative. For whatever reasonable system of free logic one chooses, there is a natural-deduction system for it, and the rules of inference involved ought to be 'capturing' the meanings of the logical operators involved. The problem is made sharper, however, by the proliferation of rules involved in free logic, and the formulation of some of them that involves so-called 'existential presuppositions'. There are also many different equivalent formulations—different sets of rules of inference—for one and the same system of free logic. Is there an identification (in the context of free logic) of 'the' introduction rule for $\exists$ (for example), and of 'the' corresponding elimination rule, which enables one to show that they are indeed in harmony, and that they succeed in specifying a unique sense for $\exists$?

2. Milne's challenge

Peter Milne (2007) identifies what he believes is a problem for an inferentialist theory of meaning for logical operators in the context of a free logic. The problem is supposed to reveal a tension between, on the one hand, the present author's invocation of free logic for constructive logicism (see Tennant 1987) and for abstractionist realism (see Tennant 2004) more generally; and, on the other hand, the inferentialist thesis that harmonious rules of inference for a logical operator successfully characterize its meaning. A diagnosis is offered below of the reason why the purported problem might be perceived as genuine; and a reasonably straightforward solution is offered. Milne's investigations provide a welcome opportunity to clarify and amend an overall position developed at different stages over quite some time. §3 provides a defence of inferentialism about free logic.

Milne also enters some misgivings about the extent to which systems of natural deduction faithfully capture the meanings of the quantifiers (especially the existential quantifier), as these are revealed in informal mathematical usage. In §5, the methods of natural deduction due to Gentzen and Prawitz are defended against Milne's criticisms.

3. In defence of inferentialism about free logic

3.1 On existence

Milne claims that the rules of $F\mathcal{U}_{NL}$ for the existential quantifier (see §7) do not succeed in conferring upon it a unique sense. In order to justify this claim, he asks his reader to consider those rules with different notation for the quantifier (Σ in place of ∃):

(Σ-I)
$$\frac{\Sigma!t \quad \varphi^x_t}{\Sigma x\varphi}$$

(Σ-E)
$$\frac{\Sigma x\varphi \qquad \begin{array}{c}\overline{\Sigma!a}^{(i)} , \overline{\varphi^x_a}^{(i)} \\ \vdots \\ \psi\end{array}}{\psi}\,(i)$$

To quote Milne:

> Let χ be any closed formula you like. These rules are truth-preserving when one reads $\Sigma x\varphi$ as $\exists x(\varphi\wedge\chi)$.

Let us call this technique the *milning* of a pair of introduction and elimination rules. Milne's phrase 'These rules', however, refers to the doctored pair (Σ-I) and (Σ-E), respectively obtained from the rules (∃-I) and (∃-E) in F$\mathcal{U}_{NL}$ by writing occurrences of ‘Σ’ in place of occurrences of ‘∃’. But what of the other two rules in which ∃ figures so prominently in F$\mathcal{U}_{NL}$—the rule of atomic denotation and the rule of functional denotation? Surely these rules, too, need their Σ-versions, if Σ is to be a genuine but deviant alternative to ∃? Let us therefore extend Milne’s competing rules by adding the following variants of the two denotation rules just mentioned:

$$\frac{A(t)}{\Sigma!t}\text{ , where }A(t)\text{ is atomic;}\qquad\text{and}\qquad\frac{\Sigma!f(t)}{\Sigma!t}$$

For the argument to follow, only the first of these two rules is needed. Consider now the following formal proofs in the expanded system. They show that, *pace* Milne, one will be able to deduce $\Sigma x\varphi$ from $\exists x\varphi$, and conversely. The proof of $\Sigma x\varphi$ from $\exists x\varphi$ is as follows:

$$\dfrac{\exists x\varphi \qquad \dfrac{{\scriptstyle(2)}\dfrac{}{\varphi^{x}_{a}} \qquad \dfrac{{\scriptstyle(2)}\dfrac{}{\exists x\, x=a}\qquad \dfrac{\dfrac{\dfrac{}{b=a}{\scriptstyle(1)}}{\Sigma!b}\quad \dfrac{}{b=a}{\scriptstyle(1)}}{\Sigma x\, x=a}}{\Sigma x\, x=a}{\scriptstyle(1)}}{\Sigma x\varphi}{\scriptstyle(2)}}{\Sigma x\varphi}$$

and the proof of the converse is obtained by interchanging ∃ and Σ. So we see that $\Sigma x\varphi$ and $\exists x\varphi$ are synonymous, after all.

Note that the rule allowing one to infer $\Sigma!t$ from an atomic sentence $A(t)$ prevents milning: for, in order for this rule to be truth-preserving when $\Sigma x\varphi$ is read as $\exists x(\varphi\&\chi)$, the sentence χ must be logically true—which is harmless.

To be fair to Milne, it should be pointed out that the step from $b=a$ to $\Sigma!b$ in the proof above is one for which his selection of Σ-rules does not provide. But then (one might say) so much the worse for that selection of Σ-rules. Without an analogue of the rule of atomic denotation, they do not capture a genuine alternative to ∃.

Lest the reader think that some genuine problem for ∃ is being downplayed here, it should be pointed out that one could milne the rules for &. Consider the following rules for a binary connective @:

$$(@\text{-I})\quad \frac{\varphi\quad\psi}{\varphi@\psi}\qquad\qquad (@\text{-E})\quad \frac{\varphi@\psi}{\varphi}\quad \frac{\varphi@\psi}{\psi}$$

With $\varphi@\psi$ interpreted as $(\varphi\&\psi)\&\chi$, these rules are truth-preserving—if, but only if, χ is a logical truth.

Milne observed, ingeniously, that the Σ-analogues of what had all along been called (∃-I) and (∃-E) in free logic allow for the deviant interpretation of $\Sigma x\varphi$ as $\exists x(\varphi\&\chi)$, where χ is any sentence. What Milne's observation prompts is the realization that the rule of existential introduction in universally free logic should be understood as consisting not only of the usual part:

$$\frac{\exists!t \quad \varphi^{x}_{t}}{\exists x\varphi}$$

but also of another part:

$$\frac{A(t)}{\exists!t}\text{ , where } A(t) \text{ is atomic,}$$

which had escaped detection as a genuine part of the introduction rule for ∃. It escaped detection because it had been given a separate title of its own, namely 'the rule of atomic denotation'. Once this rule is made part of the introduction rule for ∃, however, it prevents milning of the rules for ∃; for it is not truth-preserving on the deviant interpretation of $\exists x\varphi$ as $\exists x(\varphi\&\chi)$. The inferential meaning-theorist, it turns out, had only a classification problem with the rules of natural deduction for free logic—a problem that, as we have just seen, can be solved.

Milne goes to some lengths to stress an alleged difference between his own construal of the rule of atomic denotation, and what he takes to be the present author's construal. This reply affords the opportunity to disavow various imputed views.

> … I suspect … Tennant and I differ fundamentally in how we construe the rule of atomic denotation. … Tennant does not tell us about the rule's logical status. … I hazard the guess that Tennant … sees the rule as giving expression to an independent semantic principle. Tennant, I suspect, takes the sense of existence, of the existential quantifier, as antecedently given, as prior to the adoption of the rule. (Milne 2007, pp. 50–51)

The textual evidence that Milne offers in support of these suspicions does not ground them at all. The claim that Milne quotes—'an atomic claim will be true only if all its terms denote'—is merely a statement of the rule of atomic denotation, not some philosophically contentious 'construal' of it. Milne complains (p. 51) that he 'can see no robust sense of reality reflected in use of the rule of atomic denotation' if that rule is used 'to bolster the proof-theoretic semantics credentials of free logic'. But free logic has never sought, in any court, to present any 'proof-theoretic semantics credentials'. The question is only whether, to the extent that it stands in need of some semantics, sufficient enlightenment is to be had from within its proof-theoretic resources. If someone had to learn how to use a regimented language based on free logic, could they attain a proper grasp of the sense of each logical operator by being given rules for its correct inferential use? The answer is affirmative.

There is good independent reason to assimilate the rule of atomic denotation to the introduction rule for ∃. For, consider what the standard part of (∃-I) provides as 'the' way to warrant an assertion of a claim of the form $\exists! t$ (i.e., $\exists x\, x=t$). In order to obtain $\exists x\, x=t$ as the conclusion of an application of (the standard part of) (∃-I), one would need to be able, in turn, to warrant the premises. But one of those premises will be of the same form as the sought conclusion. Thus we would have an infinite regress. The only sensible way to end such a regress is by offering a different kind of terminus: allow $\exists! t$ to be inferred from any atomic sentence of the form A(t). To be is to be a constituent of an atomic fact. (Thus Wittgenstein: 'It is essential to things that they should be possible constituents of states of affairs.' (*Tractatus* 2.011) *Cf.* also Gödel (as quoted in Wang 1996, at p. 293): 'Mathematical logic makes explicit the central place of predication in the philosophical foundation of rational thought.') Also, allow $t=t$ to be so inferred. This will be the (new formulation of the) rule of reflexivity of identity.

3.2 On identity

The rule of reflexivity of identity is the introduction rule for the identity predicate:

(=-I) $\dfrac{A(t)}{t=t}$, where A(t) is atomic.

This introduction rule tells one that identity is the minimal reflexive (binary) relation. Hence the elimination rule, with major premise $t=u$, should 'unpack' this by saying that t will bear to u any (binary) relation that is reflexive on t:

(=-E) $\dfrac{t=u \quad \psi(t,t)}{\psi(t,u)}$

This form of (=-E) is a variant of one formulated by Martin-Löf (1971).[1] The rule of substitutivity of identicals is easily derivable using (=-E), using either the rules for → or the rules for &:

$$\dfrac{\varphi t \qquad \dfrac{t=u \qquad \dfrac{\overline{\varphi t}^{\,(1)}}{\varphi t\to\varphi t}\,(1)}{\varphi t\to\varphi u}\,(=\text{-E})}{\varphi u} \qquad\qquad \dfrac{\dfrac{t=u \qquad \dfrac{\varphi t \quad \varphi t}{\varphi t \mathbin{\&} \varphi t}}{\varphi t \mathbin{\&} \varphi u}\,(=\text{-E})}{\varphi u}$$

Conversely, (=-E) is a special case of substitutivity of identicals (Sub=):

[1] On Martin-Löf's account of (=-E) one would say that, given the major premise $t=u$, t will bear to u any reflexive (binary) relation.

$$\frac{t{=}u \quad \psi(t,t)}{\psi(t,u)}\ \text{(Sub=)}$$

To be sure, the first derivation—of (Sub=) from (=-E)—requires that the language contain either → or &. The objection can be raised that one should be able, in principle, to specify meaning-constituting introduction and elimination rules for the identity predicate that do not rely on the presence, in the language, of either of these two operators. For, surely, there is room for a grasp of identity even in a language without them. So we shall take reflexivity and substitutivity, respectively, as the introduction and elimination rules for identity:

$$(=\text{-I})\quad \frac{A(t)}{t{=}t} \qquad\qquad (=\text{-E})\quad \frac{t{=}u \quad \varphi(t)}{\varphi(u)}$$

Note, however, that by casting (=-I) in the form above, rather than in the old form

$$\frac{\exists!t}{t{=}t}$$

we avoid any milning of the rules for identity. Milne's milning of the rules for identity involved observing that analogues of the old (=-I) and of (=-E):

$$(\approx\text{-I})\quad \frac{\exists x\, x{\approx}t}{t{\approx}t} \qquad\qquad (\approx\text{-E})\quad \frac{t{\approx}u \quad \varphi(t)}{\varphi(u)}$$

are truth-preserving when $t{\approx}u$ is read as $t{=}u\&\chi$, where χ is any closed formula you like. This milning works only with the old form of (=-I). But if ≈ is subject, instead, to rules of the form

$$(\approx\text{-I})\quad \frac{A(t)}{t{\approx}t} \qquad\qquad (\approx\text{-E})\quad \frac{t{\approx}u \quad \varphi(t)}{\varphi(u)}$$

then the observation no longer holds—for (≈-I) will not be truth-preserving when t≈u is read as $t{=}u\&\chi$ (unless, harmlessly, χ is a logical truth).

The first line of defence, then, against Milne's charge that the introduction and elimination rules for ∃ in (universally) free logic underspecify its meaning is that we need a more inclusive construal of the introduction rule for ∃. To recapitulate: the amended rule is in two parts. The first part is the old rule of atomic denotation, and the second part is the 'old' (∃-I). We also adopt an amended form of reflexivity of identity (=-I). Neither the amended rules for ∃ nor those for = can be milned. The rules still generate the deducibility relation of universally free logic.

4. Systems of (universally) free logic, with identity

Milne's critique occasions a deeper consideration of the possibilities open to the natural deduction theorist. §7 gives rules for four different systems of free logic, one of which is the amended system just described. The reader is advised to look ahead at their respective statements of rules, in order better to follow the ensuing discussion. (A more comprehensive metalogical investigation of these and related systems is beyond the scope of this paper, and will be presented elsewhere.)

In order to make comparisons easier to draw, we assume that the language contains no function signs. Thus we do not have to concern ourselves with the Rule of Functional Denotation. The language may, if we wish, contain variable-binding term-forming operators, such as the definite-description operator or the set-abstraction operator. Our sortal placeholder t can then be understood as place-holding for definite descriptive terms, and/or set-abstraction terms, in addition to names and parameters. Provided only that the language contains some closed terms other than parameters, all relevant contrasts among systems can be made. Note that the language is assumed to contain the identity predicate. It can be useful (but it is not obligatory) to employ a 'thinghood' predicate in order to give expression to the existential presuppositions called for in the rules of free logic.

Four systems are generated by the presence or absence of each of two features:

($\mathcal{U}$) allowing for the empty universe; and
(**T**) using a 'thinghood' predicate.

We shall accordingly obtain four systems of free logic (with identity) that we shall call F, F$\mathcal{U}$, F**T**, and F$\mathcal{U}$**T**. All four systems have the common aim of accommodating non-denoting singular terms. The four systems are very intimately related; but only the barest details of their interrelationships can be presented here.

All four systems become classical upon addition of any of the classical rules of inference—the law of excluded middle, classical reductio ad absurdum, double-negation elimination, or dilemma. The discussion of (different kinds of) freedom in a logic is orthogonal to the debate over whether intuitionistic logic is to be preferred to classical logic, or vice versa.

F$\mathcal{U}$ is the amended version described above of the system F$\mathcal{U}_{NL}$ of universally free logic whose classical extension was proved sound and complete in Tennant 1978.

4.1 On giving up universal freedom, but retaining freedom

A second line of defence against Milne's charge merits consideration, especially because of its appeal to the abstractionist realist. The latter is committed to the necessary existence of zero, as the number attached to any empty concept (such as non-self-identity). Hence there should be no need for the abstractionist realist to strive after a *universally* free logic. It should be enough to fashion one's rules so that they provide for the possibility of non-denoting terms, such as '$\{x|x\notin x\}$'. That is, it should be enough to provide a free logic, but not necessarily a *universally* free one. We can countenance having $\exists x\ x{=}x$ as a theorem. If this is one's preference, then it can be catered for by the two systems F and F**T**. (The *universally* free systems F$\mathcal{U}$ and F$\mathcal{U}$**T**, by contrast, do not contain $\exists x\ x{=}x$ as a theorem.)

If one wishes to limit oneself to non-empty domains (by opting for a system like F or F**T**), then a syntactic distinction needs to be entered between names and parameters. No expression will be able to be both a name and a parameter. Although parameters will still be name-like (i.e., argument-place-filling, but not variables), they will have to be distinguished from names. This is because names, in a free logic, may still fail to denote. In a logic (like F or F**T**) not countenancing the possibility that the universe is empty, parameters will always be understood as though they succeed in 'temporarily standing for' individuals—however qualified this metaphor must be, given that that is not really what parameters do in the course of logical reasoning. The point is that when we use a parameter *a*, say, in phrases like 'Let *a* be any arbitrary individual…', or 'Let *a* be an arbitrary individual with property φ …', we are imagining an 'arbitrary selection' of an individual, and not of some non-existent thing. The existential pre-condition $\exists!a$ therefore becomes otiose. This is not the case, however, in a universally free logic such as F$\mathcal{U}$ or F$\mathcal{U}$**T**. Arbitrary selection within the empty domain always comes up empty-handed.

The two systems F$\mathcal{U}$ and F differ only in that F, but not F$\mathcal{U}$, treats parameters as existentially committal. We have

$$\mathrm{F}\mathcal{U} + \overline{\exists!a} \;=\; \mathrm{F}.$$

Likewise, the two systems F$\mathcal{U}$**T** and F**T** differ only in that F**T**, but not F$\mathcal{U}$**T**, treats parameters as existentially committal. We have

$$\mathrm{F}\mathcal{U}\mathbf{T} + \overline{\exists!a} \;=\; \mathrm{F}\mathbf{T}.$$

Not having to worry about the empty universe streamlines the rules, and of course renders them immune to milning.

4.2 Using a thinghood predicate

A third line of defence against Milne's charge is to introduce into the language a special monadic logical predicate **T**. The formal sentence **T**(*t*) says that *t* is a thing, or that *t* is an individual.

The new predicate **T** affords some extra nuances of expression. Both the system F**T** and the system F$\mathcal{U}$**T** have $\forall x\mathbf{T}x$ as a theorem. And all four systems have $\forall x\exists !x$ (that is, $\forall x\exists y\; x=y$) as a theorem. The formal sentence $\forall x\mathbf{T}x$ is an attractively direct regimentation of 'Everything is a thing'. The abbreviated form $\forall x\exists !x$ corresponds nicely to 'Everything exists'; while its unabbreviated form $\forall x\exists y\; x=y$ is faithful to the logico-grammatical structure discernible in 'Everything is identical to something'.

In light of what has already been observed about the interpretation of parameters when reasoning about non-empty domains, it would be natural to have the axiom (or at least the theorem) $\mathbf{T}a$. In the system F**T** the derivation of $\mathbf{T}a$ is

$$\cfrac{\cfrac{}{a=a}\,{\scriptstyle(=\text{-I})}}{\mathbf{T}a}\,{\scriptstyle(\mathbf{T}\text{-I})}$$

Whether $\mathbf{T}a$ is taken as an axiom or derived as a theorem, it affords an easy proof of the claim that everything is a thing:

$$\dfrac{\begin{array}{c}\varnothing\\ \vdots\\ \mathbf{T}a\end{array}}{\forall x\mathbf{T}x}$$

Having $\mathbf{T}a$ as an axiom or a theorem, however, requires that one rule out the possibility of the empty universe.

If one wished instead to re-formulate universally free logic by employing the predicate of thinghood, this last little proof would have to be changed slightly (for we would still want its conclusion as a theorem). The assumption $\mathbf{T}a$ would have to be permitted 'for the sake of argument' for an application of ($\forall$-I); and this application would have to be able to discharge that assumption:

$$\cfrac{\cfrac{}{\mathbf{T}a}\,{\scriptstyle(1)}}{\forall x\mathbf{T}x}\,{\scriptstyle(1)}$$

In a universally free logic, we cannot have $\mathbf{T}a$ as a theorem. It should be clear from inspection that in the two systems F$\mathcal{U}$ and F$\mathcal{U}$**T**, the sentence $\mathbf{T}a$ is not provable.

Note that (**T**-I) followed by (**T**-E) both establishes and reduces to (=-I):

$$\cfrac{\cfrac{A(t)}{\mathbf{T}(t)}\,{\scriptstyle(\mathbf{T}\text{-I})}}{t=t}\,{\scriptstyle(\mathbf{T}\text{-E})} \qquad \Rightarrow \qquad \cfrac{A(t)}{t=t}\,{\scriptstyle(=\text{-I})}$$

We can also prove that $A(t)$ logically implies $\exists !t$:

$$\dfrac{\dfrac{A(t)}{\mathbf{T}t}\,(\mathbf{T}\text{-E}) \qquad \dfrac{A(t)}{t=t}\,(\mathbf{T}\text{-I})}{\exists! t}\,(\exists\text{-I})$$

4.3 Conceptual dependencies

Inspection of the four F-systems reveals that the logical concepts of identity, thinghood and existence can be captured in that order.

The rules for identity are framed with reference only to atomic sentences, and to substitution of terms within sentences in general. These are basic syntactic notions; whence the notion of identity can be grasped *au fond*.

If one is working in a **T**-system, then the predicate **T** comes next. Its introduction rule is framed with reference only to atomic sentences, but its elimination rule involves the identity predicate—which one already grasps.

Finally, the existential quantifier ∃ can be grasped. In a **T**-system, the rule (∃-I) is framed with reference only to **T**. In the **T**-less systems, (∃-I) is framed in part with reference to sentences of the form $\exists!t$, but then these in turn are conclusions of that part of the introduction rule that involves a single premise of the form A(*t*). The rule (∃-E), moreover, involves no new conceptual ingredients.

It follows that the more liberal Dummettian picture (which Milne contrasts with the less liberal one in Tennant 1997) is right: certain logical concepts may be characterized by rules whose formulation involves certain other logical concepts. Indeed, in forthcoming work by the present author the picture is liberalized even further, so as to allow for interdependencies among coeval logical concepts that cannot be introduced in a linear order: for details, see Tennant forthcoming.

5. In defence of natural deduction

All primitive concepts are idealizations.
—Gödel (Wang 1996, at p. 300)

5.1 Milne contra *Gentzen*

Milne takes Gentzen to task for alleged confusions in the way Gentzen sets out the informal reasoning establishing a certain formal first-order sentence as a logical truth.

Gentzen's original reads as follows:

> Zweites Beispiel: $(\exists x \forall y Fxy) \supset (\forall y \exists x Fxy)$.
> Man wird so argumentieren: Es gebe ein *x*, so dass für alle *y* F*xy* gilt. Sei *a* ein solches *x*. Also gelte für alle *y*: F*ay*. Sei nun *b* irgendein beliebiger Gegenstand. Dann gilt F*ab*. Es gibt also ein *x*, nämlich *a*, so daß F*xb* gilt. *b* war beliebig, also gilt dies für alle Gegenstände, d.h.: Für alle *y* gibt es ein *x*, so daß F*xy* gilt. Damit haben wir die Behauptung. (Gentzen 1934-5, at p. 183 of the original)

Milne's first worry is that the variable '*x*' 'seems to be treated as a kind term'. Immediately after raising this worry he seeks to allay it, by means of an acceptable paraphrase. His paraphrase begins 'There is at least one thing of which this is true: that it bears F to everything.'

Milne then states a second worry, with reference to this translation of the original German: he claims that Gentzen 'states the antecedent categorically'. But Gentzen's use of 'Es gebe' rather than 'Es gibt' shows that he was not making a categorical rather than a suppositional claim. The use of the mood *Konjunktiv I* in German indicates the suppositional status of the antecedent of the conditional to be proved.

The informal locutions of logician's English (or logician's German, for that matter) are used with a certain degree of laxity. One needs to be charitable, and not read too much into them. They are, after all, in Gentzen's example, given by way of explanation or embroidery of a formal proof. Gentzen's aim was to show how the informal reasoning in effect embodied an application of the formal rule ∃-E of natural deduction. The informal version of that stretch of the argument was not being given as a justification of the rule ∃-E. Rather, it was an illustration of the rule at work in an informal example.

This passage of informal reasoning in Gentzen (like the one by Prawitz, to be considered presently) is offered as an informal rendering of a formal proof in the system of natural deduction. Thus there is a certain degree of expository self-consciousness about the way it is set out. Actual informal proofs that are given without the conscious aim of illustrating every structural detail of a formal proof are, ironically, often much terser, in taking more for granted, and not bothering to spell out all the relevant details. Yet it is the latter kind of reasoning which is supposed to be regimented (i.e., rendered more rigorous and precise) by the formal proof in a system of natural deduction. It is compatible with this theoretical aim that the formal proof will actually contain more 'deep structure' than is explicit in the informal reasoning that it regiments.

Consider, for example, informal talk of a conclusion's being 'independent' of the 'particular value' that a parameter might take in an assumption 'made for the sake of argument'. Such talk is shorthand for that formal aspect of a natural deduction that consists of the explicit discharge of the assumption containing the parameter in question, when a particular rule (such as ∃-E) is applied in order to obtain a certain conclusion. (Parameters may be understood as behaving grammatically like names. Since proofhood is a matter of form and not content, we do not even have to construe parameters as names without bearers.)

Milne discerns semantic ascent when the German word ‘gilt’ makes an appearance. English does the same with expressions such as ‘it holds of such-and-such that’. When reading informal renderings of proofs in first-order logic we have to bear in mind that these stylistic aids are not really already committing us to an object-language/metalanguage distinction. It is easier to say or write ‘there is an *x*’ than ‘there is a value for *x*’. The former makes ‘*x*’ look like a kind term; the latter does not.

5.2 Milne contra *Prawitz*

Milne also criticizes Prawitz, by reference to one of Prawitz’s examples of how the rules of natural deduction can be seen at work. According to Milne, Prawitz fails to explain how it is that an argument can be independent of the particular value of the parameter that is used for an application of ∃-E. An answer, however, is readily forthcoming, in light of the clarifications in section 5.1.

Example (in effect, Prawitz’s own): Let Π be the proof

$$\dfrac{\dfrac{\mathrm{P}ab \qquad \dfrac{\mathrm{P}ab \qquad \dfrac{\dfrac{\forall x\forall y(\mathrm{P}xy\rightarrow \mathrm{P}yx)}{\forall y(\mathrm{P}ay\rightarrow \mathrm{P}ya)}}{\mathrm{P}ab\rightarrow \mathrm{P}ba}}{\mathrm{P}ba}}{\mathrm{P}ab\&\mathrm{P}ba}}{\exists y(\mathrm{P}ay\&\mathrm{P}ya)}$$

of ∃y(P*ay*&P*ya*) from $\forall x\forall y(\mathrm{P}xy\rightarrow \mathrm{P}yx)$, P*ab*. Now let *t* be any singular term. The proof Π^{b}_{t}, that is:

$$\dfrac{\dfrac{\mathrm{P}at \qquad \dfrac{\mathrm{P}at \qquad \dfrac{\dfrac{\forall x\forall y(\mathrm{P}xy\rightarrow \mathrm{P}yx)}{\forall y(\mathrm{P}ay\rightarrow \mathrm{P}ya)}}{\mathrm{P}at\rightarrow \mathrm{P}ta}}{\mathrm{P}ta}}{\mathrm{P}at\&\mathrm{P}ta}}{\exists y(\mathrm{P}ay\&\mathrm{P}ya)}$$

is a proof of $\exists y(\mathrm{P}ay\&\mathrm{P}ya)$ from $\forall x\forall y(\mathrm{P}xy\rightarrow \mathrm{P}yx)$, P*at*.

Now consider the foregoing proof Π supplemented with a final application of ∃-E:

$$\dfrac{\exists y\mathrm{P}ay \qquad \dfrac{\dfrac{{}^{(1)}\dfrac{}{\mathrm{P}ab} \qquad \dfrac{{}^{(1)}\dfrac{}{\mathrm{P}ab} \qquad \dfrac{\dfrac{\forall x\forall y(\mathrm{P}xy\rightarrow \mathrm{P}yx)}{\forall y(\mathrm{P}ay\rightarrow \mathrm{P}ya)}}{\mathrm{P}ab\rightarrow \mathrm{P}ba}}{\mathrm{P}ba}}{\mathrm{P}ab\&\mathrm{P}ba}}{\exists y(\mathrm{P}ay\&\mathrm{P}ya)}}{\exists y(\mathrm{P}ay\&\mathrm{P}ya)}{}^{(1)}$$

It is clear that the conclusion of this proof is ‘independent’ of the particular value of the parameter *b* that instantiates the major premise ∃*y*P*ay* for ∃-E. Whatever instance *t* an asserter of the existential premise ∃*y*P*ay* might adduce as a ‘witness’, the subproof $\Pi^b{}_t$ will show that the conclusion ∃*y*(P*ay*&P*ya*) follows from ∀*x*∀*y*(P*xy*→P*yx*), P*at*.

In the example of Prawitz which Milne cites, the author (Prawitz) is assuming that his reader will be able to tell by inspection that this last logical observation holds. Appeals to one’s intuition that a conclusion has been deduced ‘independently’ of whatever particular value a parameter *b* might take (where *b* occurs in an undischarged assumption) are frequent in informal reasoning. And, as pointed out earlier, what is really meant by such talk is that the assumption containing the parameter in question is no longer needed in support of the conclusion. For that conclusion is now inferred on the basis of the appropriate existential premise (in the foregoing example, ∃*y*P*ay*), rather than on the basis of the parametric assumption (in the foregoing example, P*ab*). The latter assumption was made ‘for the sake of argument’—that is, with the intention of eventually discharging it by applying the rule ∃-E. The parameter *b* featuring in this assumption-for-the-sake-of-argument is sometimes thought of as ‘standing for’ an arbitrary individual of the existentially quantified kind (in the foregoing example, an arbitrary individual borne the relation P by *a*). This gloss, however, must be taken *cum grano salis*, since the propriety of each and every feature of a proof is a strictly formal matter.

The ‘independence’ of particular values of the instantiating parameter *b* is central to the formulation of the reduction procedure for ∃, which shows one how to get rid of any conclusion of ∃-I that also stands as the major premise for an application of ∃-E. (This is the reduction procedure for unfree logic. It has analogues in our systems of free logic, but some of the technical subtleties involved would take the present discussion too far afield.)

$$
\dfrac{(\exists\text{-I})\,\dfrac{\begin{matrix}\Omega\\ \Sigma\\ \varphi^x{}_t\end{matrix}}{\exists x\varphi} \qquad \begin{matrix}\Delta\,,\,\overline{\varphi^x{}_b}^{\,(i)}\\ \Pi\\ \psi\end{matrix}}{\psi}\,(\exists\text{-E})\,(i)
\qquad\Rightarrow\qquad
\begin{matrix}\Omega\\ \Sigma\\ \Delta\,,\,\varphi^x{}_b\ \ (=(\varphi^x{}_b)^b{}_t)\\ \Pi^b{}_t\\ \psi\end{matrix}
$$

A signal achievement of natural deduction theorists has been to systematize the structural intuitions involved in the use of parameters in informal quantified reasoning. Natural deduction provides an illuminating articulation of lines of logical dependency within proofs, and explicates the ‘parametric’ behaviour of name-like expressions that are being used with that ‘smack of generality’ prospectively conferred by phrases such as ‘Let *b* be an arbitrary φ’.

In offering a systematic codification of correct moves (such as ∃-I and ∃-E in natural deduction), the natural deduction theorist does not have to respect, slavishly, whatever he finds by way of less formal reasoning in the mathematical literature. The mathematician who studies natural deduction returns to the informal proofs of mathematics aware that many an informal passage of quantificational reasoning is poorly expressed. It would be a methodological mistake to read off from badly expressed mathematical reasoning what the correct codification of logically correct quantificational reasoning would be. Once versed in natural deduction, the reader of mathematical textbooks is equipped to clarify, re-formulate and improve upon the informal reasoning that he encounters. Such a reader would not set too much store by the sorts of examples (pp. 27-28) that Milne collected from various mathematical textbooks. Informal reasoning merely has to secure uptake by its reader; it is not material for purely mechanical checks for proofhood.

5.3 Milne contra *the present author*

Informal renderings of a passage of quantificational reasoning may differ widely. Consider the following natural deduction, with whose informal rendering in Tennant 1978 Milne expresses some dissatisfaction:

$$\dfrac{\exists x\varphi x \qquad \dfrac{\dfrac{\dfrac{{}^{(1)}}{\varphi a} \qquad \dfrac{\forall x(\varphi x\rightarrow\psi x)}{\varphi a\rightarrow\psi a}}{\psi a}}{\exists x\psi x}}{\exists x\psi x}\,{}^{(1)}$$

One can speak only in the formal sense of ‘the’ passage of reasoning that this natural deduction represents. For there are various passages of informal reasoning that would all be represented adequately by this natural deduction. It is, of course, a rather degenerate example; in informal reasoning, one might treat the passage from the premises ‘There is a φ’ and ‘All φs are ψs’ to ‘There is a ψ’ as primitive, because so obviously suasive, and not bother to supply any intervening justificatory steps. But suppose that one seeks to spell out ‘the’ details that would make incontrovertible the validity of the passage from these premises to this conclusion. The following are all possible informal solutions to the problem

> *Suppose that there is a φ, and that all φs are G. Show beyond a shadow of a doubt that it follows that there is a G.*

Solution 1. There is a φ (given). Now, since all φs are ψs (given), whatever such a φ is, it’s also a ψ. So, there is a ψ.

Solution 2. There is a φ (given). So, let *a* be an arbitrary φ. All φs are ψs (given). Thus *a* is a ψ. Hence there is a ψ.

Solution 3. There is a φ (given). So, let *a* be an arbitrary φ. All φs are ψs (given). Thus *a* is a ψ. Hence there is a ψ (namely, *a*). But *a* was an arbitrary φ. Hence there is a ψ.

Solution 4. Let *a* be an arbitrary φ. All φs are ψs (given). Thus *a* is a ψ. So there is a ψ (namely, *a*). But there is a φ (given). Hence there is a ψ, no matter which φ *a* might be.

Solution 5. Let a be an arbitrary φ. All φs are ψs (given). Thus a is a ψ. So there is a ψ (namely, a). But there is a φ (given). And our conclusion that there is a ψ would follow, independently of the particular value that the parameter a might take. Hence there is a ψ.

Solution 6. Let a be an arbitrary φ. All φs are ψs (given). Thus a is a ψ. So there is a ψ (namely, a). But there is a φ (given). Hence there is a ψ, independently of our assumption that a is a φ.

Solution 1 gestures towards some extra detail, but without bringing in an explicit parameter. **Solution 2** reaches the conclusion 'there is a ψ' without any explicit repetition thereof. It also makes the parametric assumption (that a is a φ) appear to 'follow from' the existential premise—which of course it does not. One is left to understand tacitly that the assumption corresponding to the 'let' clause is not actually needed by the stage at which one draws the final conclusion that there is a ψ. **Solution 3** provides the repetition that ones sees in the formal proof—in which 'there is a ψ' occurs once as the conclusion of the subordinate proof for ∃-E, and then once again as the overall conclusion of that step. **Solution 4** uses a 'no matter what' locution, rather than saying that the reasoning holds 'independently of the particular value that the parameter a might take'. **Solution 5**, by contrast, employs the latter locution. It also appears to mix object-level talk with metalevel talk. **Solution 6** is perhaps closest in structure to the formal proof. It sets out the subordinate proof for ∃-E first, thereby avoiding giving the reader the incorrect impression that the parametric assumption somehow 'follows from' the existential premise. It repeats the conclusion 'there is a ψ'. It also makes the discharge of the parametric assumption explicit. Thus it too is 'guilty' of apparently mixing object-level talk with metalevel talk. Yet of all the solutions, it appears to be at once the most detailed and the most fastidious in its running commentary on salient features of the argument's logical structure.

All of these informal arguments are perfectly regimented by the formal proof above. It would be a stretch to maintain that any or each of them somehow 'justifies' the individual steps within that formal proof. Rather, the formal proof is a representation of the essential structure of the logical reasoning that can be discerned in each of the informal arguments. The formal proof is both a theoretical representation of the 'deep structure' of different passages of informal reasoning, and a passage of reasoning in its own right, in a formal language equipped with rigorous and precise primitive rules of inference. Proofs in the formal system have the added advantage of being displayed graphically in two dimensions, so that the partial or tree-like ordering of logical dependencies is made especially vivid.

The inferential meaning-theorist regards the introduction and elimination rules in a formal system of natural deduction as capable (whether separately or jointly) of conferring precise senses on the logical operators involved. But these are the logical operators of the formal language. Although the latter correspond more or less exactly to certain logical expressions in ordinary language, the inferential meaning-theorist is not claiming to be providing a direct account of the senses of the latter. To the extent, however, that sentences of ordinary language can be furnished with logical forms in the formal language, and to the extent that passages of informal reasoning can be regimented by formal proofs in the system of natural deduction, one can transfer the inferential account of senses from the logical operators of the formal language to the logical expressions of ordinary language.

6. Summary and conclusion

Milne's criticisms of explanations of natural-deduction 'rules in action' given by Gentzen, Prawitz and the present author have been met. Also averted is Milne's objection that the inferentialist cannot secure a unique sense for each logical operator in free logic. The technique of milning is not efficacious against a properly thought-out re-arrangement of the rules for the quantifiers and for identity in free logic.

The author is grateful to Milne for providing the impetus to re-consider the issues carefully. The investigations above are intended to convey a clearer understanding of how the rules for free logic can be framed and grouped, so as better to serve the demands of an inferential theory of meaning.

7. Appendix: Systems of natural deduction for free logic (with no function signs)

Natural Logic's system F$\mathcal{U}_{NL}$ of universally free logic

Rule of atomic denotation

$$\frac{A(t)}{\exists! t}\text{ , where } A(t) \text{ is atomic}$$

Universal Introduction (∀-I)

$$\dfrac{\begin{array}{c}\overline{\exists! a}^{\,(i)}\\ \vdots \\ \varphi\end{array}}{\forall x \varphi^{a}_{x}}\,(i)$$

where a does not occur in any assumption, other than $\exists! a$, on which φ depends

Universal Elimination (∀-E)

$$\frac{\exists! t \quad \forall x\varphi}{\varphi^{x}_{t}}$$

Existential Introduction (∃-I)

$$\frac{\exists! t \quad \varphi^{x}_{t}}{\exists x\varphi}$$

Existential Elimination (∃-E)

$$\dfrac{\exists x\varphi \qquad \begin{array}{c}\overline{\exists! a}^{\,(i)}\ ,\ \overline{\varphi^{x}_{a}}^{\,(i)}\\ \vdots \\ \psi\end{array}}{\psi}\,(i)$$

where a does not occur in $\exists x\varphi$, in ψ, or in any assumption, other than $\exists! a$ and φ^{x}_{a}, on which the upper occurrence of ψ depends

Reflexivity of identity

$$\frac{\exists! t}{t=t}$$

Substitutivity of identity

$$\frac{t=u \quad \varphi(t)}{\varphi(u)}$$

From now on parametric conditions on *a* for (∀-I) and for (∃-E) will be taken as read.

	$\mathrm{F}\mathcal{U}$	F	$\mathrm{F}\mathcal{U}\mathbf{T}$	$\mathrm{F}\mathbf{T}$
$(\forall\text{-I})$	$\frac{}{\exists! a}(i)$ $\vdots$ $\frac{\varphi}{\forall x \varphi^{a}_{x}}(i)$	$\vdots$ $\frac{\varphi}{\forall x \varphi^{a}_{x}}(i)$	$\frac{}{\mathbf{T}a}(i)$ $\vdots$ $\frac{\varphi}{\forall x \varphi^{a}_{x}}(i)$	$\vdots$ $\frac{\varphi}{\forall x \varphi^{a}_{x}}(i)$
$(\forall\text{-E})$	$\frac{\exists! t \quad \forall x\varphi}{\varphi^{x}_{t}}$	$\frac{\exists! t \quad \forall x\varphi}{\varphi^{x}_{t}}$ $\quad \frac{\forall x\varphi}{\varphi^{x}_{a}}$	$\frac{\mathbf{T}t \quad \forall x\varphi}{\varphi^{x}_{t}}$	$\frac{\mathbf{T}t \quad \forall x\varphi}{\varphi^{x}_{t}}$ $\quad \frac{\forall x\varphi}{\varphi^{x}_{a}}$
$(\exists\text{-I})$	$\frac{\exists! t \quad \varphi xt}{\exists x\varphi}$ $\quad \frac{\mathrm{A}(t)}{\exists! t}$	$\frac{\exists! t \quad \varphi xt}{\exists x\varphi}$ $\quad \frac{\mathrm{A}(t)}{\exists! t}$ $\frac{\varphi^{x}_{a}}{\exists x\varphi}$	$\frac{\mathbf{T}t \quad \varphi xt}{\exists x\varphi}$	$\frac{\mathbf{T}t \quad \varphi xt}{\exists x\varphi}$ $\frac{\varphi^{x}_{a}}{\exists x\varphi}$
$(\exists\text{-E})$	$\frac{}{\exists! a}(i)\ ,\ \frac{}{\varphi^{x}_{a}}(i)$ $\vdots$ $\frac{\exists x\varphi \quad \psi}{\psi}(i)$	$\frac{}{\varphi^{x}_{a}}(i)$ $\vdots$ $\frac{\exists x\varphi \quad \psi}{\psi}(i)$	$\frac{}{\mathbf{T}a}(i)\ ,\ \frac{}{\varphi^{x}_{a}}(i)$ $\vdots$ $\frac{\exists x\varphi \quad \psi}{\psi}(i)$	$\frac{}{\varphi^{x}_{a}}(i)$ $\vdots$ $\frac{\exists x\varphi \quad \psi}{\psi}(i)$
$(=\text{-I})$	$\frac{\mathrm{A}(t)}{t=t}$	$\frac{\mathrm{A}(t)}{t=t}$ $\quad \frac{}{a=a}$	$\frac{\mathrm{A}(t)}{t=t}$	$\frac{\mathrm{A}(t)}{t=t}$ $\quad \frac{}{a=a}$
$(=\text{-E})$	$\frac{t=u \quad \varphi(t)}{\varphi(u)}$	$\frac{t=u \quad \varphi(t)}{\varphi(u)}$	$\frac{t=u \quad \varphi(t)}{\varphi(u)}$	$\frac{t=u \quad \varphi(t)}{\varphi(u)}$
$(\mathbf{T}\text{-I})$			$\frac{\mathrm{A}(t)}{\mathbf{T}t}$	$\frac{\mathrm{A}(t)}{\mathbf{T}t}$ $\quad \frac{}{\mathbf{T}a}$
$(\mathbf{T}\text{-E})$			$\frac{\mathbf{T}\mathrm{t}}{t=t}$	$\frac{\mathbf{T}\mathrm{t}}{t=t}$

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