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Currency Preferences, Purchasing Power Risks, and the Determination of Exchange Rates in an Optimizing Model

1. INTRODUCTION

MOST RECENT RESEARCH on the theory of exchange rate determination treats the exchange rate as the relative price of two assets: domestic money and foreign money. The value of an asset depends on the distribution of its return. However, most of the research has dealt with models in which there is no uncertainty. The present paper focuses on questions that cannot be addressed in a world of certainty, because in such a world an asset is never risky. It investigates how purchasing power *risks* affect the determination of exchange rates. This investigation is pursued in a fairly general optimizing model.

The present research is related to some recent work in international economics and in financial economics. It follows the approach of Obstfeld (1981) and Stockman (1980) in that it uses an optimizing model and takes into account the role of government transfers in the flow budget constraint of individuals. As in Calvo and Rodriguez (1977), Stockman (1980), and others, the fact that holdings of foreign monies are useful to domestic individuals is taken into account. Finally, the paper uses techniques developed in papers in the finance literature which build on Merton's (1973) asset-pricing model and use a concept of equilibrium developed in

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Lucas (1978), Cox, Ingersoll, and Ross (1978), Breeden (1979), and, especially, Jones (1980).

The plan of the paper and its main results are as follows. Section 2 presents the economic model: a two-country world in which individuals are infinitely lived and exhibit constant relative risk aversion. Cash balances are inputs in the production of consumption services. In section 3, the optimizing problem of individuals is discussed.

Section 4 presents results for the general version of the model. The ratio of foreign money holdings to domestic money holdings for a representative investor does not depend on his risk tolerance. If the concept of risk is properly defined, in equilibrium the more risky the purchasing power of the domestic currency, the larger the ratio of foreign money holdings to domestic money holdings. The exchange rate can be written as a function of the money supplies, the interest rates, and the consumption expenditures. Finally, the forward exchange rate can be written as a function of the variables that affect the exchange rate.

In section 4, the exchange rate is written as a function of variables that themselves depend on the state of the world. In section 5, the exchange rate is solved as a function of the state variables under a restrictive set of assumptions to show how, in an optimizing model, the distribution of the future spot exchange rate is a function of the distribution of the exogenous sources of uncertainty in the economy. It is shown that, ceteris paribus, the higher the covariance of the domestic money stock with the stock of capital, the lower the domestic price of the foreign currency. Section 6 offers some concluding remarks.

2. THE ECONOMIC MODEL

Consider a two-country model in which individuals hold cash balances to reduce the transaction costs they bear when they trade economic goods. These transaction costs consist only of costs associated with exchanging goods, and they need not be pecuniary costs. In an open economy, domestic residents trade not only among themselves but also with foreign residents. For the purpose of foreign trade, it is useful for domestic residents to hold cash balances denominated in foreign currencies.[1] (Of course, cash balances denominated in foreign currencies can also be used for trades among domestic residents.)

All domestic individuals maximize an intertemporal expected utility function of consumption services written as

$$E_t\left\{\int_t^{\infty} e^{-\rho\theta}\frac{1}{\gamma}S(\theta)^{\gamma}d\theta\right\}, \tag{1}$$

where γ is a fixed parameter, $-\infty < \gamma < 1$, S is the rate of consumption services,

[1]For discussion of this point, see Stockman (1980) and the references in his paper; see also Crystal (1977) and Girton and Roper (1981).

and E_t is the expectation operator conditional on information available at time t. All foreign individuals are assumed to have the expected utility function given by (1), except that their parameters ρ and γ may differ from those for domestic individuals.

At each point in time, individuals produce consumption services using the consumption good, which can be a composite commodity, and cash balances as inputs. This approach makes use of Becker's (1971) theory of household production and is similar to the approach used by Fama and Farber (1979), except that Fama and Farber assume that the domestic investors hold no foreign currencies. Furthermore, a specific functional form is used for the production function of consumption services. It is assumed to take the following form:

$$S = kQ^{\delta}[a(\Pi M)^{\beta} + b(\Pi^* M^*)^{\beta}]^{\varepsilon/\beta}, \tag{2}$$

where Q is the rate of purchase of the consumption good, M is cash balances denominated in domestic currency, and Π is the price of one unit of domestic money in terms of the consumption good. Throughout the paper, the consumption good is used as the numeraire. k, S, a, b, β, and ε are assumed to be constant parameters such that $a + b = 1$, $-\infty < \beta < 1$, $\varepsilon > 0$, and, to insure nonincreasing returns to scale, $0 < \delta + \varepsilon \leq 1$. Stars are used to designate foreign assets and prices. M^* represents foreign cash balances and Π^* is the price of one unit of foreign cash balances in terms of the consumption good. Purchasing power parity is assumed to always hold exactly. It follows that if e is the price in domestic currency of one unit of foreign currency, $\Pi^* = e\Pi$ holds all the time and the real value of foreign cash balances held by the investor, $\Pi e M^*$, is equal to $\Pi^* M^*$.

3. THE OPTIMIZATION PROBLEM OF INVESTORS

For the moment, the most general assumptions are used in describing the optimization problem of investors. More restrictive assumptions will be imposed later on.

The price Π of domestic cash balances and the price Π^* of foreign cash balances follow stochastic differential equations of the Ito type written as[2]

$$\frac{d\Pi}{\Pi} = \mu_{\Pi}\, dt + \sigma_{\Pi}\, dZ_{\Pi} \tag{3}$$

$$\frac{d\Pi^*}{\Pi^*} = \mu_{\Pi^*}\, dt + \sigma_{\Pi^*}\, dZ_{\Pi^*}. \tag{4}$$

μ_{Π} is the expected instantaneous rate of change of Π, and σ_{Π} is the instantaneous standard deviation of the rate of change of Π; dZ_{Π} is the increment of a standard Wiener process. Because of the assumption of purchasing power parity, (3) and (4) define implicitly the exchange rate dynamics. However, this does not mean that, in

[2]For an exposition of the methodology used here, see Merton (1971) or Fischer (1975).

some sense, price levels are exogenous whereas the exchange rate is endogenous. Both price levels and the exchange rate are functions of the state of the world. It must also be noted that in (3), for instance, μ_Π and σ_Π can depend on the state of the world. The dependence of the drift term and the standard deviation on the state of the world in equations like (3) holds for all stochastic differential equations used in this section. The state variables themselves are assumed to follow stochastic differential equations of the Ito type.

It is assumed that financial markets are frictionless: there are no transaction costs; all markets are always in equilibrium; perfect competition obtains on all markets; each investor has all of the available information; unlimited short-sales are allowed; there is no difference between the borrowing and the lending rate. For convenience, and without loss of generality, it is assumed that investors can hold an asset that has a nonstochastic instantaneous real rate of return, written r, and a domestic (foreign) bond that has a nonstochastic nominal rate of return, written R (R^*). If B is the price in domestic money of the domestic bond, the real instantaneous rate of return of that bond (using Ito's lemma) is

$$\frac{d(\Pi B)}{\Pi B} = (R + \mu_\Pi)\, dt + \sigma_\Pi\, dZ_\Pi\,. \tag{5}$$

There are N risky nominal assets. Using the consumption good as the numeraire, the price of a risky nominal asset is written I and evolves according to

$$\frac{dI_i}{I_i} = \mu_{I_i}\, dt + \sigma_{I_i}\, dZ_{I_i}\,. \tag{6}$$

Finally, investors receive transfers from the government (which can be positive or negative). Let T (T^*) be the *capitalized* value of the transfers for a domestic (foreign) resident. The dynamics for T can be written as

$$dT = \mu_T\, dt + \sigma_T\, dZ_T\,. \tag{7}$$

The change in investor wealth due to the existence of transfers is the sum of the change in the capitalized value of the transfers and the transfer itself, written $\tau\, dt + \sigma_\tau dZ_T$. It is assumed that each investor views transfers as unrelated to his holdings of cash balances.[3]

Define w_i to be the share of portfolio wealth $V = W - T$, that is, total wealth minus the capitalized value of transfers, invested in asset i, $0 < i \leq N$. w_M (w_{M^*}) is the share of the investor's portfolio wealth held in cash balances denominated in domestic (foreign) currency. w_B (w_{B^*}) is the share of the investor's portfolio wealth invested in the bond whose domestic (foreign) currency price is B (B^*). Finally, w_r is the fraction of the investor's wealth invested in the safe asset.

[3]Jones (1980) has a continuous-time model of a closed economy in which changes in the money supply occur through transfers.

The investor's portfolio must satisfy a stock budget constraint:

$$w_M + w_{M^*} + w_B + w_{B^*} + w_r + \sum_{i=1}^{N} w_i = 1 \tag{8}$$

The stock budget constraint can be used to eliminate w_r from the flow budget constraint, which can be written as

$$\begin{aligned} dW &= dT + \tau\, dt + \sigma_\tau dZ_\tau + dH + v(H - L)\, dt \\ &+ \sum_{i=1}^{N_t} w_i\left(\frac{dI_i}{I_i} - r\, dt\right)V + (w_M + w_B)\left(R\, dt + \frac{d\Pi}{\Pi} - r\, dt\right)V \\ &+ (w_{M^*} + w_{B^*})\left(R^* dt + \frac{d\Pi^*}{\Pi^*} - r\, dt\right)V \\ &+ rV\, dt - Q\, dt - Rw_M V\, dt - R^* w_{M^*} V\, dt\,. \end{aligned} \tag{9}$$

The optimization problem of the investor is to maximize (1) subject to (2), (8), and (9). His optimization problem can be restated in terms of finding the optimum value function $J(W, \mathbf{S}, t)$, where $\mathbf{S}$ is an $S \times 1$ vector of state variables, such that

$$J(W, \mathbf{S}, t) = \max_{\{w's,\, Q,\, L\}} E_t\left[\int_t^\infty e^{-\rho\theta} \frac{1}{\gamma} U(\theta)^\gamma\, d\theta\right]. \tag{10}$$

The portfolio strategy of the investor can be characterized in terms of the partial derivatives of the function $J(W, \mathbf{S}, t)$ without solving explicitly for that function. This approach is used in the next section to discuss the determinants of the ratio of foreign money holdings to domestic money holdings for domestic and foreign investors. In section 5, an explicit solution for the function $J(W, \mathbf{S}, t)$ is obtained for a special case.

4. EQUILIBRIUM HOLDINGS OF MONIES

The first-order conditions of the domestic individual's optimization problem are used here to discuss the determinants of the ratio of foreign money holdings to domestic money holdings for a domestic individual taking expected returns as given. Next a theory of equilibrium expected returns is used to show how the value of that ratio is determined in equilibrium. How the first-order conditions can be aggregated across investors to study the determinants of the exchange rate is then demonstrated. Finally, it is shown how the forward premium depends on the determinants of the exchange rate.

A. *Interpretation of the First-Order Conditions*

Using the first-order conditions of the domestic individual's optimization problem, we get

$$J_W R = Q^{\gamma\delta} k (a(\Pi M)^{\beta} + b(\Pi^* M^*)^{\beta})^{(\varepsilon\gamma - \beta)/\beta} \varepsilon a(\Pi M)^{\beta - 1} \tag{11}$$

$$J_W R^* = Q^{\gamma\delta} k (a(\Pi M)^{\beta} + b(\Pi^* M^*)^{\beta})^{(\varepsilon\gamma - \beta)/\beta} \varepsilon b(\Pi^* M^*)^{\beta - 1} . \tag{12}$$

Define $\Omega = (\Pi M)/(\Pi^* M^*)$. The first-order conditions for holdings of monies yield the first significant result of this paper.

PROPOSITION 1. *The ratio of holdings of real domestic cash balances to real foreign cash balances Ω does not depend on either the rate of time preference ρ or coefficient of relative risk aversion $1 - \gamma$.*

The proof is straightforward. Divide (11) by (12) and elevate to the power $1/(1 - \beta)$ to get

$$\Omega = \left(\frac{aR^*}{bR} \right)^{1/(1-\beta)} . \tag{13}$$

This completes the proof.

To understand equation (13), notice that R is the cost of holding domestic real balances. This follows from the fact that to finance one unit of domestic balances, the investor must go short one unit of the domestic nominal bond and pay interest at the rate R per unit of time. Risk turns out to be irrelevant, except insofar as it may affect R or R^*. The marginal productivity of domestic real balances is increasing in a. As $1 - \beta$ is positive, the higher a or the lower R, the higher Ω and, hence, the higher the investor's relative holdings of domestic real balances. The increase in Ω caused by an increase in a is an increasing function of β, that is, the degree to which currencies are substitutes for each other in the production of consumption services.

If markets are sufficiently complete, in the sense that there exists a perfect hedge for the risks associated with holding cash balances, the investor can separate decisions about the amount of cash balances he holds and decisions about investment proportions in his portfolio of risky assets.[4] As investors can hedge their holdings of cash balances, they do not need to take into account their risk tolerance when allocating part of their wealth across monies. A change in the variance of the purchasing power of a currency that leaves nominal interest rates and the parameters of the production function of consumption unchanged does not affect the ratio Ω. If investors hold foreign monies, it is because doing so reduces transaction costs, not because it decreases the risk of a portfolio.[5]

Proposition 1 does not hold if the production function of consumption services is not separable in Q. Another case such that proposition 1 does not hold is if there exist convertibility risks that affect bonds differently from monies and if markets are incomplete, in the sense that there exists no perfect hedge for those risks.

[4]Separation results for domestic currency holdings are obtained in Kouri (1977) and Fama and Farber (1979).

[5]The notion that investors diversify across monies has been recently stressed by Miles and Stewart (1980) and others.

B. Equilibrium Expected Returns and Holdings of Monies

The ratio of holdings of foreign real balances to holdings of domestic real balances depends on the interest rates R and R^*. To obtain the equilibrium value of Ω, one must, consequently, impose market clearing conditions on asset markets. It immediately follows that

PROPOSITION 2. *In equilibrium, interest rates R and R^* and holdings of domestic real balances per unit of foreign real balances Ω are given by*

$$R = r + (1 - \gamma)\,\mathrm{cov}(n, g) - \bar{n} \tag{14a}$$

$$R^* = r + (1 - \gamma)\,\mathrm{cov}(n^*, g) - \bar{n}^* \tag{14b}$$

$$\Omega = \left\{\frac{a[(1 - \gamma)\,\mathrm{cov}(n^*, g) + r - \bar{n}^*]}{b[(1 - \gamma)\,\mathrm{cov}(n, g) + r - \bar{n}]}\right\}^{1/(1-\beta)}, \tag{14c}$$

where n (n^*) is the rate of change of the price of one unit of domestic (foreign) money in terms of a unit of consumption services, r is the rate of interest on the safe asset, g is the rate of growth of domestic expenditures on consumption services, $\mathrm{cov}(n, g)$ *is the instantaneous covariance between changes in g and n per unit of time, and an overbar denotes an expectation.*

Proof. The proof is a straightforward extension of Stulz (1981) and, consequently, is not reproduced here. (It is, however, available from the author.)

In the following, $n(n^*)$ is called the domestic (foreign) rate of change of the purchasing power of the domestic (foreign) currency. To compute n, let q be the price of one unit of consumption services in terms of the consumption good. In this case, one unit of domestic money is worth Π/q units of consumption services and n is equal to $(q/\Pi)\,d(\Pi/q)$. The dynamics for q are given by

$$\frac{dq}{q} = s_m \frac{dR}{R} + s_{m^*} \frac{dR^*}{R^*}, \tag{15}$$

where $s_m(s_{m^*})$ is the share of expenditures on consumption services spent on holding domestic (foreign) real balances. Notice that if interest rates are constant, equations (14a) and (14b) are the continuous-time equivalent of the interest rate equation developed by Grauer and Litzenberger (1980, equation (4)). However, if interest rates are expected to increase, the nominal rate of return on a nominal bond must be higher than otherwise, as the purchasing power of the currency of the nominal bond is expected to fall more than otherwise.

Equation (14c) links the demand for real balances with purchasing power dynamics for the two currencies. As an increase in the expected rate of inflation at home implies a decrease in $\bar{n}$, it follows that an increase in the expected rate of inflation implies a decrease in Ω. As expected domestic inflation increases, investors hold relatively more foreign real balances.

If one is to enter a lottery, one would prefer a lottery that has large profits, for identical expected gains, when utility is low. As real expenditures on consumption services are low, utility is low. An asset that has low payoffs when expenditures on consumption services are low is more risky than an asset that has large payoffs when these expenditures are low in the sense that the asset will not be held if it has the same price and the same expected payoffs as the asset whose payoffs are inversely related to expenditures on consumption services.

The proper measure of purchasing power risk is the covariance of changes in purchasing power with changes in real consumption services.[6] It immediately follows that, ceteris paribus, a change in monetary policy that increases $\mathrm{cov}(n, g)$ decreases the relative holdings of domestic real balances, as it makes it more expensive to hold domestic real balances. For a given money stock, given real rate of interest and given *expected* rate of change of the purchasing power of the domestic currency, this means that the exchange rate is lower than it otherwise would have been.

C. *On Exchange Rate Determination*

This section develops an equation for the exchange rate in equilibrium. The representative foreign investor is denoted by the superscript f, except that his coefficient β is written β^*. From the first-order conditions (notice that $J_W = S^{\gamma-1}$),

$$\Pi M = \frac{\varepsilon a\, Q}{\delta R(a + b\ \Omega^{-\beta})}\,. \tag{16}$$

Money market equilibrium in the domestic country implies that

$$\Pi M^S = \mathrm{N}\ \Pi M + \mathrm{N}^f\ \Pi M^f\,, \tag{17}$$

where M^S is the domestic money stock and $N(N^f)$ is the number of domestic (foreign) investors.

Let M^{*S} be the foreign money stock. Money market equilibrium in both countries implies that

$$e = \frac{\dfrac{M^S}{R^*}\left\{\dfrac{N}{\delta}\left[\dfrac{\varepsilon b\, Q}{a\Omega^{\beta} + b}\right] + \dfrac{N^f}{\delta^f}\left[\dfrac{\varepsilon^f b^f\, Q^f}{a^f(\Omega^f)^{\beta*} + b^f}\right]\right\}}{\dfrac{M^{*S}}{R}\left\{\dfrac{N}{\delta}\left[\dfrac{\varepsilon a\, Q}{a + b\ \Omega^{-\beta}}\right] + \dfrac{N^f}{\delta^f}\left[\dfrac{\varepsilon^f a^f\, Q^f}{a^f + b^f\,(\Omega^f)^{-\beta*}}\right]\right\}}\,. \tag{18}$$

Equation (18) shows that the exchange rate can be expressed as a function of six variables, that is, the money stocks, the expenditures on the consumption good, and

[6]This is stressed, in a more restrictive model, in Stulz (1981). The present model incorporates money and transfers, which Stulz does not do. See also Breeden (1979) and Grossman and Schiller (1982).

the interest rates. Using equation (13), which defines Ω and Ω^f, one can write

$$e = e(\underset{+}{M^S}, \underset{-}{M^{*S}}, \underset{+}{R}, \underset{-}{R^*}, \underset{?}{Q}, \underset{?}{Q^f})\,. \tag{19}$$

Equation (19) states that the exchange rate is an increasing function of the domestic money stock and a decreasing function of the foreign money stock. The higher the marginal productivity of domestic money in the production of consumption services, that is, the higher a and a^f, the more valuable the domestic currency relative to the foreign currency. An increase in the domestic interest rate implies a depreciation of domestic currency; as the opportunity cost of holding domestic currency increases, the demand for currency falls and it becomes less valuable relative to foreign currency. If $b = 0$, an increase in domestic purchases of the consumption good, that is, Q, implies an appreciation of the domestic currency. However, as b becomes large, that is, as the foreign currency's marginal productivity in the production of domestic consumption services increases, an increase in Q can increase the demand for the foreign currency by domestic residents to such an extent that a depreciation of the domestic currency occurs. If b and a^f are "small," the sign of the partial derivatives of the exchange rate with respect to the money stocks, the interest rates, and the real expenditures on the consumption good are the same as those which obtain in the monetary approach to exchange rate determination.

Equation (18) makes it possible to assess how the degree to which two currencies are substitutes for each other in the production of consumption services affects the exchange rate. Suppose first that $bR > aR^*$ and $b^fR > a^fR^*$. In this case, an increase in β and β^* implies a depreciation of the domestic currency. The rationale for this result is that as β and β^* increase, the two currencies become better substitutes for each other and investors substitute the less expensive foreign currency for the more expensive domestic currency in their portfolios. The opposite result holds when $bR < aR^*$ and $b^fR < a^fR^*$. When a currency is relatively more advantageous for investors in one country and relatively less advantageous for investors in the other country, the effect on the exchange rate of an increase in β and β^* is ambiguous. Notice that if $\beta = \beta^* = 1$, the elasticity of substitution of the two currencies in the production of consumption services is infinite. In this case, the exchange rate is indeterminate if $a = b = a^f = b^f$ and $R = R^*$, as investors are indifferent as to how their holdings of real balances are distributed among currencies. However, it is not sufficient that $\beta = \beta^* = 1$ for the exchange rate to be indeterminate. If $\beta = \beta^* = 1$, it is possible that all investors hold only the foreign currency, only the domestic currency, or only the currency of their own country. For instance, if $\beta = \beta^* = 1$, $bR < aR^*$, and $b^fR < a^fR^*$, all investors hold only the domestic currency and the exchange rate is zero. Throughout the paper, however, we assume that $\beta^* \neq 1 \neq \beta$ as otherwise the properties of the equilibrium are not well defined.[7]

[7]Many authors have been concerned with the case in which $\beta^* = \beta = 1$. In general, they reach the conclusion that the exchange rate is indeterminate in the presence of perfect substitution. Canto and Miles

Equation (18) does not imply any relationship of causality between the variables on the right-hand side and the exchange rate as these variables depend on the vector of state variables. However, it shows that in a neoclassical optimizing model six variables are sufficient to characterize the state of the world for the purpose of exchange rate determination. It follows that this equation provides an extraordinary economy of information. It also shows that the equations used within the framework of the monetary approach have a sound general equilibrium foundation. Furthermore, equation (18) does not require the assumption that foreign bonds are perfect substitutes for domestic bonds.

D. *Exchange Rate Dynamics and the Forward Premium*

Equations (14a) and (14b) imply that the forward premium f, $f = R - R^*$, is a decreasing function of the covariance of changes in the rate of change of the exchange rate with changes in the rate of growth of real expenditures on consumption services, $\text{cov}(e, g)$. To understand this, notice that the expected gain on a forward contract is equal to the expected spot exchange rate minus the forward exchange rate. The lower the forward exchange rate, the higher the expected gain of a long position in a forward contract. The more risky the exchange rate, the higher the expected gain required by investors to take a long position in a forward contract and hence the lower the forward exchange rate.

Differentiating (19) using Ito's lemma yields an equation for exchange rate dynamics which implies that

$$\begin{aligned}\text{cov}(e, C^A) = \frac{1}{e}\{ & e_{M^S}\text{cov}(M^S, g) + e_{M^{*S}}\text{cov}(M^{*S}, g) \\ & + e_R(\text{cov}(R, g) + e_{R^*}\,\text{cov}(R^*, g) \\ & + e_Q\,\text{cov}(Q, g) + e_{Q^f}\,\text{cov}(Q^f, g)\}\,, \end{aligned} \tag{20}$$

where e_x is the partial derivative of the exchange rate equation with respect to x and $\text{cov}(x, g)$ is the instantaneous covariance of the rate of change of x with the rate of change of domestic consumption expenditures g per unit of time. Notice that the partial derivatives of the exchange rate equation can be computed analytically using equation (19).

It follows from (19) and (20) that both the distribution of changes in the money stocks and the distribution of the expected rate of change in the money stocks matter for the risk premium incorporated in the forward exchange rate.[8] A change in the expected rate of change in the money stock M^S, for instance, will affect the domestic

(1983) reach a different conclusion and provide references to the literature. It is interesting to note that equation (18), like the exchange rate equation used by Canto and Miles, does not imply indeterminacy when $\beta^* = \beta = 1$.

[8]Recent work by Grauer and Litzenberger (1980) discusses the relationship between monetary policy and the risk premium included in nominal interest rates. The present approach allows for changes in monetary policy and is an optimizing approach, contrary to theirs.

nominal interest rate. A positive covariance between changes in interest rates and changes in real consumption expenditures means that, ceteris paribus, unanticipated depreciations will accompany unanticipated increases in real consumption expenditures. This means that an uncovered investment in the foreign default-free bond is more risky than it would otherwise be.

5. A SPECIAL CASE

In this section the model developed so far is solved explicitly for a rather restrictive set of assumptions. The resulting gain is that the endogenous variables are obtained as an analytical solution of the state variables. It follows that it is possible to study the effects of unanticipated changes in these variables. The cost of this approach is a loss of generality. Nevertheless, the approach is useful to emphasize the general equilibrium nature of the model used. It shows how the uncertainty associated with exchange rate changes is a function of the uncertainty associated with state variables. Finally, the special case makes it possible to stress the role of monetary uncertainty as a determinant of the risk premium incorporated in the forward exchange rate.

The special case is constructed so that the only state variables are the money stocks and the stock of capital. It is assumed that the money stocks are exogenous and that

$$\frac{dM^S}{M^S} = \mu_M\, dt + \sigma_M\, dZ_M \tag{21}$$

$$\frac{dM^{*S}}{M^{*S}} = \mu_{M^*}\, dt + \sigma_{M^*}\, dZ_{M^*}\,. \tag{22}$$

Whereas earlier μ_M, σ_{M^*}, μ_{M^*}, and σ_{M^*} could be a function of the state of the world, they are assumed to be constant here. Furthermore, we need to make more assumptions about the real side of the economy. Only one commodity is produced. There is perfect competition. There is only one input, which is the commodity itself. The stock of capital evolves according to

$$\frac{dK}{K} = (\mu_K K - Q)\, dt + \sigma_K K\, dZ_K\,. \tag{23}$$

Q has the interpretation of a dividend. μ_K and σ_K are fixed.

We follow Jones (1980) by assuming that changes in money supplies occur through transfers to investors. Let Π be the price of one unit of domestic money in terms of the commodity. Domestic currency real balances are given by $m = \Pi M$ and

$$dm = M\, d\Pi + \Pi M(\mu_M\, dt + \sigma_M\, dZ_M) + \rho_{M\Pi}\sigma_M\sigma_\Pi M\Pi\, dt\,. \tag{24}$$

In this section, transfers arise only from changes in the domestic money stock and, using the notation of section 2, are given by

$$\tau dt + \sigma_\tau dt_\tau = \mu_M m\, dt + \sigma_M m\, dz_M + \rho_{M\Pi}\sigma_M\sigma_\Pi m\, dt\,. \tag{25}$$

It is assumed that markets are complete. All investors have the same utility function and the same production function of consumption services, regardless of country.[9] In such a world, all investors will hold the same portfolio of risky assets. In equilibrium, the capital stock and the money stocks have to be held. This is also true of all future transfers. It is assumed that investors maximize

$$J(W, \mathbf{S}, t) = \max_{\{w's, Q\}} E_t \int_t^\infty e^{-\rho\theta}[\delta \ln Q(\theta) + a' \ln m(\theta) + b' \ln m^*(\theta)]\, d\theta\,, \tag{26}$$

where $m^*(\theta)$ represents real balances denominated in foreign currency.[10]

As investors have a logarithmic utility function, their coefficient of relative risk aversion is equal to one and their consumption expenditures are proportional to their wealth. It follows that

$$\mu_{I_i} - r = \rho_{I_i W}\sigma_{I_i}\sigma_W \tag{27}$$

holds for any risky asset. Look now at the risky asset given by monetary wealth. Real monetary wealth is equal to the current value of the future transfers plus real balances. Let real monetary wealth be equal to Y. Notice now that the optimization problem of the investor can be written without knowing the money stocks. This suggests that Y is a function of K only. The expected instantaneous return on Y is equal to

$$Y_K(\mu_K K - Q) + \frac{1}{2} Y_{KK}K^2\sigma_K^2 + Rm + R^*m^* = rY + Y_K\rho_{KW}\sigma_K\sigma_W K, \tag{28}$$

where the right-hand side follows from equation (27). Jones (1980) provides a crucial insight for solving equations of this type by pointing out that the non-homogeneous term $Rm + R^*m^*$ has the interpretation of a dividend. The dividend yield of monetary wealth is given by the services of money. With the utility function used in this section, $Rm + R^*m^*$ is proportional to the consumption expenditures of the investor. However, the expenditures on the commodity must be equal to the

[9]These assumptions are required to ensure that a representative investor exists. If markets are not complete, investors cannot sell rights to future transfers and as future transfers differ across countries, changes in real wealth are not perfectly correlated among investors.

[10]Equation (26) is obtained by substituting equation (2) in equation (1) and setting $\gamma = \beta = 0$ and $\delta + \varepsilon = 1$. In this case, $a' = a\varepsilon$ and $b' = b\varepsilon$.

dividend paid by firms since we have assumed away investment decisions. The present value of future dividends is equal to K. Monetary services are proportional to dividends. It follows that monetary wealth is proportional to K:

$$\mathrm{Y} = \frac{a' + b'}{\delta} K . \tag{29}$$

M^S and M^{*S} are not required to solve the optimization problem of the investor. They are required only to obtain the price levels and the exchange rate. To obtain real quantities, the function $I(W)$, which represents the optimized lifetime expected utility of the investor, is required. It can be verified that this function is given by

$$I(W) = \left(\frac{\delta + a' + b'}{\rho}\right) \ln W . \tag{30}$$

From the first-order condition for holdings of domestic money (equation (11)), it follows that in equilibrium

$$\Pi M^S = \left(\frac{\rho a'}{\delta + a' + b'}\right)\frac{W}{R} = \rho\left(\frac{a'}{\delta}\right)\left(\frac{K}{R}\right) . \tag{31}$$

In this model, R and R^* are constant. Otherwise, monetary wealth would not always be the same fraction of the capital stock K. Since R is constant, the dynamics for Π are easily obtained by differentiating (31) using Ito's lemma:

$$\frac{d\Pi}{\Pi} = \frac{dK}{K} - \frac{dM^S}{M^S} - \rho_{M^SK}\sigma_{M^S}\sigma_K \,\mathrm{dt} + \sigma^2_{M^S}\, dt , \tag{32}$$

where ρ_{M^SK} is the instantaneous coefficient of correlation per unit of time between the rate of change of the domestic money stock M^S and the rate of change of the capital stock K. It follows from this that a 1 percent unanticipated increase in the domestic money stock decreases Π by 1 percent. Using equations (27) and (32), equation (14a), which gives the nominal rate of interest, now becomes

$$R = r - \overline{\Pi} + \sigma^2_K\left(\frac{a' + b'}{\delta}\right) - \rho_{M^SK}\sigma_{M^S}\sigma_K\left(\frac{a' + b'}{\delta}\right) , \tag{33}$$

where $\overline{\Pi}$ is the expected value of $(1/dt)(d\Pi/\Pi)$. Notice that the higher the covariance of the rate of growth of the money stock with the rate of growth of capital, the lower R. The real return on nominal money is negatively correlated with the money stock. In this economy, undiversifiable risks are those correlated with changes in the stock of capital. The investor prefers a high real return when the rate of change of the stock of capital is low, for a given expected real return. The expected real rate of return on a domestic nominal bond is higher if high unanticipated money growth is associated with high unanticipated growth in the capital stock. From the results of section 4, it follows that the higher the covariance of domestic money growth with growth in the capital stock, the larger the holdings of domestic real balances and the lower the domestic price of the foreign currency.

As the instantaneous forward premium f is equal to $R - R^*$, it follows, using (33) and an equivalent expression for R^*, that

$$f = \overline{\Pi}^* - \overline{\Pi} + \rho_{M^{*S}K}\sigma_{M^S}\sigma_K\left(\frac{a' + b'}{\delta}\right) - \rho_{M^S K}\sigma_{M^S}\sigma_K\left(\frac{a' + b'}{\delta}\right). \tag{34}$$

Equation (34) implies that the forward exchange rate is a decreasing (increasing) function of the covariance of the domestic (foreign) money stock with the rate of growth of the capital stock. In this model, the risk premium incorporated in the forward exchange rate depends only on the relative riskiness of the domestic and foreign monetary policies. As the covariance of changes in the domestic money stock with changes in the capital stock increases, the covariance of the real return of the domestic bond with changes in the stock of capital falls. This means that the domestic currency becomes less risky. Consequently, the forward premium f falls as individuals now require a higher expected reward to take a long position in foreign currency.[11]

6. CONCLUDING REMARKS

As a first step towards a theory of exchange rate determination within an optimizing framework with stochastic variables, this paper offers the finding that, ceteris paribus, a change in the purchasing power risks of two currencies affects today's exchange rate between the two currencies. A worthwhile goal would be to question how this affects exchange rate dynamics in more realistic models. For instance, it would be useful to see how deviations from purchasing power parity affect the results of this paper.

Methodologically, the paper uses the Bellman equation in two different ways. First, the first-order conditions are explored. The advantage of this is that it is not necessary to solve a nonlinear partial differential equation, which frequently has no known solution. However, whereas the first-order conditions make it possible to obtain useful equilibrium relationships, they are not useful in addressing questions such as the effect of an unanticipated change in the domestic money stock on the exchange rate. Solving the Bellman equation for a representative investor makes it possible to solve for the exchange rate as an analytical function of the state variables. Further research using this approach should tackle the problem of finding conditions under which foreign investors and domestic investors aggregate to a representative investor when in fact they differ.

[11]To verify that the results correspond to a rational expectations equilibrium, check that the optimal controls, price functions, and the function $I(W, S)$ are such that (1) all capital is held, (2) no bonds are held, (3) the money stocks are held, and (4) consumption of the commodity is equal to dividends. Cox, et al. (1978) justify such an approach.

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