REVIEWS

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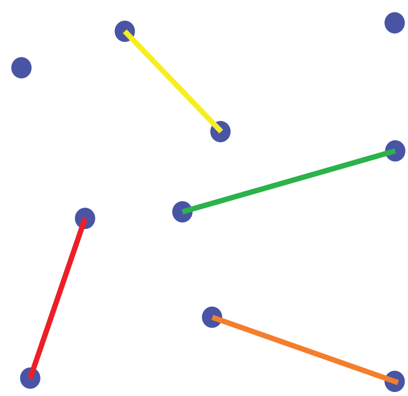
The Finite Field Distance Problem. By David Covert, MAA Press, Washington DC, 2021. 181 pp., ISBN 978-1-4704-6031-0, $65.00.

Reviewed by **Krystal Taylor**

Distance problems are incredibly simple to describe, and yet their study quickly opens up a world of mathematical ideas and questions for further research. This makes the study of distance problems an excellent entry point for students interested in starting research in mathematics, as well as a launching point for experienced mathematicians interested in learning more about the connections between combinatorics, number theory, harmonic analysis, and geometric measure theory. Dave Covert's book, *The Finite Field Distance Problem*, offers a well-written introduction to finite field analogues of some classic Euclidean problems. Before diving into the details in the setting of finite fields, we consider the classic Erdős distance problem.



This infamous problem inquires about the minimal number of distances generated by n points in the plane. The aim here is to arrange points in such a way as to maximize repeated distances and, in turn, minimize distinct distances. For example, if you were to draw three points at random on a sheet of paper and to record the distances between them, chances are that you would end up with three distinct values. It is possible, however, to choose three points in such a way that one records only one value; namely, take the vertices of an equilateral triangle. More generally, if $f(n)$ denotes the minimum number of distinct distances generated by n points in the plane, then $f(4) = 2$ (consider the vertices of a square), $f(5) = 2$ (consider the vertices of a regular pentagon), and so on. However, as one starts to try examples it becomes clear that the complexity of computing $f(n)$ increases rapidly even for small values of n. The Erdős distance conjecture, which was solved up to a log factor by Guth and Katz in [**12**], states that

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$f(n) = \Omega(n/\sqrt{\log n})$. As we will see, one can ask variants of this distance problem in both a continuous setting and in finite fields, many of which remain open.

A related problem, and one of the leading open questions in discrete geometry, is that of the Erdős unit-distance problem. Among sets of n points in the plane, it asks for the maximal number of pairs of points at a distance 1 apart. More formally, define

$$g(n) = \max_{E \subset \mathbb{R}^2,\ |E|=n} |\{(x, y) \in E \times E : \|x - y\| = 1\}|,$$

where $|\cdot|$ is used to denote cardinality. Despite many years of effort, the best upper and lower bounds on $g(n)$ are still very far apart. In 1946, Erdős constructed an example of a configuration of n points generating $n^{1+c/\log\log n}$ distances for some constant c [5]. Over 75 years later, this remains the current best lower bound for $g(n)$. The best upper bound on $g(n)$ known to date is due to Spencer, Szemeredi, and Trotter and uses an interesting combinatorial proof based on incidences between points and circles. It establishes that $g(n) = O(n^{4/3})$ [18]. In summary, for some constants $C, c > 0$, we have

$$n^{1+c/\log\log n} \leq g(n) \leq Cn^{4/3}.$$

Further progress to close the gap between the upper and lower bounds would indeed be an achievement and, at the moment, remains an open problem. Note that variants of Erdős' unit distance problem, including generalizations to higher dimensions and analogues for chains of distances, have been investigated by a number of authors (see, for instance, [8, 17] and the references therein).

Upper bounds for the Erdős unit-distance problem immediately yield lower bounds for the Erdős (distinct) distance problem. The basic idea here is that if no single distance occurs too often, then many distinct distances are guaranteed. In particular, given a set E consisting of n points in the plane, define the *distance set* of E by

$$\Delta(E) = \{\|x - y\| : x, y \in E\}.$$

Partitioning the pairs of points in $E \times E$ according to the distance between them, we can write

$$E \times E = \bigcup_{t \in \Delta(E)} \{(x, y) \in E \times E : \|x - y\| = t\}. \tag{1}$$

Since $E \times E$ consists of n^2 points, it follows by considering the cardinality of both sides of (1) that

$$n^2 = \sum_{t \in \Delta(E)} |\{(x, y) \in E \times E : \|x - y\| = t\}|\,.$$

If one were to prove, say, that the maximal number of pairs of points at a distance t apart was bounded by $Cn^{1+\epsilon}$ for some $\epsilon > 0$, re-arranging the equation above would immediately yield that

$$n^{1-\epsilon} \leq C\,|\Delta(E)|\,.$$

What happens if we ask similar questions, except instead of considering n points we consider infinitely many points? For example, how *large* does a set need to be to guarantee that it generates *many* distinct distances? While the words "large" and "many" are open to interpretation, Hausdorff dimension and Lebesgue measure are useful notions of size and structure. Roughly speaking, Lebesgue measure formalizes our notions of length, area, and volume, while Hausdorff dimension extends the usual understanding of dimension to a fractal setting.

With this in mind, the celebrated Falconer Distance Conjecture provides a version of the Erdős distance problem in the infinite setting. It states that if $E \subset \mathbb{R}^d$ is a compact set of Hausdorff dimension exceeding $\frac{d}{2}$, then the distance set of E, defined by $\Delta(E) = \{\|x - y\| : x, y \in E\}$ has positive Lebesgue measure. Falconer proved that $\frac{d+1}{2}$ suffices and provided a lattice-like example to demonstrate that $\frac{d}{2}$ is necessary [6]. His proof for $\frac{d+1}{2}$ essentially comes down to recasting the ideas following the partition in (1) above in a measure-theoretic sense. Since the appearance of Falconer's result in 1985, there has been a flurry of activity around this problem using a variety of techniques. The cutting edge result for the Falconer conjecture in the plane utilizes the decoupling theorem and is due to Guth, Iosevich, Ou, and Wang [11]. They prove that if $E \subset \mathbb{R}^2$ is a compact set of Hausdorff dimension greater than $5/4$, then the distance set, $\Delta(E)$, has positive Lebesgue measure. Their paper also includes a summary of other benchmark results along with a report on what is known in higher dimensions. There is still much progress to be made, and it seems that new methods will be needed to push the threshold to $\frac{d}{2}$.

With a little creativity, we can find a multitude of further research questions inspired by those considered thus far. In the spirit of the Erdős unit distance conjecture, rather than considering pairs of points a fixed distance apart, one might investigate the existence of triples of points forming the vertices of an equilateral triangle. This is the focus of Iosevich and Liu's work [13], in which the authors find sufficient conditions on a subset of $\mathbb{R}^d$ to guarantee that it contains the vertices of an equilateral triangle. It is a very active area of research to investigate minimal size conditions required of a set E which guarantee that E contains an affine copy of some fixed k point configuration.

In another direction, in addition to considering properties of the distance set, $\Delta(E)$, such as its Lebesgue measure or interior, one may study properties of k-chains or paths of distances within E:

$$\Delta_k(E) = \{(|x_1 - x_2|, |x_2 - x_3|, \ldots, |x_{k-1} - x_k|) : x_i \in E \text{ are distinct }\}.$$

In [1], the authors use Fourier analytic methods to show that if $E \subset \mathbb{R}^d$ with Hausdorff dimension greater than $\frac{d+1}{2}$, then $\Delta_k(E)$ has nonempty interior. A similar result, in which the Hausdorff dimension is replaced by an alternative notion of size with roots in dynamics known as Newhouse thickness, is attained in [15]. Further, in [16], the authors find sufficient conditions on a set $E \subset \mathbb{R}^d$ to guarantee that the triangle-distance set, $\{|x - y|, |y - z|, |x - z| : x, y, z \in E\}$, has nonempty interior. See also [9] and [10], where the authors consider the interior of distance sets corresponding to more general configuration sets.

In his new book, *The Finite Field Distance Problem*, David Covert explores problems analogous to those we have been discussing, but changes the setting to vector spaces over finite fields. As Covert explains, "finite fields have long been used as an uncomplicated setting in which one can play with Euclidean problems in an environment with fewer technical difficulties." The core problem is this: Let q be a prime or a power of an odd prime, and let $\mathbb{F}_q^d$ denote the d-dimensional vector space over the finite field with q elements. For $x = (x_1, x_2, \ldots, x_d) \in \mathbb{F}_q^d$, set $\|x\| = x_1^2 + x_2^2 + \cdots + x_d^d \in \mathbb{F}_q$. For different sets $E \subset \mathbb{F}_q$, we are then interested in computing the size of the distance set of E, defined by

$$\Delta(E) = \{\|x - y\| : x, y \in E\} \subset \mathbb{F}_q.$$

When $d \geq 4$ is even, it is conjectured that there exists a constant $C > 0$ independent of q so that, if $E \subset \mathbb{F}_q^d$ satisfies $|E| \geq Cq^{d/2}$, then $\Delta(E) = \mathbb{F}_q$. When $d = 2$, it is conjectured that there exists a constant $c \in (0, 1]$ independent of q so that, if $E \subset \mathbb{F}_q^2$ satisfies $|E| \geq Cq$, then $|\Delta(E)| \geq cq$.

One must be careful in the finite field setting, however, as seemingly paradoxical things can happen. For instance, an example found in Covert's book points out that if $q \equiv 1 \pmod 4$, then there exists an element $i \in \mathbb{F}_q$ such that $i^2 = -1$. Setting $E = \{(x, ix) : x \in \mathbb{F}_q\}$, we have constructed a set $E \subset \mathbb{F}_q^2$ satisfying $|E| = q$ and $\Delta(E) = \{0\}$. This example points out the necessity of the *large constant* C in the assumption of the conjecture when $d = 2$, and demonstrates that the conjecture fails if $C = 1$.

The finite fields distance problem is fairly new. While progress has been made, the conjecture remains tantalizingly unproven. The first result on the finite fields distance conjecture appeared in 2004 and is due to Bourgain, Katz, and Tao [2]. The best known result to date is due to Iosevich and Rudnev [14], who show that $|E| > 2q^{\frac{d+1}{2}}$ suffices to guarantee that $|\Delta(E)| = \mathbb{F}_q$. Covert's book delves deeper into what is known, and it includes a self-contained proof of Iosevich and Rudnev's result along with a simpler warm-up proof, examples, further developments, and a discussion of the intertwining between classic results in the Euclidean setting and finite field analogues.

In addition to offering an in-depth discussion of these and other distance problems, *The Finite Field Distance Problem* includes formulations of related important problems in combinatorics and number theory. Among my favorite parts of the book, the final chapter opens up a discussion of the sum-product problem and the Kakeya conjecture with a focus on finite fields.

In brief, the sum-product conjecture says that either the sum or the product of a finite set of integers is large. More formally, for $A \subset \mathbb{Z}$, define the sum and product sets respectively by

$$A + A = \{a + b : a, b \in A\} \quad \text{and} \quad A \cdot A = \{a \cdot b : a, b \in A\}.$$

Formally, the sum-product conjecture, posed by Erdős and Szemeredi [4], states that if $A \subset \mathbb{Z}$ with $|A| = k$, then for all $\epsilon > 0$,

$$\max\{|A + A|, |A \cdot A|\} \geq c(\epsilon)k^{2-\epsilon}.$$

While the original conjecture was posed over the integers, it is also believed to hold over real numbers, where the conjecture states: For any set $A \subset \mathbb{R}$,

$$\max\{|A + A|, |A \cdot A|\} \geq |A|^{2-o(1)}.$$

As examples, if $A = \{a, a + d, \ldots, a + (k - 1)d)\}$ is an arithmetic progression of length k, then it is known that $|A \cdot A|$ is of order k^2 and it is not hard to see that $|A + A| = 2k - 1$. On the other hand, if $A = \{a, ar, ar^2, \ldots, ar^{k-1}\}$ is a geometric progression, then $|A \cdot A| = 2k - 1$, while $|A + A| = \binom{k+1}{2}$. In either example, either the sum or product is large. Through an unexpectedly simple proof utilizing number theory and incidence geometry, Elekes [3] proved that if $A \subset \mathbb{R}$ with $|A| = k$, then $|A + A||A \cdot A| \gg k^{5/2}$, from which it follows that $\max\{|A + A|, |A \cdot A|\} \gg k^{5/4}$. His proof makes clever use of the work of Szemeredi and Trotter, mentioned above, on the number of incidences between points and lines. Further improvements on the sum-product conjecture have relied on incidence theory ever since. As discussed in Covert's book, the sum-product conjecture in finite fields possesses some idiosyncrasies not present in the Euclidean setting. For instance, if $A \subset \mathbb{F}_q$ is a subring, then $A + A = A \cdot A = A$. There are several methods, such as imposing conditions on the cardinality of A, to avoid the situation when A is a subring.

The Kakeya needle problem is one of the outstanding problems in analysis and geometric measure theory. It has roots in the study of Riemann integration and its related constructions have served as important counterexamples in harmonic analysis. The problem itself has an interesting history [7]. In 1919, Besicovitch constructed a compact subset of $\mathbb{R}^2$ with planar Lebesgue measure zero containing a unit line segment in every direction. Such a set came to be known as a Besicovitch set. Around the same time, Kakeya posed the problem of finding the area of the smallest convex set in which a unit line segment could be rotated 180° using continuous motions. It turned out that Besicovitch's construction could be modified to resolve Kakeya's problem. This, however, was far from the end of the story. The Kakeya set conjecture claims that a Besicovitch set in $\mathbb{R}^n$ must have Hausdorff dimension n. In 1971, Davies affirmed the conjecture for $n = 2$. While some breakthrough progress has been made, the conjecture remains open in dimensions 3 and higher. Interestingly, there is also a finite field analogue of the Kakeya conjecture, which was posed by Wolff in 1999 and, as discussed in Covert's book, solved by Dvir in 2009. It is not clear, however, if the techniques used for solving the conjecture in the finite field setting can be carried over to the Euclidean case. Annotated details of Dvir's proof can be found in Covert's book, along with the challenge to think more deeply about the connection between the finite field and Euclidean versions of the Kakeya set conjecture.

Through a deeper investigation of the problems described above, *The Finite Field Distance Problem* serves as an invitation to students and mathematicians alike interested in gaining a deeper understanding of distance problems in research mathematics. While the playful narrative and self-contained background make the book accessible to advanced undergraduates and graduate students, this book has plenty to offer professional mathematicians. The book serves as an excellent reference of breakthroughs in the field and includes many informative examples. Difficult material is made accessible with intuitive descriptions, well-illustrated figures, and curated equations.

In my experience of working with graduate students and advanced undergraduates, I've often heard the question of *how* exactly one gets started in the ocean of mathematical research. There can be a disconnect between what is learned in the classroom and the process of posing questions and navigating new directions. While there is plenty of general advice one can give, such as finding a mentor or reading seminal papers in a chosen field, each mathematician finds their own path. To that end, Covert's book provides an excellent aid in understanding the process of getting started in mathematical research and serves as a guide to the intriguing world of distance problems.

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100 Years Ago This Month in *The American Mathematical Monthly*
Edited by Vadim Ponomarenko

A mathematical meeting and dinner in honor of Professor Charlotte Angas Scott, on the completion of her thirty-seventh year as head of the Department of Mathematics in Bryn Mawr College, was held April 18, 1922, by Professor Scott's former students. The exercises consisted of an address of welcome by President M. Carey Thomas, an introductory address by Miss Marion Reilly, 1901, and a lecture by Professor A. N. Whitehead, professor of applied mathematics in the Imperial College of Science, South Kensington, on "Relativity and gravitation, Group tensors and their application to the formulation of physical laws." After the lecture a tea was served at the deanery to about 200 guests.

At the dinner there were present former students, members of the American Mathematical Society, and members of the Bryn Mawr College faculty. [...] In regard to Professor Scott's service to Bryn Mawr College, Professor Bascom said in part, "It is this wisdom impartial, rational, creative, articulate, that Dr. Scott possesses in a marked degree. This is the quality which makes her judgment the one sought on all important faculty matters."

—Excerpted from "Notes and News" (1922). 29(7): 275–280.

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