

# RESEARCH ARTICLE

Mathematika

# Dimension and measure of sums of planar sets and curves

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## Abstract

Considerable attention has been given to the study of the arithmetic sum of two planar sets. We focus on understanding the measure and dimension of  $A + \Gamma := \{a + v : a \in A, v \in \Gamma\}$  when  $A \subset \mathbb{R}^2$  and  $\Gamma$  is a piecewise  $C^2$  curve. Assuming  $\Gamma$  has non-vanishing curvature, we verify that:

- (a): if  $\dim_H A \leq 1$ , then  $\dim_H(A + \Gamma) = \dim_H A + 1$ ;
- (b): if  $\dim_H A > 1$ , then  $\mathcal{L}_2(A + \Gamma) > 0$ ;
- (c): if  $\dim_H A = 1$  and  $\mathcal{H}^1(A) < \infty$ , then  $\mathcal{L}_2(A + \Gamma) = 0$  if and only if  $A$  is an irregular (purely unrectifiable) 1-set.

In this article, we develop an approach using nonlinear projection theory which gives new proofs of (a) and (b) and the first proof of (c). Item (c) has a number of consequences: if a circle is thrown randomly on the plane, it will almost surely not intersect the four corner Cantor set. Moreover, the pinned distance set of an irregular 1-set has 1-dimensional Lebesgue measure equal to zero at almost every pin  $t \in \mathbb{R}^2$ .

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