Theorem. There is no positive integer $N > 1$ such that the sum

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N}$$

is an integer.

- Key idea: We will use a proof by contradiction. We will use the fact that if $\frac{a}{b} = \frac{c}{d}$, and b is even, then d must be even. We will also multiply both sides of the equation by 2^{k-1} where k is the largest power of 2 that appears among the denominators $\{1, 2, 3, \ldots, N\}$.

PROOF Proof: Let $N \in \mathbb{N}$. Set

$$S := \{k \in \mathbb{N} : \exists\, j \in \{2, 4, \ldots, N\} \text{ so that } 2^k | j\}.$$

In other words, S is the collection of powers of 2 that divide some integer between 1 and N. In order to use the Well-Ordering-Principle, we verify the hypotheses that the set S is non-empty and that the set S is bounded above.
$S \neq \emptyset$: Notice that $1 \in S$ because $N \geq 2$ implies that $2 \in \{2, 4, \ldots, N\}$, and $2^1 | 2$.
S is bounded above: Observe that $N < 2^N$. It follows that $\forall k \in S$, $k \leq N$. (If you don't like this, then think of another proof that S is bounded above...sometimes it is easier to just write your own proof then to understand one).
It follows by the Well-Ordering-Principle that the set S has a maximal element. Call it K.

By way of contradiction, assume there exists and integer $M \in \mathbb{N}$ so that

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N} = M. \tag{1}$$

Multiply both sides of equation (1) by 2^{K-1} to get:

$$2^{K-1} + \frac{2^{K-1}}{2} + \frac{2^{K-1}}{3} + \cdots + \frac{2^{K-1}}{N} = 2^{K-1}M.$$

Simplify to get an expression of the form:

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_j}{b_j} + \frac{1}{2} + \frac{a_{j+1}}{b_{j+1}} + \cdots + \frac{a_{N-1}}{b_{N-1}} = \frac{a_N}{b_N},$$

where $\{b_1, \ldots, b_N\}$ are all odd numbers. Solve for $\frac{1}{2}$, simplify both sides, and use the fact that if $\frac{a}{b} = \frac{c}{d}$, and b is even, then d must be even to get a contradiction.

HYPOTHESES For $N > 1$, let $P(N)$ be the statement that $1 + \frac{1}{2} + \cdots + \frac{1}{N}$ is not an integer. The theorem claims that if $N > 1$, then $P(N)$ is true. Notice this is necessary, for if $N = 0$ or $N = 1$, the statement $P(0)$ is not defined, and the statement $P(1)$ is not true.

CONNECTIONS This Theorem is similar to HW 5 number 6, although a different method is used in both. The well-ordering-principle is also used to prove the division algorithm.

INTERPRET Even though the sequence of partial sums of the Harmonic series diverges to infinity, the Theorem states the no partial sum will land on an integer.

NOTES The proof of this Theorem is easy to understand if one starts with a few examples. Start for instance with $N = 9$. In this case, 3 is the largest power of 2 that divides one of the denominators of $\{1, \frac{1}{2}, \frac{1}{3}, \cdots \frac{1}{9}\}$. In particular, $2^3|8$. In this case, will multiply both sides of the equation (**??**) by 2^2.