Let (X,d) be a metric space. If (x_n) is a sequence in X, then to say that (x_n) *is Cauchy* means that for each $\varepsilon > 0$, there exists $N \in \mathbf{N}$ such that for all $m, n \in \mathbf{N}$ with $m, n > N$, we have $d(x_m, x_n) < \varepsilon$. It is easy to see that if (x_n) is a convergent sequence in X, then (x_n) is Cauchy. To say that (X,d) *is complete* (as a metric space) means that each Cauchy sequence in X is convergent in X. Thus in a complete metric space, a sequence is convergent if and only if it is Cauchy.

As we know, each non-empty subset of $\mathbf{R}$ which is bounded above has a least upper bound in $\mathbf{R}$. (This property of $\mathbf{R}$ is sometimes called *Dedekind completeness.*) Using this fact, we shall soon show that $\mathbf{R}$ is also complete as a metric space, when we give it its usual metric, namely $d(x,y) = |x-y|$. For now, you may take it for granted that $\mathbf{R}$ is complete in this sense.

X4. [redacted], let X be a non-empty set, let (Y,d) be a metric space, let Z be the set of bounded functions from X into Y, and let D be the metric on Z defined by *due 2Th*

$$D(f,g) = \sup\{\, d\big(f(x), g(x)\big) : x \in X \,\}.$$

Prove that if the metric space (Y,d) is complete, then so is the metric space (Z,D).

Let (X,ρ) and (Y,σ) be metric spaces. An *isometry* from X into Y is a map $f\colon X \to Y$ such that $\sigma\big(f(x), f(x')\big) = \rho(x,x')$ for all $x, x' \in X$. Informally, an isometry is a map that preserves distances between points.

X5. Let (X,d) be a metric space. Give $\mathbf{R}$ its usual metric. Let Z be the set of bounded functions from X into $\mathbf{R}$. Give Z the metric defined by *due 2Th*

$$D(f,g) = \sup\{\, |f(\xi) - g(\xi)| : \xi \in X \,\}.$$

By problem X4, the metric space (Z,D) is complete, because $\mathbf{R}$ with its usual metric is complete as a metric space. Fix $x_0 \in X$. For each $x \in X$, define $f_x\colon X \to \mathbf{R}$ by

$$f_x(\xi) = d(\xi, x) - d(\xi, x_0).$$

(a) Prove that for each $x \in X$, we have $f_x \in Z$.
(b) Define $\Phi\colon X \to Z$ by $\Phi(x) = f_x$. Prove that Φ is an isometry from (X,d) into (Z,D).

Let (X,d) be a metric space and let $X_1 \subseteq X$. Let d_1 be the restriction of d to $X_1 \times X_1$. Then clearly d_1 is a metric on X_1. We call d_1 the *subspace metric* that X_1 inherits from (X,d) and we say that the metric space (X_1, d_1) is a *subspace* of the metric space (X,d).

Remark. The result of problem X5 shows that any metric space is isometric to a subspace of a complete metric space.

Let A and B be sets. To say that A *is equinumerous to* B means that there is a bijection[1] from A to B. Recall that ω denotes the set $\{0,1,2,\dots\}$. If $n \in \omega$, then to say that A *has* n *elements* means that either $n=0$ and A is empty or $n \in \mathbf{N}$ and A is equinumerous to $\{1,\dots,n\}$. To say that A *is finite* means that there exists $n \in \omega$ such that A has n elements. To say that A *is infinite* means that A is not finite. To say that A *is countably infinite* means that A is equinumerous to $\mathbf{N}$. To say that A *is countable* means that A is finite or countably infinite.[2] It is easy to show that if B is an infinite subset of $\mathbf{N}$, then B is equinumerous to $\mathbf{N}$. It follows that A is countable if and only if A is equinumerous to a subset of $\mathbf{N}$. To say that A *is uncountable* means that A is not countable. Cantor (1873) pointed out that the set of rational numbers is countable and proved that the set of real numbers is uncountable.

X6. Let X be a set. Prove that X contains a countably infinite subset if and only if X is equinumerous to a proper subset of itself. (Do not use the axiom of choice.)

[1] A bijection from A to B is a one-to-one map from A onto B. Another name for a bijection from A to B is a one-to-one correspondence between A and B.

[2] Warning: The definition of *countable* that I have given is the one accepted by most mathematicians, but you should watch out for the fact that Rudin uses the term *countable* to mean what I have chosen to call *countably infinite*. So for Rudin, a finite set is not countable. To me, that is just ridiculous, so I will not follow Rudin's use of the term *countable*.