Binomial Theorem:

Let a, and b be real numbers. Then for each $n \in \mathbb{N} \cup \{0\}$

$$(a+b)^n = \sum_{k=0}^{n} a^{n-k} \cdot b^k \binom{n}{k}$$

Proof: We prove the Theorem using induction.

Let $a, b \in \mathbb{R}$, and let $n \in \mathbb{N} \cup \{0\}$.

~~Base Case:~~

Let $P(n)$ be the statement

$$(a+b)^n = \sum_{k=0}^{n} a^{n-k} \cdot b^k.$$

Base Case: Certainly, $(a+b)^0 = \sum_{k=0}^{0} a^{0-k} \cdot b^0 \binom{0}{k}$ because $(a+b)^0 = 1$ and $\sum_{k=0}^{0} a^{0-k} \cdot b^0 \binom{0}{k} = 1$.

This shows $P(0)$ is true.

Inductive step: Let $n \in \mathbb{N} \cup \{0\}$ so that $P(n)$ is true. We wish to show $P(n+1)$ is true too.

Now P(n+1) is the sentence

$$\sum_{k=0}^{(n+1)} a^{n-k} \cdot b^{k} = (a+b)^{n+1}$$

Write

$$(a+b)^{n+1} = (a+b)(a+b)^{n}$$

use inductive hypothesis

$$= (a+b) \sum_{k=0}^{n} \binom{n}{k} a^{n-k} \cdot b^{k}$$

$$= a \sum_{k=0}^{n} \binom{n}{k} a^{n-k} \cdot b^{k} + b \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} a^{(n+1)-k} \cdot b^{k} + \sum_{k=0}^{n} \binom{n}{k} a^{n-k} \cdot b^{k+}$$

$$= \binom{n}{0} a^{(n+1)} + \sum_{k=1}^{n} \binom{n}{k} a^{(n+1)-k} \cdot b^{k} + \sum_{k=0}^{n}$$

$$= \binom{n}{0} a^{n+1} + \sum_{k=1}^{(n+1)} \binom{n}{k} a^{(n+1)-k} \cdot b^{k} - \binom{n}{}$$

$$= \underbrace{\binom{n}{0} a^{(n+1)} \cdot b^0}_{\text{I}} + \underbrace{\binom{n}{1} a^n \cdot b^1} + \cdots + \underbrace{\binom{n}{n} a^1 \cdot b^n}$$

$$+ \underbrace{\binom{n}{0} a^n \cdot b} + \binom{n}{1} a^{n-1} \cdot b^2 + \cdots + \underbrace{\binom{n}{n} a^0 \cdot b^{n+1}}_{\text{II}}$$

add to n+1

$$= \underbrace{\binom{n}{0} a^{n+1} \cdot b^0}_{\text{I}}$$

add to (n+1)

$$+ \left[\left(\binom{n}{0} + \binom{n}{1} \right) a^n b + \left(\left(\binom{n}{1} + \binom{n}{2} \right) a^{n-1} b^2 \right) + \cdots + \left(\binom{n}{n-1} + \binom{n}{n} \right) a^1 b^n \right]$$

$$+ \underbrace{\binom{n}{n} a^0 b^{n+1}}_{\text{II}}$$

- Use recursion relation: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$
- Use $\binom{n}{n} = 1 = \binom{n+1}{n+1}$
- Use $\binom{n}{0} = 1 = \binom{n+1}{0}$

$$= \binom{n+1}{0} a^{n+1} \cdot b$$

$$+ \left[\binom{n+1}{1} a^n \cdot b + \binom{n+1}{2} a^{n-1} b^2 + \cdots + \binom{n+1}{n} a^1 b^n \right]$$

$$+ \binom{n+1}{n+1} a^0 b^{n+1}$$

$$= \boxed{\sum_{k=0}^{(n+1)} \binom{n+1}{k} a^{(n+1)-k} \cdot b^k}.$$

We conclude that $P(n+1)$ is true too.