Logic, Mathematics and the A Priori, Part I: A Problem for Realism

by

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Abstract

This is Part I of a two-part study of the foundations of mathematics through the lenses of (i) apriority and analyticity, and (ii) the resources supplied by Core Logic.

Here we explain what is meant by apriority, as the notion applies to knowledge and possibly also to truths in general. We distinguish grounds for knowledge from grounds of truth, in light of our recent work on truthmakers. We then examine the role of apriority in the realism/anti-realism debate. We raise a hitherto unnoticed problem for any Orthodox Realist who attempts to explain the *a priori*.

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[In Part II (to appear) we shall examine the sense in which logic is *a priori*, and explain how mathematical theories, while *a priori*, can be dichotomized non-trivially into analytic and synthetic portions. We shall go on to argue that Core Logic (the system of logic that the anti-realist should espouse) contains exactly the *a-priori-because-analytically*-valid deductive principles. We shall introduce the reader to the system of Core Logic by explaining its relationship to other logical systems, and setting out its rules of inference. Important metatheorems about Core Logic will be reported, and its important features are noted, in setting out the case for Core Logic. Core Logic can serve as the basis for a foundational program that could be called *natural logicism*, an exposition of which will build on the (meta)logical ideas explained in Part II.]

1 *A priori* knowledge and *a priori* truth

Apriority is a property enjoyed only by *knowable* truths:

> φ is *a priori* true
> just in case
> there are adequate grounds for the knowledge that φ that do not involve any recourse or appeal to sensory or perceptual experience.

We shall take φ to be a thought, or proposition, or declarative sentential content.[1]

Some might object that some kind of category-mistake must be involved in seeking to speak thus of *truths* being *a priori*, rather than of *knowledge* being so. Surely, they might say, apriority is a property of items of *knowledge*? (And this is the case, they might add, because apriority attaches to the *methods* whereby we attain certain kinds of knowledge.) Thus Kitcher [1980], at p. 3:

> "A priori" is an epistemological predicate. What is primarily a priori is an item of knowledge.[fn]

But Kitcher immediately concedes (pp. 3–4)

[1] Since our concern here is, in the main, with apriority in logic and mathematics, we set aside the further finessing that might be called for in order to handle examples of *a priori* knowledge such as 'I am here now'. It may be that the pragmatics and semantics of token-reflexives force us to deal with subtly different kinds of semantic entity such as Kaplanian characters. (The author is indebted here to Joseph Salerno.)

> Of course, we can introduce a derivative use of "a priori" as a predicate of propositions: a priori propositions are those which *we could know* a priori. [Emphasis added—NT.]

Against the imagined objector just mentioned, one can point out that knowledge is factive:

$$\frac{K\varphi}{\varphi}$$

Moreover, any truth, large or small as it were, could be something we might wish to know. One is therefore easily forgiven the catachresis involved in moving from talk of the *a priori knowable* to talk of *a priori truth*; and entitled to deny that there is any kind of category-mistake involved.

The notion of *a priori* (truth and) knowledge is *not* agent-relative. An agent might know that φ on grounds that *do* involve sensory or perceptual experience; and she might remain forever ignorant of any grounds that do *not* involve sensory or perceptual experience. Her knowing in this unhappy way that φ in no way counts against φ's being *a priori* knowable (hence true).

Our resulting focus on propositions; then, among them, on the truths; then, among those, on the *a priori* truths, is not at all deviant. Compare Peacocke [1993], at p. 175:

> The standard characterization states that a truth is a priori if *it can be known* to be true in the actual world independently of any particular kind of experience or empirical information. [Emphasis added—NT.]

2 Grounds for knowledge *versus* grounds of truth

2.1 Grounds$_K$

We shall use 'grounds$_K$' in a broad (but *objective*) epistemic sense, to cover proof, justification, warrant, entitlement, reasons, etc. Such grounds are of an epistemic nature. They are humanly graspable, hence finite. They can be inspected or surveyed, effectively checked for correctness, and consensually assessed for cogency. Grounds$_K$ are *abstract types* of constructions, not necessarily tokens (i.e., physical things or events, such as visible inscriptions or audible sequences of sounds). We do *not* identify grounds$_K$ with any *psychological processes* that might produce, within a thinker, knowledge that φ. To the extent that psychological or cognitive processes are involved in *acquiring* knowledge, the interesting question is how it comes to pass that, by undergoing such processes, the thinker makes intellectual contact with appropriately non-empirical grounds$_K$ for φ. The status of φ as *a priori* turns on the nature of those non-empirical grounds$_K$, not on the nature of any internal processes in the thinker.[2]

Our initial definition can therefore be tightened so as to read as follows.

> φ is *a priori* true
> just in case
> there are adequate grounds$_K$ for the knowledge that φ that do not involve any recourse or appeal to sensory or perceptual experience.

Mathematical proofs, for example, are construct*ible*—they exist as abstract objects—but certain ones among them might never be thought of, or discovered, let alone tokened, either in speech or in writing. For all we know, there is a proof in Peano Arithmetic of Goldbach's Conjecture, which states that every even number greater than 2 is the sum of two prime numbers. But all such proofs, if any exist, have eluded us to this day; and may well elude us forever.

2.2 Grounds$_T$

Suppose there is no counterexample to Goldbach's Conjecture. Then, even if there is no proof of the Conjecture in Peano Arithmetic, some thinkers—realists—maintain it would nevertheless be *true*, courtesy of Carnap's famous ω-rule:

$$\frac{\begin{array}{cccccc} \vdots & \vdots & \vdots & & \vdots & \\ \psi(0) & \psi(1) & \psi(2) & \cdots & \psi(\underline{k}) & \cdots \end{array}}{\forall n \psi(n)}\,\mathbb{N}$$

Here, $\psi(k)$ means 'if k is even and $k > 2$, then two primes sum to k'. Clearly, ψ is a decidable property of natural numbers, since any two such primes would have to be less than k. The ω-rule can be thought of as '$\mathbb{N}$-relative $\forall$-Introduction' for the construction of arithmetical truthmakers.

In this case, the ω-rule would represent, for this supposedly true-but-Peano-unprovable Conjecture, the final step in the pedigree of a *way of*

[2]These observations entail that we would not be satisfied with the psychologistic kind of analysis of *a priori* knowledge offered by Kitcher [1980] at pp. 9–10. Our stress on making intellectual contact with the appropriate grounds$_K$ for φ also disposes of Kitcher's subsequent critique, at pp. 21–23, of any apsychologistic alternative to his proposal.

its being true in the standard model $\mathbb{N}$ of the natural numbers. *Ways of being true* are grounds of a different kind than grounds$_K$. Let us call them grounds$_T$.

Grounds$_T$ are *truthmakers*, abstract objects with formal structure. In this respect they are akin to proofs. But there is an important difference. Truthmakers are *model-relative*, and will often be infinitary, if the domain of discourse of the model in question is infinite (like the domain of natural numbers). As we have just seen, arithmetical truthmakers can branch infinitely widely, at applications of the ω-rule. Similarly, arithmetical falsitymakers can so branch, when dealing with false existentials:

$$\frac{\exists n\psi(n) \qquad \begin{array}{c} \overline{\ \psi(0)\ }^{\,(i)} \\ \vdots \\ \bot \end{array} \qquad \begin{array}{c} \overline{\ \psi(1)\ }^{\,(i)} \\ \vdots \\ \bot \end{array} \qquad \cdots}{\bot}\,{\scriptstyle (i)\ \mathbb{N}}$$

This rule is the dual of the ω-rule: it can be thought of as '$\mathbb{N}$-relative $\exists$-Elimination' for the construction of arithmetical *falsity*makers.

A formal theory of truthmakers (and falsitymakers) in a given discourse D furnishes, for every way of φ's being true in D, a truthmaker that explicates, or represents, that particular way in which φ is true in D. These truthmakers can be thought of as abstract reifications of the various grounds$_T$ of φ's truth in D. Truthmakers and falsitymakers are co-inductively definable.[3]

Truthmakers provide formal substance for the philosophical idea that truth consists in the existence of a truthmaker, or that truths enjoy grounds$_T$. Indeed, we have the following result, where $\mathcal{V}(\Pi, \varphi, M, \mathbb{D})$ means 'Π is a truthmaker for φ in the model M with domain $\mathbb{D}$':

Metatheorem 1 Modulo *a metatheory which contains the mathematics of $\overline{\overline{\mathbb{D}}}$-furcating trees of finite depth, we have, for all models M with domain $\mathbb{D}$,*

$$\exists\Pi\ \mathcal{V}(\Pi, \varphi, M, \mathbb{D}) \quad \Leftrightarrow \quad \varphi \textit{ is true in } M$$

where the right-hand side is in the sense of Tarski.

[3]For a fuller development of a formal theory of truthmakers and falsitymakers, see Tennant [2010] and Tennant [forthcoming].

3 Realism v. Anti-Realism, and the *A Priori*

In any discourse D, there are the truths; and there are the falsehoods. It is a deep and controversial question whether together they exhaust the space of propositions. The realist says they do; the anti-realist demurs.

Let us concentrate on the truths. Among these, at least some are *knowable*; indeed, many of them are *known.* It is a deep and controversial question whether *all* truths are knowable:

$$\forall\varphi(\varphi \to \Diamond K\varphi)?$$

The anti-realist says they are. The realist does not simply demur; the realist *denies* that all truths are knowable:

$$\neg\forall\varphi(\varphi \to \Diamond K\varphi).$$

Not only that, but the realist, who is a classical logician, even goes so far as to conclude (strictly classically, note!) that

$$\exists\varphi(\varphi \wedge \neg\Diamond K\varphi)$$

—without, of course, being able to furnish a justifying instance for this existential claim.

3.1 Picturing possible positions

Since a picture is worth a thousand words, we shall take the liberty of reprising below some of the pictorial aids in Chapter 6 of Tennant [1997].

We shall use the following defined symbols:

$\mathcal{D}$ $=_{df}$ the set of sentences whose truth can be established by the application of an effective decision method.[4]

$\mathcal{K}$ $=_{df}$ the set of sentences which are in principle knowable as true, even if not by way of application of an effective decision method.[5]

$\mathcal{T}$ $=_{df}$ the set of true sentences.

[4] The set $\mathcal{D}$ of effectively decidable sentences (that are true) is not to be confused with any discourse D or domain $\mathbb{D}$.

[5] Note that we say *knowable*, not 'known'.

Each of these sets is of course consistent; that is, from none of them can one deduce $\perp$ (absurdity). We wish to make clear how, on different competing accounts, these sets are interrelated (by identity or inclusion). But first we have to define one more kind of set, which we shall call a *completion*. In what follows, all sets will be sets of sentences.

When Δ is consistent, we shall denote ambiguously by Δ^+ any *completion* of Δ: that is, any consistent set Γ such that $\Delta \subseteq \Gamma$ and for every sentence φ, either φ is in Γ or $\neg\varphi$ is in Γ. In general Δ can have more than one completion.

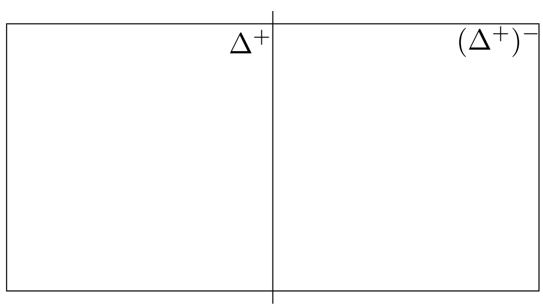
$\Delta^- =_{df}$ the set of sentences whose negations are in Δ.

Obviously, Δ^- is unique. We shall call it the *undoing* of Δ.

LEMMA: For every φ, either φ is in Δ^+ or φ is in $(\Delta^+)^-$.

> *Proof.* By definition φ is in Δ^+ or $\neg\varphi$ is in Δ^+. In the second case φ is in $(\Delta^+)^-$. The result follows.

Our Lemma tells us that for any consistent set Δ, any completion Δ^+ and its undoing $(\Delta^+)^-$ form a partition of the language. The picture would be:



where the rectangle represents the set of sentences in the language. The dividing line down the middle shows the partition effected by the completion Δ^+ (the left square) and its undoing $(\Delta^+)^-$ (the right square).

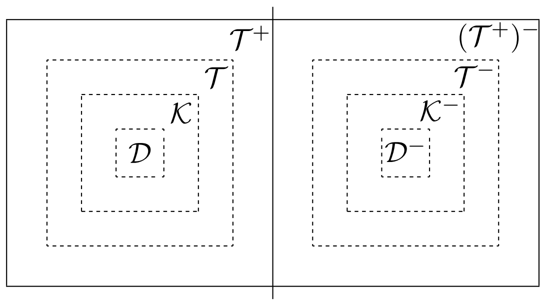
3.1.1 The Prima-Facie Picture

Now in general every decidably true sentence is knowable as true; every sentence knowable as true is true; and every true sentence will be in any completion of the set of true sentences. Thus we know we have the containments

$$\mathcal{D} \subseteq \mathcal{K} \subseteq \mathcal{T} \subseteq (\mathcal{T})^+,$$

which will govern all the possibilities to be explored below.

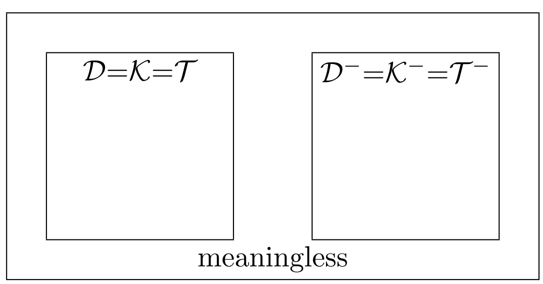
This chain of containments gives us what we may call the Prima-Facie picture:



The dashed boundaries might be thought of as still rubbery, as it were.

3.1.2 The Logical Positivists' Picture

In the 1920s the Logical Positivists of the Vienna Circle entertained a rather naïve picture, one that involved a bold answer to the question of whether any of the preceding containments might be proper:



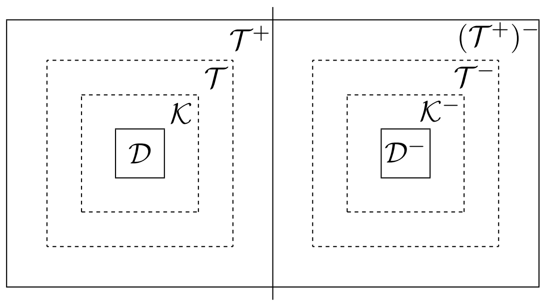
The Logical Positivists' principle of verifiability was tantamount to the requirement that any meaningful (that is, true or false) statement should have its truth-value determinable by the application of some decision procedure. Any statement for which a truth-value could not be so determined would be meaningless.

3.1.3 The Metamathematical Limitativist's Picture

Quite apart from various conceptual difficulties inherent in this naïve verificationist position (especially from the principle of compositionality), the Logical Positivists' picture could not withstand the discoveries of Gödel's completeness theorem[6] and Church's undecidability theorem[7] for first-order logic. Logical truth was undecidable—despite the fact that it was axiomatizable, hence knowable. This gave us the firm proper containment $\mathcal{D} \subset \mathcal{K}$. So the chain above had become

$$\mathcal{D} \subset \mathcal{K} \subseteq \mathcal{T} \subseteq (\mathcal{T})^{+}$$

which gives us a slightly firmer picture as follows:



3.1.4 A digression on modes of containment

We must now enquire whether any of the other containments is proper. We must be careful, however, when we talk of proper containment. Suppose we already have that $\Delta \subseteq \Gamma$, that is, that $\forall\varphi(\varphi \in \Delta \rightarrow \varphi \in \Gamma)$. For the classicist, the following are logically equivalent ways of stating that Δ is a *proper* subset of Γ ($\Delta \subset \Gamma$):

(1) assert $\exists\varphi(\varphi \in \Gamma \wedge \neg(\varphi \in \Delta))$

(2) assert $\neg\forall\varphi(\varphi \in \Gamma \rightarrow \varphi \in \Delta)$

For the intuitionist, however, (1) implies but is not implied by (2).[8] Even

[6] See Gödel [1930].

[7] See Church [1936a], Church [1936b].

[8] The following sort of objection is in error, and betrays ignorance of the logical fundamentals of intuitionism: 'The intuitionist is supposed to be a constructivist; there-

in the absence of (1) and (2), the intuitionist could

(3) refuse to assert $\forall\varphi(\varphi \in \Gamma \rightarrow \varphi \in \Delta)$

without yet denying it. For certain choices of Δ and Γ, (1) would be intuitionistically incoherent, leaving (2) and (3) as the only options.

Suppose $\Delta \subseteq \Gamma$. When (3) is the only option on the question whether this containment is proper, we shall write $\Delta \sqsubset \Gamma$. With Brouwer[9] came the option of type (3) just discussed, that $\mathcal{T} \sqsubset \mathcal{T}^+$. This is the simple refusal to assert Bivalence. The undecidability of Robinson's arithmetic Q strengthened this.[10] It gave the *anti-realist* (even if not the realist) the option of type (2) just discussed: $\mathcal{T} \subset \mathcal{T}^+$.[11] The chain of containments

$$\mathcal{D} \subset \mathcal{K} \subseteq \mathcal{T} \subseteq (\mathcal{T})^+$$

could now, it seems, be firmed up only in a limited number of ways, each one corresponding to a distinctive philosophical position.

3.1.5 The M-realist's Picture

As it turns out, making all the containments potentially proper yields the picture favoured by McDowell's so-called M-Realist.[12]

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fore, when denying a universal claim he should be prepared to provide a constructive counterexample.' All that is required for the denial of a universal claim is a constructive *reductio ad absurdum* of the universal claim itself. Such a *reductio* might well proceed by making *several* instantiations of the universal claim, whose results, collectively, lead (constructively!) to absurdity. This would suffice for the constructive *reductio* of the universal claim, without any one of those instantiations being identifiable as 'the' false one. In the same way, the intuitionist can deny a conjunction $(A \wedge B)$ without being committed either to $\neg A$ or to $\neg B$ (that is, without being committed to $(\neg A \vee \neg B)$). One can show that two propositions contradict one another without being justified in denying either of them. To be sure, in order to justify an *existential* claim, the intuitionist is required to be able to provide a constructive instance; but positive existentials are *not* straightforward duals of negative universals. The quoted objection from our imaginary objector is therefore based on a thoroughgoing and elementary misunderstanding. This misunderstanding of the intuitionist's obligations in respect of a denied universal no doubt stems from the confusion, easy for a classicist, of $\neg\forall x F(x)$ with $\exists x \neg F(x)$. But there, precisely, lies the rub.

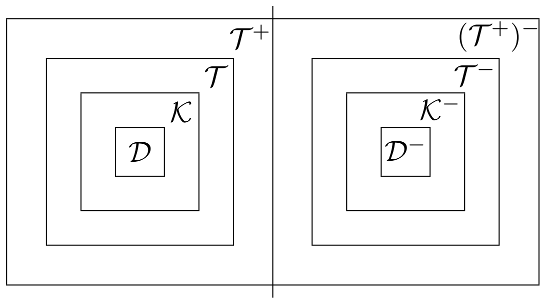
[9] See Brouwer [1912].

[10] See Tarski et al. [1953].

[11] If Δ is any arithmetical theory consistent with Q, then Δ is undecidable; whence one can deny Bivalence for the discourse of arithmetic. We are not here making the mistake of identifying knowable arithmetical truths with the theorems of any particular formal system.

[12] See McDowell [1976].

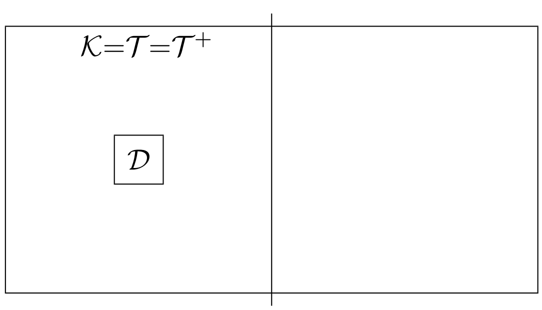
The M-Realist's picture:



The M-Realist wishes to countenance as coherent a combination of intuitionistic logic (with its refusal to grant Bivalence— $\mathcal{T} \sqsubset \mathcal{T}^+$) with 'a conception of truth conditions as possibly obtaining, or not, quite independently of the availability of appropriate evidence' ($\mathcal{K} \sqsubset \mathcal{T}$). Such a position, he maintains, would be 'essentially realist'. So he is challenging the claim that realism entails the acceptance of Bivalence. He holds to Knowledge- or Recognition-Transcendence, but not Bivalence. (In Tennant [1997] we considered this position more closely, and argued that one should reject it.)

From now on, we can ignore the right half of each picture, since it is obtained by reflecting the left half. The remaining philosophical positions are as follows.

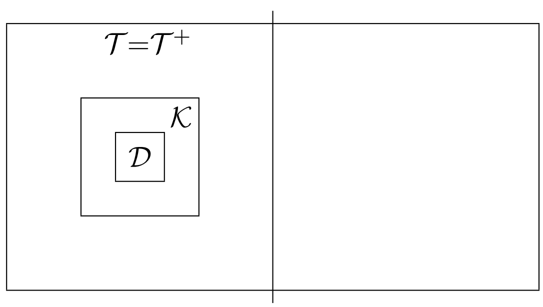
3.1.6 The Gödelian Optimist's picture



Gödelian Optimism consists in the view that truth is both bivalent and knowable. Our equation $\mathcal{T}$=$\mathcal{T}^+$ expresses bivalence. Bivalence is in effect the claim that the truths form a logically complete theory. The equation just given is a handy way of saying this. It claims that the inclusion $\mathcal{T} \subseteq \mathcal{T}^+$ is not a proper inclusion—that is, that the set $\mathcal{T}$ of truths is identical to (any of) its completion(s) $\mathcal{T}^+$. Likewise, our equation $\mathcal{K}=\mathcal{T}$ expresses the principle of knowability (that every truth is knowable). According to the Gödelian Optimist, whatever it is that makes for the (classical, bivalent) truth of a given sentence, it could in principle yield to our intellectual insights.[13] These insights might have to await further intellectual developments—discoveries of as yet unknown proofs, and formulation of as yet unthought-of methods of proof—but they could not remain, in principle, forever beyond human appreciation. That is not to say that we would ever have at our disposal a system or set of methods that would generate all the truths—for that, as we know from the essential incompleteness of arithmetic, would be impossible. The Gödelian Optimist is saying only that each truth should be knowable by some means or other; not that there is some means by which every truth can be known.

3.1.7 The Orthodox Realist's Picture

The Orthodox Realist, by contrast, is gloomier about the prospect of ever being able to establish any given truth that is deep or elusive enough. Not only does she concede that no one system could encompass all the truth; she believes, unlike her more sanguine Gödelian colleague, that some truths might never fall prey to any method of detecting them—they might escape the compass of every system. Her picture is as follows.



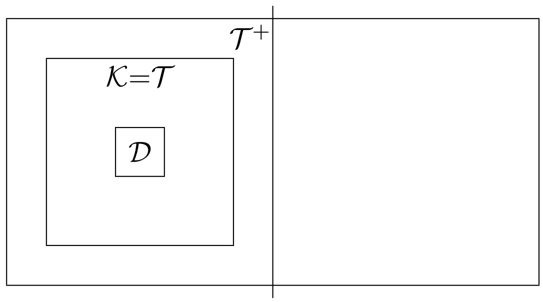
For the Orthodox Realist, every sentence is determinately true or false (i.e. Bivalence holds; in the picture, $\mathcal{T}=\mathcal{T}^+$) and there are truths that are in

[13] The term 'Optimism' is taken from Shapiro [1993], at p. 283. Shapiro calls a proposition *absolutely decidable* if 'either there is a rationally compelling argument establishing [it] or a rationally compelling argument refuting [it]'; and he defines *optimism* as 'the belief that every sentence of every unambiguous mathematical theory, or at least, say, arithmetic and analysis, is absolutely decidable.' Note that he is not assuming that the means of constructing all rationally compelling arguments are captured in any one formal system.

principle unknowable (Knowledge-Transcendence; in the picture, $\mathcal{K} \subset \mathcal{T}$).

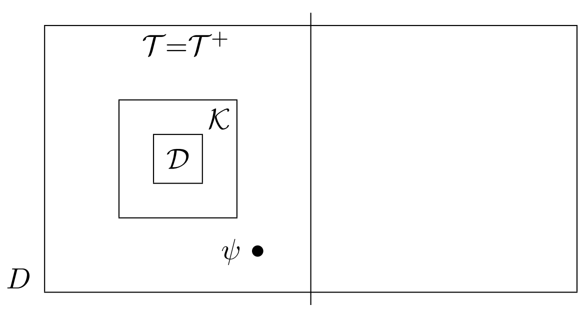
3.1.8 The Moderate Anti-Realist's Picture

The Moderate Anti-Realist opposes both of these views. For him, all truth is knowable ($\mathcal{K}$=$\mathcal{T}$); and he denies Bivalence ($\mathcal{T} \subset \mathcal{T}^+$). His picture is therefore as follows.



4 Orthodox Realism's classificatory problem with allegedly verification-transcendent truths

The Orthodox Realist, then, believes there might be verification-transcendent truths. Let us consider this claim with regard to the pure language of arithmetic, interpreted in the domain of natural numbers. The realist's claim implies, *prima facie*, that not all truths expressible in this language are *a priori*. So for the Orthodox Realist, even in a discourse D (such as arithmetic) where all the standard methods of justification are *a priori*, there is the threat of an abrupt divide. The *knowable* truths in D would be *a priori*. But what about any *un*knowable truth in D (call it ψ)?:



All of ψ's presumed *ways of being true* supposedly transcend human grasp.

Could the Orthodox Realist nevertheless be able to make sense of ψ's being *a priori*, even though *un*knowable?

But this way of pressing the matter might be too swift, resting, as it does, on the presumption that the Orthodox Realist *would* wish to make sense of ψ's being *a priori*. The aforementioned 'threat of an abrupt divide' might be one that the Orthodox Realist would simply be happy to accept.[14] For, given the definition of apriority with which we began, her quietist response might just be to say 'Sure, ψ is not an *a priori* truth, because ψ is unknowable. Aprioricity is an *epistemic* notion, remember! A truth must at least be know*able* in order to qualify as *a priori*.'

By the rules of our philosophical semantics, such quietism might be in order. But there remains the nagging worry that something about apriority will elude the Orthodox Realist. Note that ψ here is just an existential parameter for discussion of her position. It is not as though she has her hands on an actual sentence of the language of arithmetic, and is *naming* it as 'ψ' for short, for the convenience of discussion. Rather, when she uses that symbol she is speaking generally of any sentence (of arithmetic) that is true but unknowably so. She is claiming only that there is some such sentence; she is not offering any particular sentence as an example. For how could she?—she would have to be prepared to defend her claim that it *is true*, yet *unknowable*. Any successful defence of the claim that the particular sentence in question is true would immediately reduce to absurdity the claim that its truth is unknowable.

So the Orthodox Realist must be participating in this debate only at the more detached level of engagement at which she is asserting the possible existence of a sentence (*But I know not which sentence, in particular ...* she will say) that is true but unknowably so. And we are both agreeing to call such a sentence ψ for the purposes of our ongoing discussion.

For all the Orthodox Realist knows, we might already have such a sentence on our hands, in the form, say, of Goldbach's Conjecture; or, if that strikes the reader as a little too far-fetched, in the form of Riemann's Hypothesis; or the Continuum Hypothesis. The reader and the Orthodox Realist are now being asked (by the present author) to imagine that some sentence (ψ^*, say) formulating a conjecture currently under active investigation just happens to be 'such a sentence ψ' as she, the realist, thinks is a possibility. Keep in mind, though: the realist herself has no idea at all that ψ^* would happen to bear her out!

Suppose, then, that one were to suggest to the realist some radically

[14]Thanks to an anonymous referee for this suggestion.

non-a priori method of setting about discovering the truth of ψ^*. We might be able, for example, to appeal to ψ^* as an extra premise when making a testable prediction of an advanced empirical theory. The suggestion that one could proceed in such a way *would immediately jar with the realist's intuitions*, *qua* mathematician, as to what the appropriate methods would be for settling the truth-value of a conjecture such as ψ^*. For the realist is, after all, a fellow mathematician, looking at a sentence in the (pure) language of arithmetic (or complex analysis, or set theory) and casting about for an *a priori* (albeit possibly non-constructive) *proof that* ψ^*. In this regard she is just like the anti-realist or intuitionist mathematician, who would be casting about for an *a priori* (and constructive) *proof that* ψ^*.

Our question now takes the form: how can the Orthodox Realist explain and justify this 'methodological reaction' on her part to ψ^*?

Perhaps the Orthodox Realist could make sense of this after all, if one were to allow her to offer the following counterfactual conditional with (supposedly) impossible antecedent:

Γ_K: If it were possible to establish that ψ is true (in D), then one's means of doing so (one's grounds$_K$) would not involve any essential recourse or appeal to sensory or perceptual experience.

The Orthodox Realist would then be saddled with the need to articulate a semantics for counterfactuals with *impossible* antecedents. If the Orthodox Realist shrinks from this task, there could only be the following alternative way to make out the elusive apriority of ψ. Compare (Γ_K) with the corresponding claim concerning grounds$_T$:

Γ_T: If ψ is true (in D) but verification-transcendent, then φ's (ungraspable) grounds$_T$ do not involve any essential recourse or appeal to sensory or perceptual experience.

The latter claim seems intuitively clearer, and it would seem to allow the Orthodox Realist to say that all the truths of the discourse D are indeed *a priori*, even if some of them turn out to be unknowable. If the Orthodox Realist is not entitled to some sort of claim like this, then she will be saddled with an embarrassingly abrupt divide. Theorems of arithmetic will be *a priori* true; but not a single true-but-unprovable sentence of arithmetic will be able to count as *a priori* true.[15]

In order to avoid this rather embarrassing discontinuity, then, among the truths of arithmetic, the Orthodox Realist either needs to be able to

[15] This possibility appears to escape notice by Peacocke [1993], at p. 180.

make sense of counterfactuals, like (Γ_K), with impossible antecedents; *or* make sense of, and find a good argument for, the claim (Γ_T) concerning grounds$_T$. This latter task will involve, *inter alia*, arguing that the ω-rule is (in an appropriately extended sense) an *a priori* method of constructing, not finitary proofs, but infinitary arithmetical *truthmakers*, which no human mind could ever grasp. For no human mind could ever verify, of any 'application' of the ω-rule, that it was correct.

The Moderate Anti-Realist does not share this burden of the Orthodox Realist. For the Moderate Anti-Realist (and for the Gödelian Optimist, for that matter), all truths are *knowable* truths. So there is no embarrassing divide, within the realm of truths of arithmetic, say, between those that are unproblematically *a priori*, and those (the Orthodox Realist's supposedly verification-transcendent ones) that would appear to be extremely difficult to make out to be so.

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