# Fully-Coupled Characterization of Thermo-Electro-Magneto-Mechanical Materials

A Thesis

Presented in Partial Fulfillment of the Requirements for

the Degree Master of Science in the

Graduate School of The Ohio State University

By

Sarvani Piratla, B.Tech.

* * * * *

The Ohio State University

2009

Master's Examination Committee:

Dr. Stephen Bechtel, Adviser

Dr. Marcelo Dapino

Dr. Joseph Heremans

Approved by

Adviser

Graduate Program in Mechanical Engineering

# ABSTRACT

Smart materials exhibit powerful nonlinear 3D coupling and anisotropy. However, the design and associated models, experimental characterization, and control design of smart systems are in general 1D. This reduced framework severely limits applications to 1D. The research presented here enables the tailoring of composition and processing to produce multifunctional materials with targeted performance properties that are fully nonlinear and 3D. A complete combinatorial analysis of seventy-two possible energy functions and state variables that describe 3D, nonlinear, coupled behavior of thermo-electro-magneto-mechanical materials is created. Each set of state variables and corresponding energy function correlates with a given set of experiments, the independent variables being controlled and the dependent variables being the measured responses. The constitutive equations with respect to each of these energy functions are derived in a systematic way by combining the mesoscale Maxwell equations, the balance laws of mechanics, and the balance law of entropy. The application of this framework to a new class of magnetostrictive Iron-Gallium alloys (Galfenol) is investigated with the intent to implement these alloys in a new class of fuel injectors to achieve unprecedented dynamic response and performance. The possibility of determining the full 3D constitutive behavior of Galfenol from simple, 1D macroscopic experiments in combination with analytical models based on knowledge of the crystal structure is explored. In this respect, a FEM formulation is presented.

# ACKNOWLEDGMENTS

I would like to express my sincere gratitude towards my advisors, Prof. Stephen Bechtel and Prof. Marcelo Dapino, for their continuous guidance during my Masters study.

I would also like to thank Prof. Joseph Heremans for serving on my Masters examination committee.

I am grateful to my colleagues, Robert Lowe, Monon Mahboob, and especially Philip Evans for their help in addressing various theoretical issues. I am also thankful to the Mechanical Engineering Department staff for their cooperation.

I would like to acknowledge the support of IMR in funding this research.

Finally, I would like to thank my parents, two brothers and my friend Vivek Yadav for their moral support and encouragement.

# VITA

| | |
|---|---|
| August 7, 1983 | Born - Guntur, India |
| 2007 | B.Tech. Mechanical Engineering, Indian Institute of Technology Madras, Chennai, India |
| 2007 - Present | Graduate Research Associate, The Ohio State University, Columbus, OH |

# PUBLICATIONS

S. Piratla, S.E. Bechtel, and M.J. Dapino, "Thermodynamic Potentials for fully-coupled characterization of Thermo-Electro-Magneto-Mechanical Materials". *International Journal of Structural Changes in Solids-Mechanics and Applications*, draft in preparation.

# FIELDS OF STUDY

Major Field: Mechanical Engineering

Studies in:

| | |
|---|---|
| Smart Materials and Structures | Dr. Marcelo Dapino |
| Continuum Mechanics | Dr. Stephen Bechtel, Dr. Rebecca Dupaix |

# TABLE OF CONTENTS

# LIST OF FIGURES

# LIST OF TABLES

# CHAPTER 1

# INTRODUCTION

Adaptive structures based on active or smart materials (controllable material properties and response) are underutilized. Despite the observation that smart materials exhibit powerful nonlinear 3D coupling and anisotropy, the design and associated models, experimental characterization, and control of smart material based systems are in general 1D. This framework limits applications to devices executing 1D motion.

The smart material models can be categorized into 5 groups.

1. Models employing linear constitutive equations:

In [30, 31], a 3D finite-element model using linear constitutive equations is implemented. [45] listed the possible thermodynamic potentials with electric, magnetic and mechanical independent variables and postulated expressions for these potentials that resulted in linear constitutive equations.

2. Models where the constitutive equations are obtained using a Taylor series expansion of Gibbs free energy. The coefficients of these constitutive equations are either taken as constants or calculated from experimental data:

In [49, 48], the coefficients of the constitutive equations are written as Taylor series expansions of Gibbs free energy and are replaced with the empirical expressions constructed based on experimental data. [4] expands Gibbs free energy in a Taylor series, where higher-order terms are neglected and the leading order terms are taken as constants. [2] posits a Gibbs free energy that incorporates higher-order effects, e.g., fifth order in strain, fourth order in electric field, third order in temperature, fifth order in coupling terms.

3. Phenomenological models constructed based on experimental observations:

In [10], a phenomenological model is proposed by considering Jiles-Atherton mean field theory and law of approach to the anhysteretic magnetization. In [21], a 3D electromechanical constitutive law is formulated to model saturation of induced polarization with increasing electric field by assuming that electrically-induced strain depends on second-order polarization terms and an empirical hyberbolic relationship for the dielectric behavior. [15] separates the nonlinear constitutive equations into three distinct categories: phenomenological constitutive models based on thermodynamics (standard square model [47], hyberbolic tangent constitutive model, constitutive relations based on the density of domain switching), constitutive model based on noncontinuous domain switching, constitutive models with internal variables (constitutive model based on J2 flow theory, phenomenological constitutive model with anisotropic flow theory). [3] combines the magnetic anisotropy analysis of [25] with the assumption of an inverse exponential distribution of magnetized energy states.

4. Models in which part of the energy is proposed based on the crystal structure of the material. Boltzmann and probability distributions are used to obtain the magnetization and magnetostriction:

In [12], the energy has terms for magnetic anisotropy, magnetomechanical coupling, Zeeman or field energy, elastic strain energy. These energies are expressed while idealizing the complex domain structure of ferromagnetic materials as a system of non-interacting, single-domain, Stoner-Wohlfarth (S-W) particles. [13] develops a steady-state constitutive model (energy expression based on crystal structure) to obtain the expressions for magnetization and magnetostriction. [14] presents a state-space constitutive model where the energy has magnetocrystalline anisotropy and Zeeman energy terms. The expected value of magnetization is calculated with the use of a Boltzmann distribution.

5. Models having a combination of linear constitutive equations and the use of empiricism:

In [33, 32], the strain as well as the field (electric or magnetic) are split into reversible and irreversible parts. A quadratic relation is postulated between irreversible strain and irreversible field. The thermodynamic potential is a combination of terms that give rise to the linear constitutive equations and an additional nonlinear coupling term. In [40], 3D FEM is implemented with a formulation which has the linear constitutive equations along with Maxwell stress tensor.

The present research work deals with two aspects of these models.

A. The process of obtaining the constitutive equations in terms of potentials from first principles of thermodynamics.

1. Although final form of the constitutive equations in terms of potential is presented in each of the above models, the process of obtaining these equations is generally not described in depth. A general known method is to start from the local form of conservation of energy, use the reversible form of second law, and derive constitutive equations from them [44]. A rigorous treatment is presented in [17, 18, 24, 37]. Although these authors differ from one another in certain aspects, all of them start from the integral form of the governing equations of thermomechanics. Upon subjecting various quantities to invariance requirements, the local form of these equations are obtained. For the second law, either the Classius-Duhem inequality or the balance law of entropy is used.

   For the case of thermo-electro-magneto-mechanical materials, there are a number of possible thermodynamic potential functions based on the set of independent variables. In this thesis, an analysis of possible energy functions and state variables that describe 3D, nonlinear, coupled behavior of thermo-electro-magneto-mechanical materials undergoing a non-dissipative process is presented. In total, there are seventy-two material characterizations described by energy potentials arranged in eight families, each sharing the same thermomechanical independent variables. Each choice of state variables and corresponding energy function correlates with a given set of nonlinear experiments; the independent variables being controlled and the dependent variables being the measured responses. And for each such possible potential function, the constitutive equations can be derived. Seventy-two possible characterizations are worked out to obtain the constitutive equations for an unconstrained material undergoing a rate-independent process. This is done following the rigorous treatment of [17, 18, 24, 37].

2. Using the constitutive equations, an approach to evaluate the potential is laid out employing conditions as constitutive limits (e.g., constant pressure).

3. The application of this framework to a new class of magnetostrictive iron-gallium alloys (Galfenol) is investigated with the intent to implement these alloys in a new class of fuel injectors to achieve unprecedented dynamic response and performance. From the available experimental data of Galfenol, attempts are made to numerically evaluate the potential. The state of the art of the fuel injectors, failure of piezoelectric stack actuators, and the factors that qualify Galfenol for fuel injector applications are also discussed.

B. Implementing a 3D nonlinear finite element formulation employing a phenomenological model.

So far, 3D nonlinear constitutive equations are not used to solve 3D boundary value problems in the finite element method (FEM). Also, in most of the material models, the Maxwell stress tensor term is neglected.

A 3D phenomenological model given in [12] is used to obtain the fully nonlinear constitutive equations for magnetostrictive Galfenol. A boundary value problem consisting of nonlinear constitutive equations, balance law of linear momentum, Ampere's law and a set of appropriate BC's is formulated and this can be solved using a FEM formulation in the spirit of [40].

## 1.1 Outline of thesis

Chapter 2 focuses on a detailed description of the Minkowski formulation and the approach followed in [18]. A summary of the governing equations from various authors followed by detailed analyses is also presented. Chapter 3 focuses on an overview of the seventy-two possible potentials and a detailed derivation of the constitutive equations from these potentials. Chapter 4 discusses the use of smart material actuators in fuel injector applications, with an emphasis on failure of piezoelectric stack actuators and the factors that qualify Galfenol actuators for fuel injector applications. Chapter 5 presents the governing equations and 3D nonlinear constitutive equations for magnetostrictive materials and the 3D finite-element formulation.

# CHAPTER 2

# MESOSCALE CHARACTERIZATION OF DEFORMABLE THEMO-ELECTRO-MAGNETO-MECHANICAL (TEMM) MATERIALS

A general thermo-electro-magneto-mechanical (TEMM) process is described by the evolution up to the present time of thermal, electric, magnetic, and mechanical quantities. The particular choice of which quantities interrelate to describe the process, and the explicit forms of these relations, are the constitutive model for the material. For a TEMM material, the interdependence is through the following spatially and temporally varying fields,

Thermal:

$$
\begin{aligned}
&\theta \text{ – absolute temperature} \\
&\eta \text{ – specific entropy (entropy per mass = length}^2\text{ time}^{-2}\text{ temperature}^{-1}\text{)} \\
&\mathbf{Q} \text{ – Lagrangian heat flux vector (energy per area per time = mass time}^{-3}\text{)}
\end{aligned}
\tag{2.1}
$$

Electric:

$$\mathbf{e} \text{ – electric field vector (mass length time}^{-3} \text{ current }^{-1})$$

$$\mathbf{p} \text{ – polarization vector (length}^{-2} \text{ time current)}$$

$$\mathbf{d} \text{ – electric displacement vector (length}^{-2} \text{ time current)} \tag{2.2}$$

Magnetic:

$$\mathbf{h} \text{ – magnetic field vector (length}^{-1} \text{ current)}$$

$$\mathbf{m} \text{ – magnetization vector (length}^{-1} \text{ current)}$$

$$\mathbf{b} \text{ – magnetic flux vector (mass time}^{-2} \text{ current}^{-1}) \tag{2.3}$$

Mechanical:

$$\mathbf{P} \text{ – non-symmetric Piola-Kirchhoff stress tensor (force per area = mass length}^{-1} \text{ time}^{-2})$$

$$\mathbf{F} \text{ – deformation gradient tensor (dimensionless)} \tag{2.4}$$

Energetic (energy per mass = length$^2$ time$^{-2}$):

$$\varepsilon \text{ – specific internal energy}$$

$$\psi \text{ – specific Helmholtz free energy}$$

$$\phi \text{ – specific Gibbs free energy}$$

$$\chi \text{ – specific enthalpy} \tag{2.5}$$

Source terms:

$$\mathbf{j} \text{ – free current density vector (ampere per square meter = current length}^{-2})$$

$$\sigma \text{ – free charge density (coulomb per cubic meter = current time length}^{-3}) \tag{2.6}$$

The TEMM material we model has no memory, i.e. its response depends only on the values of the above quantities at the present time, with no explicit dependence on previous times or temporal rates. For example, the stress at location $\mathbf{x}$ and present time $t$ depends only on the values of deformation, temperature, thermal flux, energy, electric field, polarization, electric displacement, magnetic field, magnetization, and magnetic flux, and perhaps their spatial (but not temporal) gradients, all evaluated at the present location $\mathbf{x}$ and time $t$.

In the literature, there are many interaction models to characterize deformable thermo-electro-magneto-mechanical materials. These models introduce four different electromagnetic vector fields: $\mathbf{e}$ and $\mathbf{b}$ plus two other fields. Some models work with $\mathbf{d}$ and $\mathbf{h}$, others use $\mathbf{p}$ and $\mathbf{m}$ instead. A unique transformation from the variables $\mathbf{e}, \mathbf{b}, \mathbf{d}, \mathbf{h}$ to $\mathbf{p}, \mathbf{m}$ exists. However, for each of these interaction models, the electromagnetic stress tensor and body force in the momentum equation are not unique.

The first law of thermodynamics states that the time rate of change of the internal energy is balanced by stress power, the heat flux, and the energy supply due to heat and electromagnetic effects. Since stress is non-unique, it follows that internal energy, heat flux and electromagnetic energy supply cannot be determined uniquely either. Likewise, the electromagnetic energy supply might contain a term that is the divergence of some vector quantity, which could be absorbed in the heat flux vector. In that case, heat flux can be called as energy flux [24].

From the many interaction models that exist in the literature, the Minkowski formulation is selected. Two variations of the Minkowski formulation developed in Hutter [24] and Green & Naghdi [18] are presented in this chapter. The final form of the reduced energy equation is expressed in terms of $\mathbf{e}, \mathbf{h}, \mathbf{p}, \mathbf{m}$ in [24] and $\mathbf{e}, \mathbf{h}, \mathbf{d}, \mathbf{b}$

in [18]. This difference arises as the internal energy considered in both works is not identical.

A summary of governing equations developed by various authors is given. The equivalence of these equations is also addressed.

This work deals with magnetizable and polarizable solids, which deform elastically under the action of electromagnetic and thermal fields and which exhibit electrical and thermal conduction. Mechanical dissipation, exchange interaction and magnetic spin are not considered here. This work is done on the level of non-relativistic approximation.

A continuum theory of deformable bodies subject to electromagnetic fields amounts to the presentation of the basic electromagnetic field variables, their relations to other fields, as well as the postulation of electromagnetic body force, body couple and energy supply. Then, the Maxwell equations and the balance laws of mechanics and thermodynamics can be expressed in terms of the variables of the model that is considered. The thermodynamic arguements are used to obtain the constitutive equations in a form compatible with the second law of thermodynamics.

An important feature of this work is that it enables scientists and engineers to assess the physical relevance of a particular posited pointwise model by seeing if there exists a global statement. To be valid, pointwise equations must be derivable from integral equations. Here we deduce the pointwise equations from global statements of the first principles, modifying the rational treatment employed in thermodynamics.

## 2.1 Global form of the governing equations for electrodynamics and thermomechanics

The global form of the equations governing coupled thermo-electro-magneto-mechanical electromagnetic media are

$$
\begin{aligned}
\int_{\partial\mathcal{V}} \mathbf{b}\cdot d\mathbf{a} \;&=\; 0,\\
\frac{d}{dt}\int_{\mathcal{S}} \mathbf{b}\cdot d\mathbf{a} \;+\; \int_{\partial\mathcal{S}} \mathbf{e}^{*}\cdot d\mathbf{x} \;&=\; 0,\\
\int_{\partial\mathcal{V}} \mathbf{d}\cdot d\mathbf{a} \;&=\; \int_{\mathcal{V}} \sigma\; dv,\\
\frac{d}{dt}\int_{\mathcal{S}} \mathbf{d}\cdot d\mathbf{a} \;-\; \int_{\partial\mathcal{S}} \mathbf{h}^{*}\cdot d\mathbf{x} \;+\; \int_{\mathcal{S}} \mathbf{j}^{*}\cdot d\mathbf{a} \;&=\; 0,\\
\frac{d}{dt}\int_{\mathcal{V}} \sigma\; dv \;+\; \int_{\partial\mathcal{V}} \mathbf{j}^{*}\cdot d\mathbf{a} \;&=\; 0,
\end{aligned}
\tag{2.7}
$$

and

$$
\begin{aligned}
\frac{d}{dt}\int_{\mathcal{V}}\{\rho\varepsilon \;+\; \frac{1}{2}(\epsilon_o\mathbf{e}^{*}\cdot\mathbf{e}^{*} \;+\; \mu_o\mathbf{h}^{*}\cdot\mathbf{h}^{*}) \;+\; \frac{1}{2}\rho\mathbf{v}\cdot\mathbf{v} \;+\; \rho T\}dv \;=\; \int_{\mathcal{V}}\{\rho r^{ext} \;+\; \rho\mathbf{f}^{ext}\cdot\mathbf{v}\}dv\\
+\; \int_{\partial\mathcal{V}}\{\mathbf{T}^{T}\mathbf{v} \;-\; \mathbf{q} \;-\; \mathbf{e}^{*}\times\mathbf{h}^{*} \;+\; \frac{1}{2}(\epsilon_o\mathbf{e}^{*}\cdot\mathbf{e}^{*} \;+\; \mu_o\mathbf{h}^{*}\cdot\mathbf{h}^{*})\mathbf{v} + R\}\cdot da,
\end{aligned}
\tag{2.8}
$$

or

$$
\begin{aligned}
\frac{d}{dt}\int_{\mathcal{V}}\rho dv \;&=\; 0,\\
\frac{d}{dt}\int_{\mathcal{V}}\rho\mathbf{v}dv \;&=\; \int_{\mathcal{V}}\rho(\mathbf{f}^{ext}+\mathbf{f}^{e})dv \;+\; \int_{\partial\mathcal{V}}\mathbf{t}\cdot d\mathbf{a},\\
\frac{d}{dt}\int_{\mathcal{V}}\rho(\mathbf{x}\times\mathbf{v})dv \;&=\; \int_{\mathcal{V}}\rho(\mathbf{x}\times(\mathbf{f}^{ext}+\mathbf{f}^{e})+\mathbf{L}^{e})dv \;+\; \int_{\partial\mathcal{V}}(\mathbf{x}\times\mathbf{t})\cdot d\mathbf{a},\\
\frac{d}{dt}\int_{\mathcal{V}}\rho(\varepsilon+\frac{1}{2}\mathbf{v}\cdot\mathbf{v})dv \;&=\; \int_{\mathcal{V}}(\rho r^{ext}+\rho r^{e}+\rho(\mathbf{f}^{ext}+\mathbf{f}^{e})\cdot\mathbf{v})dv \;+\; \int_{\partial\mathcal{V}}(\mathbf{t}\cdot\mathbf{v}-h)da,
\end{aligned}
\tag{2.9}
$$

which must hold for all material surfaces $\mathcal{S}$ with closed boundary $\partial\mathcal{S}$ and any material volume $\mathcal{V}$ with a closed boundary surface $\partial\mathcal{V}$ and outward normal $\mathbf{n}$.

In equations (2.8), (2.9), $\rho$ is the mass density, $\mathbf{v}$ is the velocity of the mass particle, $\mathbf{f}^{ext}$ is the body force due to an externally applied field, $r^{ext}$ is the specific heat supply rate, $\mathbf{t} = \mathbf{Tn} = \frac{\mathbf{PF}^T\mathbf{n}}{det\mathbf{F}}$ is the surface traction vector, $\mathbf{T}$ is the stress tensor, $h = \mathbf{q} \cdot \mathbf{n} = \frac{\mathbf{FQ}}{det\mathbf{F}} \cdot \mathbf{n}$ is the heat flux, $\mathbf{q}$ is the heat flux vector, $\mu_o$ is the permeability of the free space, $\epsilon_o$ is the permittivity of the free space, and $\rho T$, R are quantities to be expressed as functions of electromagnetic fields. The force $\mathbf{f}^e$, the couple $\mathbf{L}^e$, and the energy supply rate $r^e$ are the effects of the electromagnetic fields on the thermomechanical problem.

The quantities $\mathbf{e}^*$, $\mathbf{h}^*$, $\mathbf{j}^*$ are the effective electric field strength, magnetic field strength, and conductive current respectively, i.e., field and current per area acting on the deformed body. The relation between the effective fields $\mathbf{e}^*$, $\mathbf{h}^*$, $\mathbf{j}^*$ and the primitive fields $\mathbf{e}$, $\mathbf{h}$, $\mathbf{j}$ is part of the characterization of the particular material. Several electromechanical interaction models for deformable matter have been presented in the literature, deduced from various degrees of first-principle based modeling of microscale behavior and empiricism. These interaction models include the Chu, Minkowski, Lorentz, and statistical formulations.

Chu formulation:

$$\mathbf{e}^* = \mathbf{e} + \mathbf{v} \times \mu_o \mathbf{h}, \qquad \mathbf{h}^* = \mathbf{h} - \mathbf{v} \times \epsilon_o \mathbf{e}, \qquad \mathbf{j}^* = \mathbf{j} - \sigma \mathbf{v}, \tag{2.10}$$

Minkowski formulation:

$$\mathbf{e}^* = \mathbf{e} + \mathbf{v} \times \mathbf{b}, \qquad \mathbf{h}^* = \mathbf{h} - \mathbf{v} \times \mathbf{d}, \qquad \mathbf{j}^* = \mathbf{j} - \sigma \mathbf{v}, \tag{2.11}$$

Lorentz formulation:

$$\mathbf{e}^* = \mathbf{e} + \mathbf{v} \times \mathbf{b}, \qquad \mathbf{h}^* = \frac{1}{\mu_o}\mathbf{b} - \mathbf{m} - \mathbf{v} \times \epsilon_o \mathbf{e}, \qquad \mathbf{j}^* = \mathbf{j} - \sigma \mathbf{v}, \tag{2.12}$$

Statistical formulation:

$$\mathbf{e}^* = \mathbf{e} + \mathbf{v} \times \mathbf{b}, \qquad \mathbf{h}^* = \frac{1}{\mu_o}\mathbf{b} - \epsilon_o \mathbf{v} \times \mathbf{e} - \mathbf{m} - \mathbf{v} \times \mathbf{p}, \qquad \mathbf{j}^* = \mathbf{j} - \sigma \mathbf{v}, \tag{2.13}$$

where $\mathbf{e}, \mathbf{h}, \mathbf{j}$ in (2.10) are the Chu's electric field strength, magnetic field strength, and free current density and the Minkowskian electric field strength, magnetic field strength, and free current density in (2.11) and so on.

In this work, we employ the Minkowski formulation. In this, a rigid body moves with constant velocity $\mathbf{v}$ relative to the laboratory frame, and a reference frame called the rest frame is attached to the moving body. The equations of electrodynamics in the coordinates of both the frames should be invariant. By postulating that the energy balance is invariant under a super posed rigid body motion, the equations of balance laws of mass and momenta are derived [24]. Although the Minkowski formulation is based on rigid body motion, it has been applied to deformable bodies in motion [37]. The physical motivations of the other three formulas can be found in [22, 46, 19, 16, 23, 38, 39].

The balance law of entropy we employ is due to [18, 17]:

$$\frac{d}{dt}\int_{\mathcal{V}} \rho\eta \; dv \;\; + \;\; \int_{\partial\mathcal{V}} k \; da - \;\; \int_{\mathcal{V}} \rho s \; dv \;\; = \;\; \int_{\mathcal{V}} \rho\xi \;\;\; dv, \tag{2.14}$$

where $k$ is the entropy flux, $s$ is the external specific entropy supply rate, and $\xi$ is the entropy production. In this work, we adopt the constitutive assumptions

$$k = \frac{\mathbf{q}}{\theta} \cdot \mathbf{n}, \qquad s = \frac{r^{ext}}{\theta}. \tag{2.15}$$

## 2.2 Pointwise form of the governing equations for electrodynamics and thermomechanics

With smoothness assumptions on the integrands in the equations (2.7), (2.8),(2.9), and (2.14), the area integrals convert to volume integrals using the divergence theorem. Line integrals convert to area integrals using the Stokes theorem, and the time derivatives are taken inside the integrals using the transport theorem (the vectorial generalization of Leibniz's rule). Hence the equations (2.7), (2.8), (2.9), and (2.14) reduce either to

$$\int_{\mathcal{V}} \phi(\mathbf{x}, t)\ dv = 0, \qquad \text{or} \qquad \int_{\mathcal{S}} \phi(\mathbf{x}, t) \cdot d\mathbf{a} = 0. \tag{2.16}$$

Then, since the above integrals are valid for any arbitrary susbset $\mathcal{V}$ or $\mathcal{S}$ of the present configuration, and the variables are assumed to be continuous, use of localization theorem[1] gives the pointwise field equations. The pointwise Maxwell equations resulting from the global equations (2.7) are:

$$\begin{aligned} \nabla \cdot \mathbf{b} &= \mathbf{0}, \\ \nabla \times \mathbf{e}^* &= -\frac{\partial \mathbf{b}}{\partial \mathbf{t}} - \nabla \times (\mathbf{b} \times \mathbf{v}), \end{aligned}$$

[1]If $\phi$ is a continuous scalar or tensor-valued field in $\mathcal{R}$ and

$$\int_{\mathcal{V}} \phi(\mathbf{x}, t)\ dv = 0 \ \text{ for all subsets } \mathcal{V} \text{ of } \mathcal{R},$$

for every part $\mathcal{V}$ then it is necessary and sufficient that

$$\phi = 0 \qquad \text{for all points } \mathbf{x} \text{ in } \mathcal{R}$$

$$
\begin{aligned}
\nabla \cdot \mathbf{d} &= \sigma, \\
\nabla \times \mathbf{h}^* &= \frac{\partial \mathbf{d}}{\partial \mathbf{t}} + \nabla \times (\mathbf{d} \times \mathbf{v}) + \sigma \mathbf{v} + \mathbf{j}^*, \\
\nabla \cdot \mathbf{j}^* + \frac{\partial \sigma}{\partial \mathbf{t}} + \nabla \cdot (\sigma \mathbf{v}) &= \mathbf{0}.
\end{aligned} \tag{2.17}
$$

There are two ways of obtaining the pointwise balance laws of mechanics. One way is to obtain the equations from (2.9) and the second way is to obtain the equations from the global energy balance law (2.8). Using the same procedure as described above for obtaining pointwise Maxwell equations, the pointwise balance laws of mechanics (2.19) are obtained from (2.9). The same set of equations are obtained from (2.8) as described in the Minkowski formulation: In order to determine the unknown quantities $\rho T$ and $R$ in (2.8), one subjects the local form of (2.8) to a Euclidean transformation. The values of $\rho T$ and $R$ are obtained from invariance requirements and substituted into the global form. From this the local form can be derived, which with the use of the Maxwell equations and the constitutive equations may be written as

$$
\begin{aligned}
(\dot{\rho} + \rho \nabla \cdot \mathbf{v})(\varepsilon + \frac{1}{2}\mathbf{v} \cdot \mathbf{v}) + \rho \dot{\epsilon} - \mathbf{j}^* \cdot \mathbf{e}^* - \mathbf{e}^* \cdot \mathring{\mathbf{p}} - \mu_o \mathbf{h}^* \cdot \mathring{\mathbf{m}}^* + \nabla \cdot \mathbf{q} - \rho r^{ext} - \\
\{\mathbf{T} + \mathbf{e}^* \otimes \mathbf{p} + \mu_o \mathbf{h}^* \otimes \mathbf{m}^*\} \cdot \mathbf{L} + \{\rho \dot{\mathbf{v}} - \nabla \cdot \mathbf{T} - \rho \mathbf{f}^{ext} - \\
\sigma \mathbf{e}^* - \mathbf{j}^* \times \mathbf{b} - (\nabla \mathbf{e}^*)^T \mathbf{p} - \mu_o (\nabla \mathbf{h}^*)^T \mathbf{m}^* - (\mathring{\mathbf{d}} \times \mathbf{b} + \mathbf{d} \times \mathring{\mathbf{b}})\} \cdot \mathbf{v} = 0.
\end{aligned} \tag{2.18}
$$

The invariance requirements under which $\rho$, $\epsilon$, $\mathbf{T}$, $(\dot{\mathbf{v}} - \mathbf{f}^{ext})$, $\mathbf{q}$, $\rho r^{ext}$, $Q$, $\mathbf{j}^*$, $\mathbf{e}^*$, $\mathbf{p}$, $\mathbf{h}^*$, $\mathbf{m}^*$, $\mathbf{d}$, $\mathbf{b}$ are assumed to transform as objective quantities then yield the pointwise equations for mass, linear momentum, angular momentum, and energy :

$$
\dot{\rho} + \rho \nabla \cdot \mathbf{v} = 0,
$$

$$\rho\dot{\mathbf{v}} \;=\; \rho\left(\mathbf{f^{ext}} \;+\; \mathbf{f^{e}}\right) \;+\; \nabla\cdot\mathbf{T},$$

$$\mathbf{T_{[ij]}} \;=\; \rho\mathbf{L^{e}_{ij}},$$

$$\rho\dot{\varepsilon} \;=\; \rho r^{ext} \;+\; \rho r^{e} \;-\; \nabla\cdot\mathbf{q} \;+\; \mathbf{T}\cdot\mathbf{L}. \qquad (2.19)$$

The force $\mathbf{f}^{e}$, the couple $\mathbf{L}^{e}$, and the energy supply rate $r^{e}$ are the effects of the electromagnetic fields on the thermomechanical problem. The nature of these coupling terms is another important part of the material characterization. As an example, for the Minkowski formulation,

$$\rho\mathbf{f^{e}} \;=\; \sigma\mathbf{e}^{*} \;+\; \mathbf{j}^{*}\times\mathbf{b} \;+\; \mathbf{p}\cdot\nabla\mathbf{e}^{*} \;+\; \mu_{\mathbf{o}}\mathbf{m}^{*}\cdot\nabla\mathbf{h}^{*} \;+\; \mathring{\mathbf{d}}\times\mathbf{b}+\mathbf{d}\times\mathring{\mathbf{b}},$$

$$\rho\mathbf{L}^{e} \;=\; \frac{1}{2}\left(\mathbf{p}\otimes\mathbf{e}^{*} \;-\; \mathbf{e}^{*}\otimes\mathbf{p} \;+\; \mu_{o}[\mathbf{m}^{*}\otimes\mathbf{h}^{*} \;-\; \mathbf{h}^{*}\otimes\mathbf{m}^{*}]\right),$$

$$\rho r^{e} \;=\; \mathbf{j}^{*}\cdot\mathbf{e}^{*} \;+\; \rho\mathbf{e}^{*}\cdot\frac{d}{dt}(\frac{\mathbf{p}}{\rho}) \;+\; \rho\mu_{o}\mathbf{h}^{*}\cdot\frac{\mathbf{d}}{\mathbf{dt}}(\frac{\mathbf{m}^{*}}{\rho}), \qquad (2.20)$$

where

$\mathbf{m}^{*} \;=\; \mathbf{m} \;+\; \mathbf{v}\times\mathbf{p},$

$\mathring{\mathbf{a}} = \frac{\partial\mathbf{a}}{\partial t} + \mathbf{v}\nabla\cdot\mathbf{a} + \nabla\times(\mathbf{a}\times\mathbf{v}),\ \forall\ \mathbf{a},$

$[\mathbf{a}\otimes\mathbf{b}]_{ij} = a_i b_j.$

The coupling terms $\mathbf{f}^{e}$, $\mathbf{L}^{e}$, $r^{e}$ for the Chu, statistical, and Lorentz formulations can be found in [24].

The pointwise entropy balance equation from (2.14):

$$\rho\dot{\eta} \;+\; \nabla\cdot(\frac{\mathbf{q}}{\theta}) \;-\; \rho(\frac{r^{ext}}{\theta}+\xi) \;=\; 0. \qquad (2.21)$$

Substituting $\mathbf{e}^{*}$ from the Minkowski formulation in $(2.17)_2$, one obtains

$$\nabla\times(\mathbf{e}+\mathbf{v}\times\mathbf{b}) \;=-\; \frac{\partial\mathbf{b}}{\partial t} \;-\; \nabla\times(\mathbf{b}\times\mathbf{v}), \qquad (2.22)$$

which reduces to

$$\nabla \times \mathbf{e} = -\frac{\partial \mathbf{b}}{\partial t}.$$

Similarly, substituting $\mathbf{h}^*$ and $\mathbf{j}^*$ from the Minkowski formulation in $(2.17)_4$,

$$\nabla \times (\mathbf{h} - \mathbf{v} \times \mathbf{d}) \;\; = \;\; \frac{\partial \mathbf{d}}{\partial t} \;\; + \;\; \nabla \times (\mathbf{d} \times \mathbf{v}) \;\; + \;\; \sigma \mathbf{v} \;\; + \;\; (\mathbf{j} - \sigma \mathbf{v}),$$

reduces to

$$\nabla \times \mathbf{h} = \mathbf{j} + \frac{\partial \mathbf{d}}{\partial t}. \tag{2.23}$$

The pointwise Maxwell equations for an undeformed body are obtained by setting $\mathbf{v} = 0$ in (2.17),

$$\begin{aligned} \nabla \cdot \mathbf{b} &= 0, \\ \nabla \times \mathbf{e} &= -\frac{\partial \mathbf{b}}{\partial t}, \\ \nabla \cdot \mathbf{d} &= \sigma, \\ \nabla \times \mathbf{h} &= \mathbf{j} + \frac{\partial \mathbf{d}}{\partial t}. \end{aligned} \tag{2.24}$$

Note that (2.24) are same as the Minkowski Maxwell equations. The Maxwell equations in Minkowski formulation are in terms of $\mathbf{e}$, $\mathbf{b}$, $\mathbf{d}$, $\mathbf{h}$ and those corresponding to Statistical formulation are in terms of $\mathbf{e}$, $\mathbf{b}$, $\mathbf{p}$, $\mathbf{m}$. On using the relations (given below) that connect $\mathbf{p}$, $\mathbf{m}$ to $\mathbf{d}$, $\mathbf{h}$, one can find that the Maxwell equations from Statistical and Minkowski formulation are same.

$$\mathbf{p} = \mathbf{d} - \epsilon_{\mathbf{o}} \mathbf{e}, \qquad\qquad \mu_{\mathbf{o}} \mathbf{m} = \mathbf{b} - \mu_{\mathbf{o}} \mathbf{h}, \tag{2.25}$$

Upon satisfying the invariance property, (2.25) reduces to

$$\mathbf{p} = \mathbf{d} - \epsilon_{\mathbf{o}} \mathbf{e}^*, \qquad\qquad \mu_{\mathbf{o}} \mathbf{m}^* = \mathbf{b} - \mu_{\mathbf{o}} \mathbf{h}^*. \tag{2.26}$$

The Maxwell equations for the Lorentz and Chu formulations do not resemble those for the undeformed body.

For example, $(2.17)_4$ for the Lorentz formulation is

$$\mu_o^{-1}\nabla \times \mathbf{b} - \epsilon_o \frac{\partial \mathbf{e}}{\partial t} = \mathbf{j} + \frac{\partial \mathbf{p}}{\partial t} + \nabla \times (\mathbf{p} \times \mathbf{v}) + \nabla \times \mathbf{m}, \tag{2.27}$$

and $(2.17)_2$ for the Chu formulation is

$$\nabla \times \mathbf{e} + \mu_o \frac{\partial \mathbf{h}}{\partial t} = -\frac{\partial(\mu_o \mathbf{m})}{\partial t} - \nabla \times (\mu_o \mathbf{m} \times \mathbf{v}). \tag{2.28}$$

Although the form of Maxwell equations for the undeformed case and Minkowski formulation are same, there is difference in the constitutive relation, as we now demonstrate: Temporarily ignoring thermomechanical dependence until the next chapter, constitutive assumptions in the electromagnetic problem must relate $\mathbf{d}, \mathbf{b}$ to $\mathbf{e}, \mathbf{h}$ in the moving frame. The most general nonlinear couple dependence is

$$\begin{aligned} \mathbf{d} &= \hat{\mathbf{d}}(\mathbf{e}, \mathbf{h}), \\ \mathbf{b} &= \hat{\mathbf{b}}(\mathbf{e}, \mathbf{h}), \\ \mathbf{j} &= \hat{\mathbf{j}}(\mathbf{e}, \mathbf{h}). \end{aligned} \tag{2.29}$$

To satisfy invariance, the constitutive equations must be cast in the invariant frame,

$$\mathbf{d}' = \hat{\mathbf{d}}(\mathbf{e}', \mathbf{h}'), \qquad \mathbf{b}' = \hat{\mathbf{b}}(\mathbf{e}', \mathbf{h}'), \qquad \mathbf{j}' = \hat{\mathbf{j}}(\mathbf{e}', \mathbf{h}'), \tag{2.30}$$

The relation between the variables in the moving frame and the laboratory frame is given by the Lorentz transformation as follows

$$\mathbf{e}' = \mathbf{e} + \mathbf{v} \times \mathbf{b}, \qquad \mathbf{h}' = \mathbf{h} - \mathbf{v} \times \mathbf{d}, \tag{2.31}$$

$$\mathbf{d}' = \mathbf{d} + \mathbf{v} \times \frac{\mathbf{h}}{c^2}, \qquad \mathbf{b}' = \mathbf{b} - \mathbf{v} \times \frac{\mathbf{e}}{c^2}, \tag{2.32}$$

$$\mathbf{j}' = \mathbf{j} - \sigma\mathbf{v}, \qquad \sigma' = \sigma - \mathbf{v}\cdot\frac{\mathbf{j}}{c^2}. \tag{2.33}$$

The consequence of the invariance requirement is that $\mathbf{e}$, $\mathbf{h}$, and $\mathbf{v}$ must appear in the constitutive assumption only in the combination

$$\begin{aligned}
\mathbf{d} + \mathbf{v}\times\frac{\mathbf{h}}{c^2} &= \hat{\mathbf{d}}(\mathbf{e}+\mathbf{v}\times\mathbf{b}, \mathbf{h}-\mathbf{v}\times\mathbf{d}),\\
\mathbf{b} - \mathbf{v}\times\frac{\mathbf{e}}{c^2} &= \hat{\mathbf{b}}(\mathbf{e}+\mathbf{v}\times\mathbf{b}, \mathbf{h}-\mathbf{v}\times\mathbf{d}),\\
\mathbf{j} - \sigma\mathbf{v} &= \hat{\mathbf{j}}(\mathbf{e}+\mathbf{v}\times\mathbf{b}, \mathbf{h}-\mathbf{v}\times\mathbf{d}),
\end{aligned} \tag{2.34}$$

or

$$\mathbf{d} = \tilde{\mathbf{d}}(\mathbf{e},\mathbf{h},\mathbf{v}), \qquad \mathbf{b} = \tilde{\mathbf{b}}(\mathbf{e},\mathbf{h},\mathbf{v}), \qquad \mathbf{j} = \tilde{\mathbf{j}}(\mathbf{e},\mathbf{h},\mathbf{v}), \tag{2.35}$$

As a special case, the constitutive equations for linear isotropic materials are

$$\mathbf{d} = \epsilon\mathbf{e}, \qquad \mathbf{b} = \mu\mathbf{h}, \qquad \mathbf{j} = \nu\mathbf{e}. \tag{2.36}$$

According to Minkowski formulation, the constitutive equations for linear isotropic materials in a moving frame are

$$\mathbf{d}' = \epsilon\mathbf{e}', \qquad \mathbf{b}' = \mu\mathbf{h}', \qquad \mathbf{j}' = \nu\mathbf{e}'. \tag{2.37}$$

Expressing the above variables in the coordinates of the laboratory frame,

$$\begin{aligned}
\mathbf{d} &= \epsilon\mathbf{e} + (\epsilon\mu - \epsilon_{\mathbf{o}}\mu_{\mathbf{o}})\mathbf{v}\times\mathbf{h},\\
\mathbf{b} &= \mu\mathbf{h} + (\epsilon_{\mathbf{o}}\mu_{\mathbf{o}} - \epsilon\mu)\mathbf{v}\times\mathbf{e},\\
\mathbf{j} - \sigma\mathbf{v} &= \nu(\mathbf{e}+\mathbf{v}\times\mathbf{b}).
\end{aligned} \tag{2.38}$$

Neglecting $\epsilon_o\mu_o = \frac{1}{c^2}$, the above constitutive equations reduce to

$$\mathbf{d} = \epsilon\mathbf{e} + \epsilon\mu(\mathbf{v}\times\mathbf{h}),$$

$$\mathbf{b} = \mu\mathbf{h} - \epsilon\mu(\mathbf{v} \times \mathbf{e}),$$

$$\mathbf{j} - \sigma\mathbf{v} = \nu(\mathbf{e} + \mathbf{v} \times \mathbf{b}). \tag{2.39}$$

In the Minkowski formulation of electro-magnetism in a deforming material that we employ, the effective fields through deforming areas through the rigid body motion given by the Lorentz transformation:

$$\mathbf{e}^* = \mathbf{e}', \qquad \mathbf{h}^* = \mathbf{h}', \qquad \mathbf{j}^* = \mathbf{j}' \tag{2.40}$$

The governing equations (2.17) – (2.21) in the spatial description. A material description is generally more useful in describing the deformation of solids, because the boundary conditions for solids are usually prescribed on the undeformed body, which is generally the body in its reference configuration. In the material description, the governing equations of mechanics and the entropy balance law [24] in pointwise form are

$$\rho_o = \rho J,$$

$$\rho_o \dot{\mathbf{v}} = \rho_o(\mathbf{f}^{ext} + \mathbf{f}^e) + \nabla \cdot \mathbf{P},$$

$$P_{[i\alpha}F_{j]\alpha} = \rho_o L^e_{ij},$$

$$\rho_o \dot{\varepsilon} = \rho_o(r^e + r^{ext}) - \nabla \cdot \mathbf{Q} + \mathbf{P} \cdot \dot{\mathbf{F}}, \tag{2.41}$$

$$\rho_o \dot{\eta} \;+\; \nabla \cdot (\frac{\mathbf{Q}}{\theta}) \;-\; \rho_o(\frac{r^{ext}}{\theta} + \xi) \;=\; 0, \tag{2.42}$$

where

$$\mathbf{P} = J\mathbf{T}\mathbf{F}^{-T}, \qquad \mathbf{Q} = J\mathbf{F}^{-1}\mathbf{q}, \qquad \bar{\sigma} = J\sigma, \qquad \bar{\mathbf{j}} = J\mathbf{F}^{-1}\mathbf{j}^*, \qquad \bar{\mathbf{p}} = J\mathbf{F}^{-1}\mathbf{p},$$
$$\bar{\mathbf{e}} = \mathbf{F}^T\mathbf{e}^*, \qquad \bar{\mathbf{d}} = J\mathbf{F}^{-1}\mathbf{d}, \qquad \bar{\mathbf{h}} = \mathbf{F}^T\mathbf{h}^*, \qquad \bar{\mathbf{b}} = J\mathbf{F}^{-1}\mathbf{b}, \qquad \bar{\mathbf{m}} = J\mathbf{F}^{-1}\mathbf{m}^*, \tag{2.43}$$

$$\rho_o \mathbf{f}^e = \mathbf{F}^{-T}(\bar{\sigma}\bar{\mathbf{e}}+\bar{\mathbf{j}}\times\bar{\mathbf{b}}+(\nabla\bar{\mathbf{e}}^T)\bar{\mathbf{p}}+\mu_o(\nabla\bar{\mathbf{h}}^T)\bar{\mathbf{m}}+\bar{\mathbf{d}}\times\dot{\bar{\mathbf{b}}}+\dot{\bar{\mathbf{d}}}\times\bar{\mathbf{b}})+\nabla\mathbf{F}^{-\mathbf{T}}\mathbf{F}(\bar{\mathbf{p}}\otimes\bar{\mathbf{e}}+\mu_{\mathbf{o}}\bar{\mathbf{m}}\otimes\bar{\mathbf{h}}),$$

$$\rho_o L^e_{ij} = \mathbf{F}_{[i\alpha}\mathbf{F}^{-1}_{\beta j]}(\bar{\mathbf{p}}\otimes\bar{\mathbf{e}}+\mu_o\bar{\mathbf{m}}\otimes\bar{\mathbf{h}}),$$

$$\rho_o r^e = \bar{\mathbf{j}}\cdot\bar{\mathbf{e}}+\dot{\bar{\mathbf{p}}}\cdot\bar{\mathbf{e}}+\mu_o\dot{\bar{\mathbf{m}}}\cdot\bar{\mathbf{h}}+\mathbf{F}^{-\mathbf{T}}(\bar{\mathbf{e}}\otimes\bar{\mathbf{p}}+\mu_{\mathbf{o}}\bar{\mathbf{h}}\otimes\bar{\mathbf{m}})\cdot\dot{\mathbf{F}}. \tag{2.44}$$

## 2.3 Jump conditions

In order to model physical problems where finite jumps arise in any of the electromagnetic field quantities or mechanical quantities in the governing equations, we allow a surface of discontinuity in the body. Since the presence of such a surface violates the smoothness assumptions, we cannot apply the transport theorem and the localization theorem to the integral forms of the governing equations and hence the pointwise equations which we presented above do not hold. We derive jump conditions by using a new transport theorem which allows for the surface of discontinuity, and the localization theorem by assuming that the jump is a continous function on the surface of disconstinuity. If the surface of discontinuity is material, then these jump conditions serve as boundary conditions.

The jump conditions for the Maxwell equations and the balance laws of mass, momentum, and energy [24] are

$$\epsilon_{\alpha\beta\gamma}[[\bar{e}_\beta]]N_\gamma \ + \ [[\bar{b}_\alpha W_N]] \ = \ 0, \qquad [[\bar{b}_\alpha]]N_\alpha \ = \ 0,$$

$$\epsilon_{\alpha\beta\gamma}[[\bar{h}_\beta]]N_\gamma \ + \ [[\bar{d}_\alpha W_N]] \ = \ 0, \qquad [[\bar{d}_\alpha]]N_\alpha \ = \ 0,$$

$$[[\rho_o W_N)]] \ = \ 0,$$

$$[[\rho_o v_i W_N]] - [[T_{i\alpha} + F^{-1}_{\beta i}(\bar{d}_\alpha\bar{e}_\beta + \bar{b}_\alpha\bar{h}_\beta) - \frac{1}{2}\mid\bar{j}\mid F^{-1}_{\alpha i}C^{-1}_\beta\gamma(\epsilon_o\bar{e}_\beta\bar{e}_\gamma + \mu_o\bar{h}_\beta\bar{h}_\gamma)]]N_\alpha = 0$$

$$[[(\frac{1}{2}\rho_o v_i v_i + \rho_o \varepsilon + \frac{1}{2} \mid \bar{j} \mid C^{-1}_{\alpha\beta}(\epsilon_o \bar{e}_\alpha \bar{e}_\beta + \mu_o \bar{h}_\alpha \bar{h}_\beta))W_N]] - [[v_i T_{i\alpha} - Q_\alpha - e_{\alpha\beta\gamma}\bar{e}_\beta \bar{h}_\gamma$$

$$+F^{-1}_{\beta i}(\bar{d}_\alpha \bar{e}_\beta + \bar{b}_\alpha \bar{h}_\beta)v_i - \frac{1}{2} \mid \bar{j} \mid F^{-1}_{\alpha i} C^{-1}_\beta \gamma(\epsilon_o \bar{e}_\beta \bar{e}_\gamma + \mu_o \bar{h}_\beta \bar{h}_\gamma)v_i]]N_\alpha = 0 \tag{2.45}$$

## 2.4 Alternative formulation for governing equations

In section (2.1-2.2), we present the equation for conservation of energy governing TEMM materials involving the electromagnetic variables $\mathbf{e}$, $\mathbf{h}$, $\mathbf{p}$, and $\mathbf{m}$. In this section, we present an alternative formulation that has the equation for conservation of energy involving the electromagnetic variables $\mathbf{e}$, $\mathbf{h}$, $\mathbf{d}$, and $\mathbf{b}$ following the work of [18]. We do this in order to accomodate the family of potentials that involves various combinations of the electromagnetic variables presented in both formulations.

Let $\zeta_t$ be an arbitrary material subset of body $\mathcal{B}$ at time t in the current configuration. For any subset $\zeta_t$ of $\mathcal{B}$, denote $K(\zeta_t)$ as the kinetic energy in $\zeta_t$, $H(\zeta_t)$ as the heat energy in $\zeta_t$, $E(\zeta_t)$ as the electromagnetic energy in $\zeta_t$, $R(\zeta_t)$ as the external rate of supply of mechanical work to $\zeta_t$, $Q(\zeta_t)$ as the external rate of supply of heat to $\zeta_t$, $T(\zeta_t)$ as the external rate of supply of electromagnetic energy to $\zeta_t$, and $W(\zeta_t)$ as the internal rate of supply of energy (mechanical, thermal and electromagnetic) within $\zeta_t$. These terms are

$$K(\zeta_t) = \int_{\mathcal{V}} \frac{1}{2}\rho \mathbf{v}\cdot\mathbf{v}dv, \quad H(\zeta_t) = \int_{\mathcal{V}} \rho\eta\theta dv, \quad E(\zeta_t) = \int_{\mathcal{V}} (\mathbf{d}\cdot\mathbf{e}^* + \mathbf{b}\cdot\mathbf{h}^*)dv, \tag{2.46}$$

$$R(\zeta_t) = \int_{\mathcal{V}} \rho\hat{\mathbf{b}}\cdot\mathbf{v}dv + \int_{\partial\mathcal{V}} \mathbf{t}\cdot\mathbf{v}d\mathbf{a}, \qquad Q(\zeta_t) = \int_{\mathcal{V}} \rho s\theta dv - \int_{\partial\mathcal{V}} k\theta d\mathbf{a}, \tag{2.47}$$

$$T(\zeta_t) = -\int_{\mathcal{V}} e\mathbf{e}^*\cdot\mathbf{v}dv + \int_{\partial\mathcal{V}} \{(\mathbf{h}^*\times\mathbf{e}^*).\mathbf{n} + \mathbf{t}_e\cdot\mathbf{v}\}d\mathbf{a}, \tag{2.48}$$

$$W(\zeta_t) = \int_{\mathcal{V}} (\rho\mathbf{f}\cdot\mathbf{v} + \rho\xi\theta - \mathbf{j}^*\cdot\mathbf{e}^* + \rho w)dv, \tag{2.49}$$

where $\rho$, $\eta$, s, k, $\xi$, w, $\theta$, e are scalar-valued functions, $\mathbf{d}$, $\mathbf{b}$, $\mathbf{j}^*$, $\mathbf{t}_e$, $\hat{\mathbf{b}}$, $\mathbf{t}$, $\mathbf{f}$ are vector-valued functions of $(\mathbf{X}, t)$, and $\mathbf{t}$, $\mathbf{t}_e$ and k depend on the outward unit normal $\mathbf{n}$. The vector $\mathbf{t}_e$ is defined by

$$\mathbf{t}_e = \mathbf{T}_e\mathbf{n}, \qquad \mathbf{T}_e = \mathbf{e}^* \otimes \mathbf{d} + \mathbf{h}^* \otimes \mathbf{b}. \tag{2.50}$$

It was observed that the integrands in $(2.46)_{2,3}$, $(2.47)_{1,2}$, and (2.48) are linear in the independent variables

$$\{\mathbf{v},\ \ \theta,\ \ \mathbf{e}^*,\ \ \mathbf{h}^*\}, \tag{2.51}$$

but no explicit statement has been made in regard to the dependence of the scalar $w$ in (2.49). Since the first three terms in the integrand (2.49) are already linear in the variables (2.51), with a reasonable generality, $w$ is assumed to be linear in the set of variables

$$\{\dot{\theta},\ \ \frac{\partial \theta}{\partial x},\ \ \frac{\partial \mathbf{v}}{\partial x},\ \ \dot{\mathbf{e}}^*,\ \ \dot{\mathbf{h}}^*\}, \tag{2.52}$$

and has the form

$$\rho w = \rho\eta_1\dot{\theta} + \mathbf{p}_1 \cdot \frac{\partial \theta}{\partial x} - \mathbf{T}_1 \cdot \frac{\partial \mathbf{v}}{\partial \mathbf{x}} + \mathbf{d}_1 \cdot \dot{\mathbf{e}}^* + \mathbf{b}_1 \cdot \dot{\mathbf{h}}^*, \tag{2.53}$$

where $\mathbf{p}_1 = \mathbf{p}_1(\mathbf{X}, t)$, $\mathbf{d}_1 = \mathbf{d}_1(\mathbf{X}, t)$, $\mathbf{b}_1 = \mathbf{b}_1(\mathbf{X}, t)$ are vector valued functions of $(\mathbf{X}, t)$, $\mathbf{T}_1 = \mathbf{T}_1(\mathbf{X}, t)$ is a second order tensor valued function.

The first law of thermodynamics in presence of electromagnetic effects states that [18]:

(i) For any subset $\zeta_t$ of the body $\mathcal{B}$ in the current configuration, the rate of change of kinetic energy $K(\zeta_t)$, heat energy $H(\zeta_t)$, and electromagnetic energy $E(\zeta_t)$ is balanced by the external rate of supply of mechanical work $R(\zeta_t)$, the external rate of supply of electromagnetic energy $T(\zeta_t)$, the external rate of supply of heat $Q(\zeta_t)$, and internal rate of supply of energy (mechanical, heat and electromagnetic) $W(\zeta_t)$,

(ii) The total heat energy Q, mechanical work R and rate of supply of electromagnetic energy T supplied to or extracted from $\zeta_t$ in a cycle is zero. Expressing statement $(i)$ mathematically,

$$-\frac{d}{dt}[K(\zeta_t) + H(\zeta_t) + E(\zeta_t)] + R(\zeta_t) + Q(\zeta_t) + T(\zeta_t) + W(\zeta_t) = 0, \tag{2.54}$$

or equivalently

$$\begin{aligned} \mathcal{E}(t) = -\frac{d}{dt}\int_{\mathcal{V}}[\rho(\frac{1}{2}\mathbf{v}\cdot\mathbf{v} + \mathbf{d}\cdot\mathbf{e}^* + \mathbf{b}\cdot\mathbf{h}^*)]dv + \\ \int_{\mathcal{V}}[\rho(\hat{\mathbf{b}}\cdot\mathbf{v} + \mathbf{f}\cdot\mathbf{v} + s\theta + \xi\theta + w) - e\mathbf{e}^*\cdot\mathbf{v} - \mathbf{j}^*\cdot\mathbf{e}^*]dv + \\ \int_{\partial\mathcal{V}}[\mathbf{t}\cdot\mathbf{v} - k\theta + (\mathbf{h}^*\times\mathbf{e}^*)\cdot\mathbf{n} + \mathbf{t}_e\cdot\mathbf{v}]d\mathbf{a} = 0. \end{aligned} \tag{2.55}$$

The integrands in (2.55) are functions of (2.51) with coefficients that are independent of these variables, although $\rho w$ does depend on their space and time derivatives. The energy balance (2.55) is valid for every choice of the variables (2.51); but, because of the expression in (2.53), in general $\mathcal{E}(t)$ will change its form. However, if (2.51) is replaced by

$$\{\mathbf{v} + \bar{\mathbf{c}},\ \ \theta + \bar{c},\ \ \mathbf{e}^* + e_1,\ \ \mathbf{h}^* + h_1\}, \tag{2.56}$$

$\mathcal{E}(t)$ is form invariant. Thus introducing (2.56) into (2.55), the equation is valid for every choice of $\{\bar{\mathbf{c}},\ \bar{c},\ \mathbf{e_1},\ \mathbf{h_1}\}$ and hence the integral expressions were obtained. With the usual smoothness assumptions and boundedness properties, the application of these integral expressions to an arbitrary tetrahedron resulted in

$$\mathbf{t} = \mathbf{T}\mathbf{n}, \qquad k = \mathbf{p}\cdot\mathbf{n}, \tag{2.57}$$

where $\mathbf{T}$ is a second order tensor function and $\mathbf{p}$ is a vector function of $(\mathbf{X}, t)$. Upon substituting (2.57) in the integral expressions (2.55) and applying the divergence theorem, transport theorem, and the localization theorem, we obtain their corresponding local forms. We obtain local form of energy equation (2.55) using a similar approach. Then the statement $(ii)$ is used to obtain

$$W(\zeta_t) = -\frac{d}{dt}\boldsymbol{\Psi}(\zeta_t), \qquad \boldsymbol{\Psi}(\zeta_t) = \int_{\mathcal{P}} \rho\psi dv. \tag{2.58}$$

Combining the local form of energy equation and (2.58), the reduced energy equation is obtained. Then, the unknown quantities are identified as shown in the following table.

| | |
|---|---|
| $\rho$ | mass density in the current configuration of $\mathcal{B}$ |
| $\hat{\mathbf{b}}$ | external body force per unit mass |
| $\rho\mathbf{f}_e + e\mathbf{e}^*$ | internal body force per unit mass in $\mathcal{V}$ due to electromagnetic fields |
| $\mathbf{t}$ | surface force (or the stress vector) per unit area over $\partial\mathcal{V}$ |
| $\mathbf{f}$ | internal force per unit mass in $\mathcal{V}$ |
| $s$ | external rate of supply of entropy per unit mass |
| $\xi$ | internal rate of supply of entropy per unit mass |
| $k$ | flux of entropy per unit area across $\partial\mathcal{V}$ |
| $\mathbf{T}$ | stress tensor per unit area $\partial\mathcal{V}$ |
| $\mathbf{p}$ | entropy flux vector per unit area of $\partial\mathcal{V}$ |
| $\eta$ | density of entropy per unit mass |
| $s\theta(=r)$ | external rate of supply of heat per unit mass |
| $k\theta(=h)$ | flux of heat per unit area across $\partial\mathcal{V}$ |
| $\eta\theta$ | heat density per unit mass |
| $\psi$ | Helmholtz free energy per unit mass |
| $\mathbf{d}$ | electric displacement |
| $\mathbf{b}$ | magnetic induction |
| e | free charge |

Table 2.1: Quantities - Minkowski formulation

Under non-relativistic approximation, the reduced energy equation was considered invariant under a special Galilean transformation,

$$\mathbf{x}^{+} = \bar{\mathbf{a}}t + \mathbf{Q}\mathbf{x}, \qquad t^{+} = t, \tag{2.59}$$

where $\bar{\mathbf{a}}$ is a constant vector and $\mathbf{Q}$ is a constant proper orthogonal tensor. The above identified variables under this transformation are substituted back into the reduced energy equation, and it is found that $\mathbf{f} = \mathbf{0}$. The final forms of the integral expressions are

$$\begin{aligned}
\frac{d}{dt}\int_{\mathcal{P}} \rho dv &= 0, \\
\frac{d}{dt}\int_{\mathcal{P}} \rho \mathbf{v} dv &= \int_{\mathcal{P}} \{\rho \hat{\mathbf{b}} - e\mathbf{e}^{*}\} dv + \int_{\partial\mathcal{P}} (\mathbf{t} + \mathbf{t}_{e}) da, \\
\frac{d}{dt}\int_{\mathcal{P}} \rho \eta dv &= \int_{\mathcal{P}} \rho(s + \xi) dv - \int_{\partial\mathcal{P}} k da, \\
\frac{d}{dt}\int_{\mathcal{P}} \mathbf{d} dv &= -\int_{\mathcal{P}} (e\mathbf{v} + \mathbf{j}^{*}) dv + \int_{\partial\mathcal{P}} \{\mathbf{n} \times \mathbf{h}^{*} + (\mathbf{n}.\mathbf{d})\mathbf{v}\} d\mathbf{a}, \\
\frac{d}{dt}\int_{\mathcal{P}} \mathbf{b} dv &= \int_{\partial\mathcal{P}} \{\mathbf{e}^{*} \times \mathbf{n} + (\mathbf{n} \cdot \mathbf{b})\mathbf{v}\} d\mathbf{a}, \\
-\int_{\mathcal{P}} e dv + \int_{\partial\mathcal{P}} \mathbf{n}.\mathbf{d} \;\; d\mathbf{a} &= 0, \\
\int_{\partial\mathcal{P}} \mathbf{n}.\mathbf{b} \;\; d\mathbf{a} &= 0,
\end{aligned} \tag{2.60}$$

and their corresponding local forms are

$$
\begin{aligned}
&\dot{\rho} + \rho div\mathbf{v} = 0, \\
&\rho\dot{\mathbf{v}} = \rho(\hat{\mathbf{b}} + \mathbf{f}_e) + div\mathbf{T}, \\
&\rho\dot{\eta} = \rho(s + \xi) - div\mathbf{p}, \\
&curl\mathbf{h}^* = \dot{\mathbf{d}} + \mathbf{d}div\mathbf{v} - \mathbf{L}\mathbf{d} + \mathbf{j}^*, \\
&-curl\mathbf{e}^* = \dot{\mathbf{b}} + \mathbf{b}div\mathbf{v} - \mathbf{L}\mathbf{b}, \\
&div\mathbf{d} = e, \\
&div\mathbf{b} = 0,
\end{aligned} \tag{2.61}
$$

where

$$
\rho\mathbf{f}_e = \frac{\partial\mathbf{e}^*}{\partial\mathbf{x}}\mathbf{d} + \frac{\partial\mathbf{h}^*}{\partial\mathbf{x}}\mathbf{b}, \tag{2.62}
$$

and the reduced energy equation is

$$
-\rho(\dot{\psi} + \eta\dot{\theta}) + (\mathbf{T} + \mathbf{T}_e)\cdot\mathbf{L} - \mathbf{p}\cdot\frac{\partial\theta}{\partial\mathbf{x}} - \rho\xi\theta - \bar{\mathbf{d}}\cdot\dot{\bar{\mathbf{e}}} - \bar{\mathbf{b}}\cdot\dot{\bar{\mathbf{h}}} + \mathbf{j}^*\cdot\mathbf{e}^* = \mathbf{0}, \tag{2.63}
$$

where

$$
\bar{\mathbf{e}} = \mathbf{F}^T\mathbf{e}^*, \;\; \bar{\mathbf{d}} = \mathbf{F}^{-1}\mathbf{d}, \;\; \bar{\mathbf{h}} = \mathbf{F}^T\mathbf{h}^*, \;\; \bar{\mathbf{b}} = \mathbf{F}^{-1}\mathbf{b}. \tag{2.64}
$$

## 2.5 Summary of governing equations from different authors

The governing equations developed by five authors Pao [37], Hutter [24], Naghdi83 [17], Naghdi95 [18], Smith [44] are summarized and the differences in these equations are presented in the sections that follow.

### 2.5.1 Conservation of mass

Pao, Hutter, Naghdi83, Naghdi95 presented same equation for conservation of mass:

**Integral form**:

$$\frac{d}{dt}\int_{\mathcal{V}} \rho dv = 0 \tag{2.65}$$

**Pointwise form**:

$$\frac{d\rho}{dt} + \rho div\mathbf{v} = 0 \tag{2.66}$$

### 2.5.2 Conservation of linear momentum

| | |
|---|---|
| Integral form | |
| Pao | $\frac{d}{dt}\int_{\mathcal{V}} \rho\mathbf{v}dv = \int_{\partial\mathcal{V}} \mathbf{t}\cdot d\mathbf{a} + \int_{\mathcal{V}} \rho\mathbf{f}^e dv$ |
| Hutter and Naghdi83 | $\frac{d}{dt}\int_{\mathcal{V}} \rho\mathbf{v}dv = \int_{\partial\mathcal{V}} \mathbf{t}\cdot d\mathbf{a} + \int_{\mathcal{V}} \rho(\mathbf{f}^{ext} + \mathbf{f}^e)dv$ |
| Naghdi95 | $\frac{d}{dt}\int_{\mathcal{V}} \rho\mathbf{v}dv = \int_{\partial\mathcal{V}}(\mathbf{t} + \mathbf{t^e})\cdot d\mathbf{a} + \int_{\mathcal{V}}(\rho\mathbf{f}^{ext} - e\mathbf{e}^*)dv$ |
| Pointwise form | |
| Pao | $\rho\dot{\mathbf{v}} = \rho\mathbf{f}^e + div\mathbf{T}$ |
| Hutter and Naghdi83 | $\rho\dot{\mathbf{v}} = \rho(\mathbf{f}^e + \mathbf{f}^{ext}) + div\mathbf{T}$ |
| Naghdi95 | $\rho\dot{\mathbf{v}} = \rho(\mathbf{f}^e + \mathbf{f}^{ext}) + div\mathbf{T}$ |
| $\mathbf{t} = \mathbf{Tn}$ | |
| Pao | $\rho\mathbf{f}^e = \sigma\mathbf{e} + \mathbf{j}\times\mathbf{b} - \frac{1}{2}[\nabla\mathbf{b}\cdot\mathbf{h} - \nabla\mathbf{h}\cdot\mathbf{b} - \nabla\mathbf{e}\cdot\mathbf{d} + \nabla\mathbf{d}\cdot\mathbf{e}]$ |
| Hutter and Naghdi83 | $\rho\mathbf{f}^e = \sigma\mathbf{e}^* + \mathbf{j}^*\times\mathbf{b} + \mathbf{p}\cdot\nabla\mathbf{e}^* + \mu_{\mathbf{o}}\mathbf{m}^*\cdot\nabla\mathbf{h}^* + \mathring{\mathbf{d}}\times\mathbf{b} + \mathbf{d}\times\mathring{\mathbf{b}}$ |
| Naghdi95 | $\mathbf{t}^e = \mathbf{T}^e\mathbf{n}, \qquad \mathbf{T}^e = \mathbf{e}^*\otimes\mathbf{d} + \mathbf{h}^*\otimes\mathbf{b}$<br>$\int_{\mathcal{V}} \rho\mathbf{f}^e dv = \int_{\partial\mathcal{V}} \mathbf{t}^e\cdot d\mathbf{a} - \int_{\mathcal{V}} e\mathbf{e}^* dv$<br>$\rho\mathbf{f}^e = \frac{\partial\mathbf{e}^*}{\partial\mathbf{x}}\mathbf{d} + \frac{\partial\mathbf{h}^*}{\partial\mathbf{x}}\mathbf{b}$ |

Table 2.2: Conservation of linear momentum

The equation for conservation of linear momentum from Hutter and Naghdi83 are same. The balance laws for Hutter are deduced from an energy balance law. According to [24], the balane laws presented in Pao are not invariant in the non-relativistic sense, and hence they can never be deduced from an energy balance. Hence, the equations from Hutter do not agree with those of Pao. Also, Pao did not take external body force into consideration. The conservation of linear momentum equation that is considered in this work is from Hutter and Naghdi83.

### 2.5.3 Conservation of angular momentum

| | |
|---|---|
| Integral form | |
| Pao | $\frac{d}{dt}\int_{\mathcal{V}}(\mathbf{x}\times\rho\mathbf{v})dv = \int_{\partial\mathcal{V}}(\mathbf{x}\times\mathbf{t})\cdot d\mathbf{a} + \int_{\mathcal{V}}\rho(\mathbf{c}^e + \mathbf{x}\times\mathbf{f}^e)dv$ |
| Hutter and Naghdi83 | $\frac{d}{dt}\int_{\mathcal{V}}(\mathbf{x}\times\rho\mathbf{v})dv = \int_{\partial\mathcal{V}}(\mathbf{x}\times\mathbf{t})\cdot d\mathbf{a} + \int_{\mathcal{V}}\rho(\mathbf{c}^e + \mathbf{x}\times(\mathbf{f}^{ext} + \mathbf{f}^e)dv$ |
| Pointwise form | |
| Pao, Hutter, and Naghdi83 | $T_{[ij]} = \rho L^e_{ij}$ |
| $L^e_{ij} = \frac{1}{2}e_{ijk}c^e_k$ or<br>$\mathbf{L}^e\mathbf{u} = \mathbf{c}^e\times\mathbf{u}$ for every vector $\mathbf{u}$ | |
| Pao | $\rho\mathbf{L}^e = \frac{1}{2}(\mathbf{e}\otimes\mathbf{d} + \mathbf{h}\otimes\mathbf{b} - \mathbf{d}\otimes\mathbf{e} - \mathbf{b}\otimes\mathbf{h})$ |
| Hutter | $\rho\mathbf{L}^e = \frac{1}{2}(\mathbf{p}\otimes\mathbf{e}^* - \mathbf{e}^*\otimes\mathbf{p} + \mu_o[\mathbf{m}^*\otimes\mathbf{h}^* - \mathbf{h}^*\otimes\mathbf{m}^*])$, |
| Naghdi83 | $\rho\mathbf{L}^e = \frac{1}{2}(\mathbf{e}^*\otimes\mathbf{d} + \mathbf{h}^*\otimes\mathbf{b} - \mathbf{d}\otimes\mathbf{e}^* - \mathbf{b}\otimes\mathbf{h}^*)$ |
| $(\rho\mathbf{L}^e)_{\text{Naghdi83}} = (\rho\mathbf{L}^e)_{\text{Hutter}}$ | |

Table 2.3: Conservation of angular momentum

The equations for conservation of angular momentum given by Hutter and Naghdi83 are same. Although the form given by Pao looks similar to Hutter or Naghdi83, the expression given by Pao is in reference frame coordinates where as the later is in moving frame coordinates. In this work, the conservation of angular momentum equation from Hutter or Naghdi83 is adopted.

### 2.5.4 Conservation of energy

| | |
|---|---|
| Integral form | |
| Pao | $\frac{d}{dt}\int_{\mathcal{V}}(\rho\varepsilon+\frac{1}{2}\rho\mathbf{v}\cdot\mathbf{v})dv=\int_{\partial\mathcal{V}}(\mathbf{t}\cdot\mathbf{v}-\mathbf{q}\cdot\mathbf{n})da+\int_{\mathcal{V}}\rho(r^q+r^e+\mathbf{f}^e\cdot\mathbf{v})dv$ |
| Hutter and Naghdi83 | $\frac{d}{dt}\int_{\mathcal{V}}(\rho\varepsilon+\frac{1}{2}\rho\mathbf{v}\cdot\mathbf{v})dv=\int_{\partial\mathcal{V}}(\mathbf{t}\cdot\mathbf{v}-\mathbf{q}\cdot\mathbf{n})da+\int_{\mathcal{V}}\rho(r^{ext}+r^e+(\mathbf{f}^{ext}+\mathbf{f}^e)\cdot\mathbf{v})dv$ |
| Hutter-Minkowski formulation | $\frac{d}{dt}\int_{\mathcal{V}}\{\rho\varepsilon+\frac{1}{2}(\epsilon_o\mathbf{e}^*\cdot\mathbf{e}^*+\mu_o\mathbf{h}^*\cdot\mathbf{h}^*)+\frac{1}{2}\rho\mathbf{v}\cdot\mathbf{v}\}dv=\int_{\mathcal{V}}\rho\{r^{ext}+\mathbf{f}^{ext}\cdot\mathbf{v}\}dv$<br>$+\int_{\partial\mathcal{V}}\{\mathbf{T}\mathbf{v}-\mathbf{q}-\mathbf{e}^*\times\mathbf{h}^*+\frac{1}{2}(\epsilon_o\mathbf{e}^*\cdot\mathbf{e}^*+\mu_o\mathbf{h}^*\cdot\mathbf{h}^*)\mathbf{v}\}\cdot d\mathbf{a}$ |
| Naghdi95 | $\frac{d}{dt}\int_{\mathcal{V}}[\rho(\frac{1}{2}\mathbf{v}\cdot\mathbf{v}+\eta\theta)+\mathbf{d}\cdot\mathbf{e}^*+\mathbf{b}\cdot\mathbf{h}^*]dv=\int_{\partial\mathcal{V}}[\mathbf{t}\cdot\mathbf{v}-k\theta+(\mathbf{h}^*\times\mathbf{e}^*)\cdot\mathbf{n}+\mathbf{t}^e\cdot\mathbf{v}]da$<br>$+\int_{\mathcal{V}}[\rho(\mathbf{f}^{ext}\cdot\mathbf{v}+s\theta+\zeta\theta+w)-e\mathbf{e}^*\cdot\mathbf{v}-\mathbf{j}^*\cdot\mathbf{e}^*]dv$ |
| Pointwise form | |
| Pao | $\rho\dot{\varepsilon}=\mathbf{T}\cdot\mathbf{L}-div\mathbf{q}+\rho r^e$ |
| Hutter and Naghdi83 | $\rho\dot{\varepsilon}=\mathbf{T}\cdot\mathbf{L}-div\mathbf{q}+\rho r^e+\rho r^{ext}$ |
| Naghdi95 | $-\rho(\dot{\psi}+\eta\dot{\theta})+\mathbf{T}\cdot\mathbf{L}-\mathbf{p}\cdot\frac{\partial\theta}{\partial\mathbf{x}}-\rho\zeta\theta-\rho\mathbf{f}\cdot\mathbf{v}-\mathbf{d}\cdot\dot{\mathbf{e}}^*-\mathbf{b}\cdot\dot{\mathbf{h}}^*+\mathbf{j}^*\cdot\mathbf{e}^*=0$ |
| Smith | $dU=\sigma\cdot d\epsilon+E\cdot dP+TdS-TdS_{irr}$ |
| electromagnetic energy supply rate $r^e$ | |
| Pao | $\rho r^e=\mathbf{j}^*\cdot\mathbf{e}^*+\frac{1}{2}[\mathbf{h}\cdot\frac{d\mathbf{b}}{dt}-\mathbf{b}\cdot\frac{d\mathbf{h}}{dt}+\mathbf{d}\cdot\frac{d\mathbf{e}}{dt}-\mathbf{e}\cdot\frac{d\mathbf{d}}{dt}]$ |
| Hutter-Minkowski | $\rho r^e=\mathbf{j}^*\cdot\mathbf{e}^*+\rho\mathbf{e}^*\cdot\frac{d}{dt}(\frac{\mathbf{p}}{\rho})+\rho\mu_o\mathbf{h}^*\cdot\frac{\mathbf{d}}{\mathbf{dt}}(\frac{\mathbf{m}^*}{\rho}).$ |
| Naghdi83 | $\rho r^e=\mathbf{j}^*\cdot\mathbf{e}^*+\mathbf{e}^*\cdot\dot{d}+\mathbf{h}^*\cdot\dot{\mathbf{b}}+(\mathbf{e}^*\cdot\mathbf{d}+\mathbf{h}^*\cdot\mathbf{b}-\frac{1}{2}(\epsilon_o\mathbf{e}^*\cdot\mathbf{e}^*+\mu_o\mathbf{h}^*\cdot\mathbf{h}^*))div\mathbf{v}$ |
| Naghdi95 | $\rho w-\rho\eta\dot{\theta}+\mathbf{T}\cdot\mathbf{L}-\mathbf{p}\cdot\frac{\partial\theta}{\partial\mathbf{x}}-\mathbf{d}\cdot\dot{\mathbf{e}}^*-\mathbf{b}\cdot\dot{\mathbf{h}}^*=0$<br>$\rho w+\rho\dot{\psi}+\rho\mathbf{f}\cdot\mathbf{v}+\rho\theta\zeta-\mathbf{j}^*\cdot\mathbf{e}^*=0$ |

Table 2.4: Conservation of energy

The basic form of equations for conservation of energy is same for Pao, Hutter, Naghdi83. The expression for electromagnetic energy rate is different from one author to other including Hutter and Naghdi83. Since Minkowski formulation is followed in this work, the global energy balance law from Hutter and Naghdi95 are presented. In all the equations for conservation of energy, the electromagnetic energy rate is represented by $r^e$ and it is later postulated with respect to a particular formulation in question. But in Minkowski formulation, the electromagnetic energy term is intrinsic in the global energy balance law. In Hutter, by subjecting the global energy balance law to invariance requirements, the pointwise form of conservation of mass, momentum, angular momentum and energy are obtained. In the same way, in Naghdi95, by subjecting the global energy balance law to invariance requirements, first the integral equations for mass, momentum, energy and also Maxwell equations are obtained. Then, the pointwise equations are obtained from the use of divergence, transport and localization theorems. Comparing the Minkowski formulation from Hutter and Naghdi95, it appears that Naghdi95 gave a holistic picture as balance laws of mechanics as well as Maxwell equations are derived from its global energy balance law where as only pointwise equations of balance laws of mechanics are obtained from Hutter. The global energy balance law from Hutter when compared to Naghdi95 has an additional term $\frac{1}{2}(\epsilon_o \mathbf{e}^* \cdot \mathbf{e}^* \;+\; \mu_o \mathbf{h}^* \cdot \mathbf{h}^*)$ on both left hand side as well as right hand side of the equation. Due to this reason, the final form of reduced energy equation from Hutter has the electromagnetic coupling effect in terms of $\mathbf{e}, \mathbf{p}, \mathbf{h}, \mathbf{m}$. The reduced energy equation from Naghdi95 has the electromagnetic coupling effect in terms of $\mathbf{e}, \mathbf{d}, \mathbf{h}, \mathbf{b}$. For the characterization of TEMM materials in the next chapter, all the electromagnetic terms $\mathbf{e}, \mathbf{p}, \mathbf{d}, \mathbf{h}, \mathbf{m}, \mathbf{b}$ are taken into consideration. Hence, depending on the state variables in question, the switching from global energy balance law of Hutter to Naghdi95 is done. In order to check the equivalence of Minkowski formulation given by Hutter and Naghdi95, the additional term as mentioned above is added to Naghdi95's global energy balance law. In such a case, the integral equations of balance laws of mechanics and Maxwell equations cannot be deduced from it.

### 2.5.5 Balance law for entropy

| | |
|---|---|
| Integral form | |
| Hutter, Naghdi83, and Naghdi95 | $\frac{d}{dt}\int_{\mathcal{V}} \rho\eta dv = \int_{\mathcal{V}} \rho(s+\xi)dv - \int_{\partial\mathcal{V}} kda$ |
| Pointwise form | |
| Pao and Hutter | $\rho\dot{\eta} \ + \ \nabla\cdot(\frac{\mathbf{q}}{\theta}) \ - \ \rho(\frac{r^{ext}}{\theta}) \ \geq \ 0$ |
| Naghdi83 and Naghdi95 | $\rho\dot{\eta} \ + \ \nabla\cdot(\frac{\mathbf{q}}{\theta}) \ - \ \rho(\frac{r^{ext}}{\theta}+\xi) \ = \ 0$ |
| Smith | $dS = \frac{dQ}{T} + dS_{irr}$ |

Table 2.5: Balance law of entropy

Naghdi83 and Naghdi95 represented the irreversible part of second law with the term $\xi$. Pao and Hutter replaced this irreversible term with an inequality sign. This is generally known as the Classius-Duhem inequality. Although Smith presented the second law which has the irreversible component in it, he neglected it and considered a reversible second law for the purpose of obtaining linear constitutive equations.

# CHAPTER 3

# CHARACTERIZATION OF RATE-INDEPENDENT TEMM MATERIALS

Chapter 2 presents the governing equations of balance laws of mechanics and Maxwell equations describing a thermo-electro-magneto-mechanical (TEMM) material. This chapter gives overview of thermodynamic potentials and presents a detailed derivation of constitutive equations necessary to evaluate thermodynamic potentials.

The quantities (2.1)-(2.5) are divided into four sets:

1. the fundamental potential function (a scalar function which provides a complete description of the TEMM state)

2. independent variables (the arguments of the fundamental potential function)

3. primary dependent variables (determined by the independent variables through derivatives of the potential function)

4. secondary dependent variables (determined algebraically from the independent and primary dependent variables)

The different choices of this division constitute different formulations of the same characterization of a rate-independent TEMM material.

The potential function can be any one of the four energies, temperature, or entropy [42]. If the potential is Helmholtz free energy $\psi$, the thermomechanical independent variables must be deformation and temperature[2]. Any one of the three electrical vectors and any one of the three magnetic vectors complete the set. Therefore, there are nine possible Helmholtz free energies associated with a TEMM material. For example, if we choose electric field $\mathbf{e}$ and magnetization $\mathbf{m}$ to be the independent

[2]Consider $\varepsilon$ as a function of the mechanical independent variable $\mathbf{F}$, thermal independent variable $\eta$, electrical independent variable $\mathbf{e}$ and magnetic independent variable $\mathbf{m}$ denoted by

$$\varepsilon = \tilde{\varepsilon}(\mathbf{F}, \eta, \bar{\mathbf{e}}, \bar{\mathbf{m}}), \tag{3.1}$$

With regards to this potential, the thermal constitutive equation

$$\frac{\partial \varepsilon}{\partial \eta} = -\theta, \tag{3.2}$$

which gives directly

$$\theta = \tilde{\theta}(\mathbf{F}, \eta, \bar{\mathbf{e}}, \bar{\mathbf{m}}). \tag{3.3}$$

Assuming the function is invertible, one can obtain

$$\eta = \hat{\eta}(\mathbf{F}, \theta, \bar{\mathbf{e}}, \bar{\mathbf{m}}) \tag{3.4}$$

Substituting (3.4) into (3.1), one gets

$$\begin{aligned} \varepsilon &= \tilde{\varepsilon}(\mathbf{F}, \eta = \hat{\eta}(\mathbf{F}, \theta, \bar{\mathbf{e}}, \bar{\mathbf{m}}), \bar{\mathbf{e}}, \bar{\mathbf{m}}), \\ &= \hat{\varepsilon}(\mathbf{F}, \theta, \bar{\mathbf{e}}, \bar{\mathbf{m}}). \end{aligned} \tag{3.5}$$

Therefore internal energy $\varepsilon$ can post facto be expressed as a function of temperature so as to compute $c = \frac{\partial \hat{\varepsilon}}{\partial \theta}$ for instance.

electromagnetic variables, then the Helmholtz free energy is

$$\psi = \psi^{F\theta em}(\mathbf{F}, \theta, \mathbf{e}, \mathbf{m}), \tag{3.6}$$

where $\psi$ is the value of function $\psi^{F\theta em}$ and the superscript $F\theta em$ denotes the particular choice of independent variables. For this particular potential, the primary dependent variables are stress, entropy, polarization and magnetic field.

$$\begin{aligned}
\rho_o \frac{\partial \psi^{F\theta em}}{\partial \mathbf{F}} &= \mathbf{P} + \mathbf{F}^{-T}(\bar{\mathbf{e}} \otimes \bar{\mathbf{p}} + \mu_o \bar{\mathbf{h}} \otimes \bar{\mathbf{m}}), \\
\frac{\partial \psi^{F\theta em}}{\partial \theta} &= -\eta, \\
\rho_o \frac{\partial \psi^{F\theta em}}{\partial \bar{\mathbf{e}}} &= -\mathbf{p}, \\
\rho_o \frac{\partial \psi^{F\theta em}}{\partial \bar{\mathbf{m}}} &= \mu_o \mathbf{h}.
\end{aligned} \tag{3.7}$$

where $\rho_o$ is the mass density in the reference configuration, $\rho = \rho_o \det \mathbf{F}$ is the mass density in the current configuration, and $\mu_o$ is the permeability of free space. If the potential is Gibbs free energy, the independent variables include stress and temperature.

The different choices of this division constitute different characterizations of a rate-independent TEMM material. To generate the possible characterizations:

1. one of stress $\mathbf{P}$ or deformation $\mathbf{F}$ is selected as the mechanical independent variable, with the other identified as the mechanical dependent variable; (conjugates)

2. one of temperature $\theta$, entropy $\eta$, or energies $\varepsilon$, $\psi$, $\phi$, $\chi$ is selected as the thermal independent variable, another as the potential function and the remaining quantities are identified as thermal dependent variables. The particular choice

of thermal and mechanical independent variables dictates which energy is employed:

deformation $\mathbf{F}$, entropy $\eta \Leftrightarrow$ internal energy $\varepsilon$

deformation $\mathbf{F}$, temperature $\theta \Leftrightarrow$ Helmholz free energy $\psi$

stress $\mathbf{P}$, entropy $\eta \Leftrightarrow$ enthalpy $\psi$

stress $\mathbf{P}$, temperature $\theta \Leftrightarrow$ Gibbs free energy

3. one of $\mathbf{e}$, $\mathbf{p}$ or $\mathbf{d}$ is selected as the electrical independent variable, with the other two considered as the electrical dependent variables;

4. one of $\mathbf{m}$, $\mathbf{h}$ or $\mathbf{b}$ is selected as the magnetic independent variable, with the other two considered as the magnetic dependent variables.

Hence from (i) – (iv) the possible characteristics of rate independent TEMM material are grouped according to their potential functions.

The seventy-two characterizations are arranged in eight families, each sharing the same thermomechanical independent variables, and hence the same thermodynamic potential. For example, the formulations of Family1 share the same thermomechanical independent variables deformation and entropy, so that the potential function must be internal energy $\varepsilon$. The notation $\varepsilon^{F\eta em}$, $\varepsilon^{F\eta pm}$, ...., $\varepsilon^{F\eta db}$ in Family1 is to distinguish the nine different energy potential functions that appear in Family 1. The utility of having seventy-two potentials is to characterize potentials based on limits/special cases and to design experiments for full $3D$ characterization

Potential functions

| Family1 | Family2 | Family3 | Family4 | Family5 | Family6 | Family7 | Family8 |
|---|---|---|---|---|---|---|---|
| $\varepsilon^{F\eta pm}$ | $\eta^{F\varepsilon pm}$ | $\psi^{F\theta pm}$ | $\theta^{F\psi pm}$ | $\chi^{P\eta pm}$ | $\eta^{P\chi pm}$ | $\phi^{P\theta pm}$ | $\theta^{P\phi pm}$ |
| $\varepsilon^{F\eta em}$ | $\eta^{F\varepsilon em}$ | $\psi^{F\theta em}$ | $\theta^{F\psi em}$ | $\chi^{P\eta em}$ | $\eta^{P\chi em}$ | $\phi^{P\theta em}$ | $\theta^{P\phi em}$ |
| $\varepsilon^{F\eta ph}$ | $\eta^{F\varepsilon ph}$ | $\psi^{F\theta ph}$ | $\theta^{F\psi ph}$ | $\chi^{P\eta ph}$ | $\eta^{P\chi ph}$ | $\phi^{P\theta ph}$ | $\theta^{P\phi ph}$ |
| $\varepsilon^{F\eta eh}$ | $\eta^{F\varepsilon eh}$ | $\psi^{F\theta eh}$ | $\theta^{F\psi eh}$ | $\chi^{P\eta eh}$ | $\eta^{P\chi eh}$ | $\phi^{P\theta eh}$ | $\theta^{P\phi eh}$ |
| $\varepsilon^{F\eta dm}$ | $\eta^{F\varepsilon dm}$ | $\psi^{F\theta dm}$ | $\theta^{F\psi dm}$ | $\chi^{P\eta dm}$ | $\eta^{P\chi dm}$ | $\phi^{P\theta dm}$ | $\theta^{P\phi dm}$ |
| $\varepsilon^{F\eta pb}$ | $\eta^{F\varepsilon pb}$ | $\psi^{F\theta pb}$ | $\theta^{F\psi pb}$ | $\chi^{P\eta pb}$ | $\eta^{P\chi pb}$ | $\phi^{P\theta pb}$ | $\theta^{P\phi pb}$ |
| $\varepsilon^{F\eta dh}$ | $\eta^{F\varepsilon dh}$ | $\psi^{F\theta dh}$ | $\theta^{F\psi dh}$ | $\chi^{P\eta dh}$ | $\eta^{P\chi dh}$ | $\phi^{P\theta dh}$ | $\theta^{P\phi dh}$ |
| $\varepsilon^{F\eta eb}$ | $\eta^{F\varepsilon eb}$ | $\psi^{F\theta eb}$ | $\theta^{F\psi eb}$ | $\chi^{P\eta eb}$ | $\eta^{P\chi eb}$ | $\phi^{P\theta eb}$ | $\theta^{P\phi eb}$ |
| $\varepsilon^{F\eta db}$ | $\eta^{F\varepsilon db}$ | $\psi^{F\theta db}$ | $\theta^{F\psi db}$ | $\chi^{P\eta db}$ | $\eta^{P\chi db}$ | $\phi^{P\theta db}$ | $\theta^{P\phi db}$ |

Table 3.1: Potential Functions

## 3.1 Background

For a set of $n$ independent variables and corresponding conjugates, there will be $2^n$ potentials. Consider the case when $n = 3$, and the conjugate pairs are given by $(1a, 2a, 3a)$ and $(1b, 2b, 3b)$. There are eight possible potentials, and each potential will yield three constitutive relations. The potentials are denoted $\varepsilon_i$.

$$\begin{aligned}
\varepsilon_1 &= \varepsilon_1(1a, 2a, 3a) \\
\varepsilon_2 &= \varepsilon_2(1a, 2a, 3b) \\
\varepsilon_3 &= \varepsilon_3(1a, 2b, 3a) \\
\varepsilon_4 &= \varepsilon_4(1a, 2b, 3b) \\
\varepsilon_5 &= \varepsilon_5(1b, 2a, 3a) \\
\varepsilon_6 &= \varepsilon_6(1b, 2a, 3b) \\
\varepsilon_7 &= \varepsilon_7(1b, 2b, 3a) \\
\varepsilon_8 &= \varepsilon_8(1b, 2b, 3b)
\end{aligned}$$

Constitutive relations:

$$
\begin{array}{lll}
1b = \left.\dfrac{\partial \varepsilon_1}{\partial 1a}\right|_{2a,3a} & 2b = \left.\dfrac{\partial \varepsilon_1}{\partial 2a}\right|_{1a,3a} & 3b = \left.\dfrac{\partial \varepsilon_1}{\partial 3a}\right|_{1a,2a} \\
1b = \left.\dfrac{\partial \varepsilon_2}{\partial 1a}\right|_{2a,3b} & 2b = \left.\dfrac{\partial \varepsilon_2}{\partial 2a}\right|_{1a,3b} & 3a = \left.\dfrac{\partial \varepsilon_2}{\partial 3b}\right|_{1a,2a} \\
1b = \left.\dfrac{\partial \varepsilon_3}{\partial 1a}\right|_{2b,3a} & 2a = \left.\dfrac{\partial \varepsilon_3}{\partial 2b}\right|_{1a,3a} & 3b = \left.\dfrac{\partial \varepsilon_3}{\partial 3a}\right|_{1a,2b} \\
1b = \left.\dfrac{\partial \varepsilon_4}{\partial 1a}\right|_{2b,3b} & 2a = \left.\dfrac{\partial \varepsilon_4}{\partial 2b}\right|_{1a,3b} & 3a = \left.\dfrac{\partial \varepsilon_4}{\partial 3b}\right|_{1a,2b} \\
1a = \left.\dfrac{\partial \varepsilon_5}{\partial 1b}\right|_{2a,3a} & 2b = \left.\dfrac{\partial \varepsilon_5}{\partial 2a}\right|_{1b,3a} & 3b = \left.\dfrac{\partial \varepsilon_5}{\partial 3a}\right|_{1b,2a} \\
1a = \left.\dfrac{\partial \varepsilon_6}{\partial 1b}\right|_{2a,3b} & 2b = \left.\dfrac{\partial \varepsilon_6}{\partial 2a}\right|_{1b,3b} & 3a = \left.\dfrac{\partial \varepsilon_6}{\partial 3b}\right|_{1b,2a} \\
1a = \left.\dfrac{\partial \varepsilon_7}{\partial 1b}\right|_{2b,3a} & 2a = \left.\dfrac{\partial \varepsilon_7}{\partial 2b}\right|_{1b,3a} & 3b = \left.\dfrac{\partial \varepsilon_7}{\partial 3a}\right|_{1b,2b} \\
1a = \left.\dfrac{\partial \varepsilon_8}{\partial 1b}\right|_{2b,3b} & 2a = \left.\dfrac{\partial \varepsilon_8}{\partial 2b}\right|_{1b,3b} & 3a = \left.\dfrac{\partial \varepsilon_8}{\partial 3b}\right|_{1b,2b}
\end{array}
$$

## 3.2 Unconstrained material

The conservation of energy equation in material description (2.41) is

$$\rho_o\dot{\varepsilon} \;=( \;\mathbf{P}+\;\mathbf{F}^{-T}(\bar{\mathbf{e}}\otimes\bar{\mathbf{p}}\;+\;\mu_o\bar{\mathbf{h}}\otimes\bar{\mathbf{m}}))\cdot\dot{\mathbf{F}}\;-\;\nabla\cdot Q\;+\;\rho_o r^{ext}\;+\;\bar{\mathbf{j}}\cdot\bar{\mathbf{e}}\;+\;\dot{\bar{\mathbf{p}}}\cdot\bar{\mathbf{e}}\;+\;\mu_o\dot{\bar{\mathbf{m}}}\cdot\bar{\mathbf{h}}. \tag{3.8}$$

The entropy balance equation (2.42) multiplied by temperature is

$$\rho_o\dot{\eta}\theta\;+\;\nabla\cdot Q\;-\;\frac{Q}{\theta}\frac{\partial\theta}{\partial\mathbf{X}}\;-\;\rho_o r^{ext}-\rho_o\xi\theta\;=\;0. \tag{3.9}$$

Combining (3.8) and (3.9),

$$-\rho_o\dot{\varepsilon}\;+\;(\mathbf{P}+\mathbf{F}^{-T}(\bar{\mathbf{e}}\otimes\bar{\mathbf{p}}\;+\;\mu_o\bar{\mathbf{h}}\otimes\bar{\mathbf{m}}))\cdot\dot{\mathbf{F}}\;+\;\rho_o\dot{\eta}\theta\;+\;\dot{\bar{\mathbf{p}}}\cdot\bar{\mathbf{e}}\;+\;\mu_o\dot{\bar{\mathbf{m}}}\cdot\bar{\mathbf{h}}\;+\;B\;=\;0, \tag{3.10}$$

where

$$B=\bar{\mathbf{j}}\cdot\bar{\mathbf{e}}\;-\;\frac{\mathbf{Q}}{\theta}\frac{\partial\theta}{\partial\mathbf{X}}\;-\;\rho_o\xi\theta. \tag{3.11}$$

The relation between various potential energies is as follows:

$$\rho_o\psi=\rho_o\varepsilon-\rho_o\eta\theta, \tag{3.12}$$

$$\rho_o\chi=\rho_o\varepsilon-\mathbf{P}\cdot\mathbf{F}, \tag{3.13}$$

$$\rho_o\phi=\rho_o\varepsilon-\mathbf{P}\cdot\mathbf{F}-\rho_o\eta\theta. \tag{3.14}$$

Substituting (3.12) in (3.10),

$$-\rho_o\dot{\psi}\;+\;(\mathbf{P}+\;\mathbf{F}^{-T}(\bar{\mathbf{e}}\otimes\bar{\mathbf{p}}\;+\;\mu_o\bar{\mathbf{h}}\otimes\bar{\mathbf{m}}))\cdot\dot{\mathbf{F}}\;-\;\rho_o\eta\dot{\theta}\;+\;\bar{\mathbf{e}}\cdot\dot{\bar{\mathbf{p}}}\;+\;\mu_o\bar{\mathbf{h}}\cdot\dot{\bar{\mathbf{m}}}\;+\;B\;=\;0. \tag{3.15}$$

Substituting (3.13) in (3.10),

$$-\rho_o\dot{\chi}\;-\;\dot{\mathbf{P}}\cdot\mathbf{F}\;+\;\rho_o\dot{\eta}\theta\;+\;\bar{\mathbf{e}}\cdot\dot{\bar{\mathbf{p}}}\;+\;\mu_o\bar{\mathbf{h}}\cdot\dot{\bar{\mathbf{m}}}+\mathbf{F^{-1}LF}\bar{\mathbf{p}}\cdot\bar{\mathbf{e}}+\mu_\mathbf{o}\mathbf{F^{-1}LF}\bar{\mathbf{m}}\cdot\bar{\mathbf{h}}\;+\;B\;=\;0. \tag{3.16}$$

Substituting (3.14) in (3.10),

$$-\rho_o\dot{\phi} \;-\; \dot{\mathbf{P}}\cdot\mathbf{F} \;-\; \rho_o\eta\dot{\theta} \;+\; \bar{\mathbf{e}}\cdot\dot{\bar{\mathbf{p}}} \;+\; \mu_o\bar{\mathbf{h}}\cdot\dot{\bar{\mathbf{m}}}+\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{p}}\cdot\bar{\mathbf{e}}+\mu_{\mathbf{o}}\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{m}}\cdot\bar{\mathbf{h}} \;+\; B \;=\; 0. \tag{3.17}$$

### 3.2.1 Family 1

**Case (a)** :

Consider $\varepsilon = \varepsilon(\mathbf{F},\eta,\bar{\mathbf{p}},\bar{\mathbf{m}})$, one can write

$$\rho_o\dot{\varepsilon}^{F\eta pm} \;=\; \rho_o\frac{\partial\varepsilon^{F\eta pm}}{\partial\mathbf{F}}\dot{\mathbf{F}} \;+\; \rho_o\frac{\partial\varepsilon^{F\eta pm}}{\partial\eta}\dot{\eta} \;+\; \rho_o\frac{\partial\varepsilon^{F\eta pm}}{\partial\bar{\mathbf{p}}}\dot{\bar{\mathbf{p}}} \;+\; \rho_o\frac{\partial\varepsilon^{F\eta pm}}{\partial\bar{\mathbf{m}}}\dot{\bar{\mathbf{m}}}. \tag{3.18}$$

Substituting (3.18) into (3.10), and rearranging gives

$$\begin{aligned}-(\rho_o\frac{\partial\varepsilon^{F\eta pm}}{\partial\mathbf{F}} - (\mathbf{P}+\mathbf{F}^{-T}(\bar{\mathbf{e}}\otimes\bar{\mathbf{p}} \;+\; \mu_o\bar{\mathbf{h}}\otimes\bar{\mathbf{m}}))\cdot\dot{\mathbf{F}} \;-\; (\rho_o\frac{\partial\varepsilon^{F\eta pm}}{\partial\eta} - \rho_o\theta)\dot{\eta} \\ -\; (\rho_o\frac{\partial\varepsilon^{F\eta pm}}{\partial\bar{\mathbf{p}}} - \bar{\mathbf{e}})\cdot\dot{\bar{\mathbf{p}}} \;-\; (\rho_o\frac{\partial\varepsilon^{F\eta pm}}{\partial\bar{\mathbf{m}}} - \mu_o\bar{\mathbf{h}})\cdot\dot{\bar{\mathbf{m}}} \;+\; B \;=\; 0.\end{aligned} \tag{3.19}$$

The constitutive equations thus have the form

$$\begin{aligned}\rho_o\frac{\partial\varepsilon^{F\eta pm}}{\partial\mathbf{F}} &= \mathbf{P}+\mathbf{F}^{-T}(\bar{\mathbf{e}}\otimes\bar{\mathbf{p}} \;+\; \mu_o\bar{\mathbf{h}}\otimes\bar{\mathbf{m}}), \\ \frac{\partial\varepsilon^{F\eta pm}}{\partial\eta} &= \theta, \\ \rho_o\frac{\partial\varepsilon^{F\eta pm}}{\partial\bar{\mathbf{p}}} &= \bar{\mathbf{e}}, \\ \rho_o\frac{\partial\varepsilon^{F\eta pm}}{\partial\bar{\mathbf{m}}} &= \mu_o\bar{\mathbf{h}},\end{aligned} \tag{3.20}$$

and the residual equality is

$$\bar{\mathbf{j}}\cdot\bar{\mathbf{e}} \;-\; \frac{Q}{\theta}\cdot\frac{\partial\theta}{\partial\mathbf{X}} \;-\; \rho_o\xi\theta \;=\; 0. \tag{3.21}$$

**Case (b)** :

Consider $\varepsilon = \varepsilon(\mathbf{F},\eta,\bar{\mathbf{e}},\bar{\mathbf{m}})$, one can write

$$\rho_o\dot{\varepsilon}^{F\eta em} \;=\; \rho_o\frac{\partial\varepsilon^{F\eta em}}{\partial\mathbf{F}}\dot{\mathbf{F}} \;+\; \rho_o\frac{\partial\varepsilon^{F\eta em}}{\partial\eta}\dot{\eta} \;+\; \rho_o\frac{\partial\varepsilon^{F\eta em}}{\partial\bar{\mathbf{e}}}\dot{\bar{\mathbf{e}}} \;+\; \rho_o\frac{\partial\varepsilon^{F\eta em}}{\partial\bar{\mathbf{m}}}\dot{\bar{\mathbf{m}}}. \tag{3.22}$$

The corresponding Legendre transformation is

$$\rho_o \varepsilon^{F\eta em} = \rho_o \varepsilon - \bar{\mathbf{p}} \cdot \bar{\mathbf{e}}, \tag{3.23}$$

$$\rho_o \dot{\varepsilon}^{F\eta em} = \rho_o \dot{\varepsilon} - \bar{\mathbf{p}} \cdot \dot{\bar{\mathbf{e}}} - \dot{\bar{\mathbf{p}}} \cdot \bar{\mathbf{e}}. \tag{3.24}$$

Substituting (3.24) into (3.10),

$$-\rho_o \dot{\varepsilon}^{F\eta em} \; + \; (\mathbf{P} + \mathbf{F}^{-T}(\bar{\mathbf{e}} \otimes \bar{\mathbf{p}} \; + \; \mu_o \bar{\mathbf{h}} \otimes \bar{\mathbf{m}})) \cdot \dot{\mathbf{F}} \; + \; \rho_o \dot{\eta} \theta \; - \; \bar{\mathbf{p}} \cdot \dot{\bar{\mathbf{e}}} \; + \; \mu_o \dot{\bar{\mathbf{m}}} \cdot \bar{\mathbf{h}} \; + \; B \; = \; 0. \tag{3.25}$$

Substituting (3.22) into (3.25), and rearranging gives

$$\begin{aligned} -(\rho_o \frac{\partial \varepsilon^{F\eta em}}{\partial \mathbf{F}} - (\mathbf{P} + \mathbf{F}^{-T}(\bar{\mathbf{e}} \otimes \bar{\mathbf{p}} \; + \; \mu_o \bar{\mathbf{h}} \otimes \bar{\mathbf{m}})) \cdot \dot{\mathbf{F}} \; - \; (\rho_o \frac{\partial \varepsilon^{F\eta em}}{\partial \eta} - \rho_o \theta) \dot{\eta} \\ - \; (\rho_o \frac{\partial \varepsilon^{F\eta em}}{\partial \bar{\mathbf{e}}} + \bar{\mathbf{p}}) \cdot \dot{\bar{\mathbf{e}}} \; - \; (\rho_o \frac{\partial \varepsilon^{F\eta em}}{\partial \bar{\mathbf{m}}} - \mu_o \bar{\mathbf{h}}) \cdot \dot{\bar{\mathbf{m}}} \; + \; B \; = \; 0. \end{aligned} \tag{3.26}$$

The constitutive equations thus have the form

$$\begin{aligned} \rho_o \frac{\partial \varepsilon^{F\eta em}}{\partial \mathbf{F}} &= \mathbf{P} + \mathbf{F}^{-T}(\bar{\mathbf{e}} \otimes \bar{\mathbf{p}} \; + \; \mu_o \bar{\mathbf{h}} \otimes \bar{\mathbf{m}}), \\ \frac{\partial \varepsilon^{F\eta em}}{\partial \eta} &= \theta, \\ \rho_o \frac{\partial \varepsilon^{F\eta em}}{\partial \bar{\mathbf{e}}} &= -\bar{\mathbf{p}}, \\ \rho_o \frac{\partial \varepsilon^{F\eta em}}{\partial \bar{\mathbf{m}}} &= \mu_o \bar{\mathbf{h}}. \end{aligned} \tag{3.27}$$

**Case (c)** :

Consider $\varepsilon = \varepsilon(\mathbf{F}, \eta, \bar{\mathbf{p}}, \bar{\mathbf{h}})$, one can write

$$\rho_o \dot{\varepsilon}^{F\eta ph} \; = \; \rho_o \frac{\partial \varepsilon^{F\eta ph}}{\partial \mathbf{F}} \dot{\mathbf{F}} \; + \; \rho_o \frac{\partial \varepsilon^{F\eta ph}}{\partial \eta} \dot{\eta} \; + \; \rho_o \frac{\partial \varepsilon^{F\eta ph}}{\partial \bar{\mathbf{p}}} \dot{\bar{\mathbf{p}}} \; + \; \rho_o \frac{\partial \varepsilon^{F\eta ph}}{\partial \bar{\mathbf{h}}} \dot{\bar{\mathbf{h}}}. \tag{3.28}$$

The corresponding Legendre transformation is

$$\rho_o \varepsilon^{F\eta ph} = \rho_o \varepsilon - \mu_o \bar{\mathbf{m}} \cdot \bar{\mathbf{h}}, \tag{3.29}$$

$$\rho_o \dot{\varepsilon}^{F\eta ph} = \rho_o \dot{\varepsilon} - \mu_o \bar{\mathbf{m}} \cdot \dot{\bar{\mathbf{h}}} - \mu_{\mathbf{o}} \dot{\bar{\mathbf{m}}} \cdot \bar{\mathbf{h}}. \tag{3.30}$$

Substituting (3.30) into (3.10),

$$-\rho_o \dot{\varepsilon}^{F\eta ph} \;+\; (\mathbf{P} + \mathbf{F}^{-T}(\bar{\mathbf{e}} \otimes \bar{\mathbf{p}} \;+\; \mu_o \bar{\mathbf{h}} \otimes \bar{\mathbf{m}})) \cdot \dot{\mathbf{F}} \;+\; \rho_o \dot{\eta} \theta \;+\; \dot{\bar{\mathbf{p}}} \cdot \bar{\mathbf{e}} \;-\; \mu_o \bar{\mathbf{m}} \cdot \dot{\bar{\mathbf{h}}} \;+\; B \;=\; 0. \tag{3.31}$$

Substituting (3.28) into (3.31), and from rearranging, one arrives at the constitutive equations

$$\begin{aligned} \rho_o \frac{\partial \varepsilon^{F\eta ph}}{\partial \mathbf{F}} &= \mathbf{P} + \mathbf{F}^{-T}(\bar{\mathbf{e}} \otimes \bar{\mathbf{p}} \;+\; \mu_o \bar{\mathbf{h}} \otimes \bar{\mathbf{m}}), \\ \frac{\partial \varepsilon^{F\eta ph}}{\partial \eta} &= \theta, \\ \rho_o \frac{\partial \varepsilon^{F\eta ph}}{\partial \bar{\mathbf{p}}} &= \bar{\mathbf{e}}, \\ \rho_o \frac{\partial \varepsilon^{F\eta ph}}{\partial \bar{\mathbf{h}}} &= -\mu_o \bar{\mathbf{m}}. \end{aligned} \tag{3.32}$$

**Case (d)** :

Consider $\varepsilon = \varepsilon(\mathbf{F}, \eta, \bar{\mathbf{d}}, \bar{\mathbf{m}})$, one can write

$$\rho_o \dot{\varepsilon}^{F\eta dm} \;=\; \rho_o \frac{\partial \varepsilon^{F\eta dm}}{\partial \mathbf{F}} \dot{\mathbf{F}} \;+\; \rho_o \frac{\partial \varepsilon^{F\eta dm}}{\partial \eta} \dot{\eta} \;+\; \rho_o \frac{\partial \varepsilon^{F\eta dm}}{\partial \bar{\mathbf{d}}} \dot{\bar{\mathbf{d}}} \;+\; \rho_o \frac{\partial \varepsilon^{F\eta dm}}{\partial \bar{\mathbf{m}}} \dot{\bar{\mathbf{m}}}. \tag{3.33}$$

The corresponding Legendre transformation is

$$\rho_o \dot{\varepsilon}^{F\eta dm} = \rho_o \dot{\varepsilon} + \epsilon_o J \mathbf{F}^{-T} \bar{\mathbf{e}} \cdot (-\mathbf{L}^T \mathbf{F}^{-T} \bar{\mathbf{e}} + \mathbf{F}^{-T} \dot{\bar{\mathbf{e}}}) + \epsilon_o \mathbf{F}^{-T} \bar{\mathbf{e}} \cdot \mathbf{F}^{-T} \bar{\mathbf{e}} \; div \mathbf{v}. \tag{3.34}$$

Substituting (3.34) into (3.10),

$$-\rho_o \dot{\varepsilon}^{F\eta dm} \;+\; (\mathbf{P} + \mathbf{F}^{-T}(\bar{\mathbf{e}} \otimes \bar{\mathbf{d}} \;+\; \mu_o \bar{\mathbf{h}} \otimes \bar{\mathbf{m}})) \cdot \dot{\mathbf{F}} \;+\; \rho_o \dot{\eta} \theta \;+\; \bar{\mathbf{e}} \cdot \dot{\bar{\mathbf{d}}} \;+\; \mu_o \dot{\bar{\mathbf{m}}} \cdot \bar{\mathbf{h}} \;+\; B \;=\; 0. \tag{3.35}$$

| Material Description | Spatial Description |
|---|---|
| $\epsilon_o J\mathbf{F}^{-T}\bar{\mathbf{e}}\cdot(-\mathbf{L}^T\mathbf{F}^{-T}\bar{\mathbf{e}}+\mathbf{F}^{-T}\dot{\bar{\mathbf{e}}})+\epsilon_o\mathbf{F}^{-T}\bar{\mathbf{e}}\cdot\mathbf{F}^{-T}\bar{\mathbf{e}}\ div\mathbf{v}$ | $\epsilon_o\mathbf{e}^*\cdot\dot{\mathbf{e}}^*+\epsilon_o\mathbf{e}^*\cdot\mathbf{e}^* div\mathbf{v}$ |
| $\mu_o J\mathbf{F}^{-T}\bar{\mathbf{h}}\cdot(-\mathbf{L}^T\mathbf{F}^{-T}\bar{\mathbf{h}}+\mathbf{F}^{-T}\dot{\bar{\mathbf{h}}})+\mu_o\mathbf{F}^{-T}\bar{\mathbf{h}}\cdot\mathbf{F}^{-T}\bar{\mathbf{h}}\ div\mathbf{v}$ | $\mu_o\mathbf{h}^*\cdot\dot{\mathbf{h}}^*+\mu_o\mathbf{h}^*\cdot\mathbf{h}^* div\mathbf{v}$ |

Substituting (3.33) into (3.35), and from rearranging one arrives at the constitutive equations

$$\begin{aligned}
\rho_o\frac{\partial\varepsilon^{F\eta dm}}{\partial\mathbf{F}} &= \mathbf{P}+\mathbf{F}^{-T}(\bar{\mathbf{e}}\otimes\bar{\mathbf{d}}\ +\ \mu_o\bar{\mathbf{h}}\otimes\bar{\mathbf{m}}), \\
\frac{\partial\varepsilon^{F\eta dm}}{\partial\eta} &= \theta, \\
\rho_o\frac{\partial\varepsilon^{F\eta dm}}{\partial\bar{\mathbf{d}}} &= \bar{\mathbf{e}}, \\
\rho_o\frac{\partial\varepsilon^{F\eta dm}}{\partial\bar{\mathbf{m}}} &= \mu_o\bar{\mathbf{h}}.
\end{aligned} \tag{3.36}$$

**Case** (**e**) :

Consider $\varepsilon=\varepsilon(\mathbf{F},\eta,\bar{\mathbf{p}},\bar{\mathbf{b}})$, one can write

$$\rho_o\dot{\varepsilon}^{F\eta pb} = \rho_o\frac{\partial\varepsilon^{F\eta pb}}{\partial\mathbf{F}}\dot{\mathbf{F}}\ +\ \rho_o\frac{\partial\varepsilon^{F\eta pb}}{\partial\eta}\dot{\eta}\ +\ \rho_o\frac{\partial\varepsilon^{F\eta pb}}{\partial\bar{\mathbf{p}}}\dot{\bar{\mathbf{p}}}\ +\ \rho_o\frac{\partial\varepsilon^{F\eta pb}}{\partial\bar{\mathbf{b}}}\dot{\bar{\mathbf{b}}}. \tag{3.37}$$

The corresponding Legendre transformation is

$$\rho_o\dot{\varepsilon}^{F\eta pb}=\rho_o\dot{\varepsilon}+\mu_o J\mathbf{F}^{-T}\bar{\mathbf{h}}\cdot(-\mathbf{L}^T\mathbf{F}^{-T}\bar{\mathbf{h}}+\mathbf{F}^{-T}\dot{\bar{\mathbf{h}}})+\mu_o\mathbf{F}^{-T}\bar{\mathbf{h}}\cdot\mathbf{F}^{-T}\bar{\mathbf{h}}\ div\mathbf{v}. \tag{3.38}$$

Substituting (3.38) into (3.10),

$$-\rho_o\dot{\varepsilon}^{F\eta pb}\ +\ (\mathbf{P}+\mathbf{F}^{-T}(\bar{\mathbf{e}}\otimes\bar{\mathbf{p}}\ +\ \bar{\mathbf{h}}\otimes\bar{\mathbf{b}}))\cdot\dot{\mathbf{F}}\ +\ \rho_o\dot{\eta}\theta\ +\ \bar{\mathbf{e}}\cdot\dot{\bar{\mathbf{p}}}\ +\ \mu_o\bar{\mathbf{h}}\cdot\dot{\bar{\mathbf{b}}}+\ B\ =\ 0. \tag{3.39}$$

Substituting (3.33) into (3.35), and from rearranging one arrives at the constitutive equations

$$\begin{aligned}
\rho_o \frac{\partial \varepsilon^{F\eta pb}}{\partial \mathbf{F}} &= \mathbf{P} + \mathbf{F}^{-T}(\bar{\mathbf{e}} \otimes \bar{\mathbf{p}} \ + \ \bar{\mathbf{h}} \otimes \bar{\mathbf{b}}), \\
\frac{\partial \varepsilon^{F\eta pb}}{\partial \eta} &= \theta, \\
\rho_o \frac{\partial \varepsilon^{F\eta pb}}{\partial \bar{\mathbf{p}}} &= \bar{\mathbf{e}}, \\
\rho_o \frac{\partial \varepsilon^{F\eta pb}}{\partial \bar{\mathbf{b}}} &= \mu_o \bar{\mathbf{h}}.
\end{aligned} \tag{3.40}$$

### 3.2.2 Family 2

**Case (a)** :

Consider $\eta = \eta(\mathbf{F}, \varepsilon, \bar{\mathbf{e}}, \bar{\mathbf{m}})$ The Legendre transformation is

$$\rho_o \dot{\eta}^{F\varepsilon em}\theta = \rho_o \dot{\eta}\theta + \bar{\mathbf{p}} \cdot \dot{\bar{\mathbf{e}}} + \dot{\bar{\mathbf{p}}} \cdot \bar{\mathbf{e}}. \tag{3.41}$$

Substituting (3.41) into (3.10),

$$\rho_o \dot{\eta}^{F\varepsilon em}\theta \ + \ (\mathbf{P} + \mathbf{F}^{-T}(\bar{\mathbf{e}} \otimes \bar{\mathbf{p}} \ + \ \mu_o \bar{\mathbf{h}} \otimes \bar{\mathbf{m}})) \cdot \dot{\mathbf{F}} \ - \rho_o \dot{\varepsilon} \ - \ \bar{\mathbf{p}} \cdot \dot{\bar{\mathbf{e}}} \ + \ \mu_o \dot{\bar{\mathbf{m}}} \cdot \bar{\mathbf{h}} \ + \ B \ = \ 0. \tag{3.42}$$

The corresponding constitutive equations are

$$\begin{aligned}
\rho_o \theta \frac{\partial \eta^{F\varepsilon em}}{\partial \mathbf{F}} &= -(\mathbf{P} + \mathbf{F}^{-T}(\bar{\mathbf{e}} \otimes \bar{\mathbf{p}} \ + \ \mu_o \bar{\mathbf{h}} \otimes \bar{\mathbf{m}})), \\
\frac{\partial \eta^{F\varepsilon em}}{\partial \varepsilon} &= \frac{1}{\theta}, \\
\rho_o \theta \frac{\partial \eta}{\partial \bar{\mathbf{e}}} &= \bar{\mathbf{p}}, \\
\rho_o \theta \frac{\partial \eta}{\partial \bar{\mathbf{m}}} &= -\mu_o \bar{\mathbf{h}}.
\end{aligned} \tag{3.43}$$

**Case (b) :**

Consider $\eta = \eta(\mathbf{F}, \varepsilon, \bar{\mathbf{d}}, \bar{\mathbf{m}})$ The Legendre transformation is

$$\rho_o \dot{\eta}^{F\varepsilon dm}\theta = \rho_o \dot{\eta}\theta - \epsilon_o J \mathbf{F}^{-T}\bar{\mathbf{e}} \cdot (-\mathbf{L}^T \mathbf{F}^{-T}\bar{\mathbf{e}} + \mathbf{F}^{-T}\dot{\bar{\mathbf{e}}}) - \epsilon_o \mathbf{F}^{-T}\bar{\mathbf{e}} \cdot \mathbf{F}^{-T}\bar{\mathbf{e}}\ div\mathbf{v}. \quad (3.44)$$

Substituting (3.44) into (3.10),

$$\rho_o \dot{\eta}^{F\varepsilon dm}\theta \ + \ (\mathbf{P} + \mathbf{F}^{-T}(\bar{\mathbf{e}} \otimes \bar{\mathbf{d}} \ + \ \mu_o \bar{\mathbf{h}} \otimes \bar{\mathbf{m}})) \cdot \dot{\mathbf{F}} \ - \rho_o \dot{\varepsilon} \ - \ \bar{\mathbf{e}} \cdot \dot{\bar{\mathbf{d}}} \ + \ \mu_o \dot{\bar{\mathbf{m}}} \cdot \bar{\mathbf{h}} \ + \ B \ = \ 0. \quad (3.45)$$

The corresponding constitutive equations are

$$\begin{aligned} \rho_o \theta \frac{\partial \eta^{F\varepsilon em}}{\partial \mathbf{F}} &= -(\mathbf{P} + \mathbf{F}^{-T}(\bar{\mathbf{e}} \otimes \bar{\mathbf{d}} \ + \ \mu_o \bar{\mathbf{h}} \otimes \bar{\mathbf{m}})), \\ \frac{\partial \eta^{F\varepsilon em}}{\partial \varepsilon} &= \frac{1}{\theta}, \\ \rho_o \theta \frac{\partial \eta}{\partial \bar{\mathbf{d}}} &= -\bar{\mathbf{e}}, \\ \rho_o \theta \frac{\partial \eta}{\partial \bar{\mathbf{m}}} &= -\mu_o \bar{\mathbf{h}}. \end{aligned} \quad (3.46)$$

The terms which will get added in the Family 1's legendre transformation will get subracted in the Family 2's legendre transformation. In this way one can obtain the constitutive equations for all the cases in Family 2.

### 3.2.3 Family 3

For family 3 potentials, one needs to subract $\eta\theta$ from the corresponding family 1 potentials. For example

Consider the case $\psi = \psi(\mathbf{F}, \theta, \bar{\mathbf{d}}, \bar{\mathbf{b}})$ The Legendre transformation is

$$\psi^{\mathbf{F}\theta db} = \varepsilon^{\mathbf{F}\eta db} - \eta\theta, \quad (3.47)$$

$$\rho_o \dot{\psi}^{\mathbf{F}\theta db} = \rho_o \dot{\varepsilon}^{\mathbf{F}\eta db} - \rho_o \dot{\eta}\theta - \rho_o \eta\dot{\theta}. \quad (3.48)$$

Substituting (3.48) in the corresponding equation for $\varepsilon^{\mathbf{F}\eta db}$,

$$\rho_o\dot{\psi}^{\mathbf{F}\theta db} = \rho_o\dot{\varepsilon} + \epsilon_o J\mathbf{F}^{-T}\bar{\mathbf{e}}\cdot(-\mathbf{L}^T\mathbf{F}^{-T}\bar{\mathbf{e}} + \mathbf{F}^{-T}\dot{\bar{\mathbf{e}}}) + \epsilon_{\mathbf{o}}\mathbf{F}^{-\mathbf{T}}\bar{\mathbf{e}}\cdot\mathbf{F}^{-\mathbf{T}}\bar{\mathbf{e}}\ \mathbf{div v} \tag{3.49}$$
$$+\mu_o J\mathbf{F}^{-T}\bar{\mathbf{h}}\cdot(-\mathbf{L}^T\mathbf{F}^{-T}\bar{\mathbf{h}} + \mathbf{F}^{-T}\dot{\bar{\mathbf{h}}}) + \mu_{\mathbf{o}}\mathbf{F}^{-\mathbf{T}}\bar{\mathbf{h}}\cdot\mathbf{F}^{-\mathbf{T}}\bar{\mathbf{h}}\ \mathbf{div v} - \rho_{\mathbf{o}}\dot{\eta}\theta - \rho_{\mathbf{o}}\eta\dot{\theta}.$$

Substituting (3.50) into (3.10),

$$-\rho_o\dot{\psi}^{F\theta db} + (\mathbf{P} + \mathbf{F}^{-T}(\bar{\mathbf{e}}\otimes\bar{\mathbf{d}} + \bar{\mathbf{h}}\otimes\bar{\mathbf{b}}))\cdot\dot{\mathbf{F}} - \rho_o\eta\dot{\theta} + \bar{\mathbf{e}}\cdot\dot{\bar{\mathbf{d}}} + \mu_o\bar{\mathbf{h}}\cdot\dot{\bar{\mathbf{b}}} + B = 0. \tag{3.50}$$

The constitutive equations are

$$\begin{aligned}
\rho_o\frac{\partial\psi^{F\theta db}}{\partial\mathbf{F}} &= \mathbf{P} + \mathbf{F}^{-T}(\bar{\mathbf{e}}\otimes\bar{\mathbf{d}} + \mu_o\bar{\mathbf{h}}\otimes\bar{\mathbf{b}}), \\
\frac{\partial\psi^{F\theta db}}{\partial\theta} &= -\eta, \\
\rho_o\frac{\partial\psi^{F\theta db}}{\partial\bar{\mathbf{d}}} &= \bar{\mathbf{e}}, \\
\rho_o\frac{\partial\psi^{F\theta db}}{\partial\bar{\mathbf{b}}} &= \mu_o\bar{\mathbf{h}}.
\end{aligned} \tag{3.51}$$

### 3.2.4 Family 4

The terms that get added in Family 2's legendre transformation get subracted in Family 4's legendre transformation.

For example, Consider the case $\theta = \theta(F, \psi, e, h)$ The Legendre transformation for this case is

$$\rho_o\eta\dot{\theta}^{F\psi eh} = \rho_o\eta\dot{\theta} - \bar{\mathbf{p}}\cdot\dot{\bar{\mathbf{e}}} - \dot{\bar{\mathbf{p}}}\cdot\bar{\mathbf{e}} - \mu_{\mathbf{o}}\bar{\mathbf{m}}\cdot\dot{\bar{\mathbf{h}}} - \mu_{\mathbf{o}}\dot{\bar{\mathbf{m}}}\cdot\bar{\mathbf{h}}, \tag{3.52}$$

where as the corresponding Legendre tranformation for family 2 is

$$\rho_o\dot{\eta}^{F\varepsilon eh}\theta = \rho_o\dot{\eta}\theta + \bar{\mathbf{p}}\cdot\dot{\bar{\mathbf{e}}} + \dot{\bar{\mathbf{p}}}\cdot\bar{\mathbf{e}} + \mu_{\mathbf{o}}\bar{\mathbf{m}}\cdot\dot{\bar{\mathbf{h}}} + \mu_{\mathbf{o}}\dot{\bar{\mathbf{m}}}\cdot\bar{\mathbf{h}}. \tag{3.53}$$

Substituting (3.52) into (3.15),

$$-\rho_o\eta\dot{\theta}^{F\psi eh} \;+\; (\mathbf{P}+\mathbf{F}^{-T}(\bar{\mathbf{e}}\otimes\bar{\mathbf{p}} \;+\; \mu_o\bar{\mathbf{h}}\otimes\bar{\mathbf{m}}))\cdot\dot{\mathbf{F}} \;-\rho_o\dot{\psi} \;-\; \bar{\mathbf{p}}\cdot\dot{\bar{\mathbf{e}}} \;-\; \mu_o\bar{\mathbf{m}}\cdot\dot{\bar{\mathbf{h}}} \;+\; B \;=\; 0. \tag{3.54}$$

The constitutive equations are

$$\begin{aligned}\rho_o\eta\frac{\partial\theta^{F\psi eh}}{\partial\mathbf{F}} &= (\mathbf{P}+\mathbf{F}^{-T}(\bar{\mathbf{e}}\otimes\bar{\mathbf{p}}+\mu_o\bar{\mathbf{h}}\otimes\bar{\mathbf{m}})),\\ \frac{\partial\theta^{F\psi eh}}{\partial\psi} &= -\frac{1}{\eta},\\ \rho_o\eta\frac{\partial\theta^{F\psi eh}}{\partial\bar{\mathbf{e}}} &= -\bar{\mathbf{p}},\\ \rho_o\eta\frac{\partial\theta^{F\psi eh}}{\partial\bar{\mathbf{h}}} &= -\mu_o\bar{\mathbf{m}}.\end{aligned} \tag{3.55}$$

### 3.2.5 Family 5

For family 5 potentials, one needs to subract $\mathbf{P}\cdot\mathbf{F}$ from the corresponding family 1 potentials. For example

$$\rho_o\chi^{P\eta dh} = \rho_o\varepsilon^{F\eta dh} - \mathbf{P}\cdot\mathbf{F}, \tag{3.56}$$

$$\rho_o\dot{\chi}^{P\eta dh} = \rho_o\dot{\varepsilon}^{F\eta dh} - \mathbf{P}\cdot\dot{\mathbf{F}} - \mathbf{F}\cdot\dot{\mathbf{P}}. \tag{3.57}$$

Substituting (3.57) into the corresponding equation for $\varepsilon^{F\eta dh}$,

$$\begin{aligned}\rho_o\dot{\chi}^{P\eta dh} = \rho_o\dot{\varepsilon} + \epsilon_o J\mathbf{F}^{-T}\bar{\mathbf{e}}\cdot(-\mathbf{L}^T\mathbf{F}^{-T}\bar{\mathbf{e}}+\mathbf{F}^{-T}\dot{\bar{\mathbf{e}}}) + \epsilon_{\mathbf{o}}\mathbf{F}^{-\mathbf{T}}\bar{\mathbf{e}}\cdot\mathbf{F}^{-\mathbf{T}}\bar{\mathbf{e}}\;\mathbf{div}\mathbf{v} - \\ \mu_o\bar{\mathbf{m}}\cdot\dot{\bar{\mathbf{h}}} - \mu_o\dot{\bar{\mathbf{m}}}\cdot\bar{\mathbf{h}} - \mathbf{P}\cdot\dot{\mathbf{F}} - \mathbf{F}\cdot\dot{\mathbf{P}}.\end{aligned} \tag{3.58}$$

Substituting (3.59) into (3.10)

$$\begin{aligned}-\rho_o\dot{\chi}^{P\eta dh} \;-\; \mathbf{F}\cdot\dot{\mathbf{P}} \;+\; \rho_o\dot{\eta}\theta \;+\; \bar{\mathbf{e}}\cdot\dot{\bar{\mathbf{d}}} \;-\; \mu_{\mathbf{o}}\bar{\mathbf{m}}\cdot\dot{\bar{\mathbf{h}}} \;+\; \mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{d}}\cdot\bar{\mathbf{e}} \;+\; \mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{m}}\cdot\bar{\mathbf{h}} \\ +\; B \;=\; 0.\end{aligned} \tag{3.59}$$

The constitutive equations are

$$
\begin{aligned}
\rho_o \frac{\partial \chi^{P\eta dh}}{\partial \mathbf{P}} &= -\mathbf{F}, \qquad (3.60)\\
\frac{\partial \chi^{P\eta dh}}{\partial \eta} &= \theta,\\
\rho_o \frac{\partial \chi^{P\eta dh}}{\partial \bar{\mathbf{d}}} &= \bar{\mathbf{e}},\\
\rho_o \frac{\partial \chi^{P\eta dh}}{\partial \bar{\mathbf{h}}} &= -\mu_o \bar{\mathbf{m}}.
\end{aligned}
$$

The residual equality is

$$
\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{d}} \cdot \bar{\mathbf{e}} \;+\; \mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{m}} \cdot \bar{\mathbf{h}} \;-\; \frac{Q}{\theta}\frac{\partial \theta}{\partial \mathbf{X}} \;-\; \rho_o \xi \theta \;+\; \bar{\mathbf{j}} \cdot \bar{\mathbf{e}} \;=\; 0. \qquad (3.61)
$$

### 3.2.6 Family 6

The form of Family 6's Legendre transformation equation is same as Family 2's. The only difference is, the Legendre transformation of Family 6 is substituted in (3.16) where as the one for Family 2 is substituted in (3.10). For example, Consider the case $\eta = \eta(P, \chi, e, b)$: The Legendre transformation for this case is

$$
\rho_o \dot{\eta}^{P\chi eb}\theta = \rho_o \dot{\eta}\theta - \mu_o J \mathbf{F}^{-T}\bar{\mathbf{h}} \cdot (-\mathbf{L}^T \mathbf{F}^{-T}\bar{\mathbf{h}} + \mathbf{F}^{-T}\dot{\bar{\mathbf{h}}}) - \mu_{\mathbf{o}} \mathbf{F}^{-\mathbf{T}}\bar{\mathbf{h}} \cdot \mathbf{F}^{-\mathbf{T}}\bar{\mathbf{h}} \;\mathbf{div v} + \bar{\mathbf{p}} \cdot \dot{\bar{\mathbf{e}}} + \dot{\bar{\mathbf{p}}} \cdot \bar{\mathbf{e}}, \qquad (3.62)
$$

where as the corresponding tranformation for family 2 is

$$
\rho_o \dot{\eta}^{F\varepsilon eb}\theta = \rho_o \dot{\eta}\theta - \mu_o J \mathbf{F}^{-T}\bar{\mathbf{h}} \cdot (-\mathbf{L}^T \mathbf{F}^{-T}\bar{\mathbf{h}} + \mathbf{F}^{-T}\dot{\bar{\mathbf{h}}}) - \mu_{\mathbf{o}} \mathbf{F}^{-\mathbf{T}}\bar{\mathbf{h}} \cdot \mathbf{F}^{-\mathbf{T}}\bar{\mathbf{h}} \;\mathbf{div v} + \bar{\mathbf{p}} \cdot \dot{\bar{\mathbf{e}}} + \dot{\bar{\mathbf{p}}} \cdot \bar{\mathbf{e}}. \qquad (3.63)
$$

Substituting (3.62) into(3.16),

$$
\rho_o \dot{\eta}^{P\chi eb}\theta \;-\; \rho_o \dot{\chi} \;-\; \mathbf{F} \cdot \dot{\mathbf{P}} \;-\; \bar{\mathbf{p}} \cdot \dot{\bar{\mathbf{e}}} \;+\; \mu_o \bar{\mathbf{h}} \cdot \dot{\bar{\mathbf{b}}} \;+ \mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{p}} \cdot \bar{\mathbf{e}} \;+\; \mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{b}} \cdot \bar{\mathbf{h}} \;+\; B \;=\; 0. \qquad (3.64)
$$

The constitutive equations are

$$
\begin{aligned}
\rho_o\theta\frac{\partial\eta^{P\chi eb}}{\partial\mathbf{P}} &= \mathbf{F}, \\
\frac{\partial\eta^{P\chi eb}}{\partial\chi} &= \frac{1}{\theta}, \\
\rho_o\theta\frac{\partial\eta^{P\chi eb}}{\partial\bar{\mathbf{e}}} &= \bar{\mathbf{p}}, \\
\rho_o\theta\frac{\partial\eta^{P\chi eb}}{\partial\bar{\mathbf{b}}} &= -\mu_o\bar{\mathbf{h}}.
\end{aligned}
\tag{3.65}
$$

The residual equality is

$$
\mathbf{F}^{-1}\mathbf{L}\mathbf{F}\bar{\mathbf{p}}\cdot\bar{\mathbf{e}} \;+\; \mathbf{F}^{-1}\mathbf{L}\mathbf{F}\bar{\mathbf{b}}\cdot\bar{\mathbf{h}} \;-\; \frac{Q}{\theta}\frac{\partial\theta}{\partial\mathbf{X}} \;-\; \rho_o\xi\theta \;+\; \bar{\mathbf{j}}\cdot\bar{\mathbf{e}} \;=\; 0. \tag{3.66}
$$

### 3.2.7 Family 7

Because the temperature $\theta$, stress $\mathbf{P}$, electric field $\bar{\mathbf{e}}$, magnetic field $\bar{\mathbf{h}}$ are easier to control as independent variables in characterization of thermoelastic materials, it is usual practice to use them as the independent variables and the Gibbs free energy as the potential. In order to get the legendre transformation for Family 7, we need to subract $\mathbf{P}\cdot\mathbf{F}+\rho_o\eta\theta$ from the corresponding potentials of Family 1.

For example, consider the case $\phi=\phi(P,\theta,e,h)$ The Legendre transformation is

$$
\rho_o\phi^{P\theta eh} = \rho_o\varepsilon^{F\eta eh} - \mathbf{P}\cdot\mathbf{F} - \rho_o\eta\theta, \tag{3.67}
$$

$$
\rho_o\dot{\phi}^{P\theta eh} = \rho_o\dot{\varepsilon}^{F\eta eh} - \mathbf{P}\cdot\dot{\mathbf{F}} - \mathbf{F}\cdot\dot{\mathbf{P}} - \rho_o\dot{\eta}\theta - \rho_o\eta\dot{\theta}. \tag{3.68}
$$

Substituting (3.68) into the corresponding equation for $\varepsilon^{F\eta eh}$,

$$
\rho_o\dot{\phi}^{P\theta eh} = \rho_o\dot{\varepsilon} - \bar{\mathbf{p}}\cdot\dot{\bar{\mathbf{e}}} - \dot{\bar{\mathbf{p}}}\cdot\bar{\mathbf{e}} - \mu_{\mathbf{o}}\bar{\mathbf{m}}\cdot\dot{\bar{\mathbf{h}}} - \mu_{\mathbf{o}}\dot{\bar{\mathbf{m}}}\cdot\bar{\mathbf{h}} - \mathbf{P}\cdot\dot{\mathbf{F}} - \mathbf{F}\cdot\dot{\mathbf{P}} - \rho_{\mathbf{o}}\dot{\eta}\theta - \rho_{\mathbf{o}}\eta\dot{\theta}. \tag{3.69}
$$

Substituting (3.69) into(3.10),

$$-\rho_o\dot{\phi}^{P\theta em} - \mathbf{F}\cdot\dot{\mathbf{P}} - \rho_o\eta\dot{\theta} - \bar{\mathbf{p}}\cdot\dot{\bar{\mathbf{e}}} - \mu_{\mathbf{o}}\bar{\mathbf{m}}\cdot\dot{\bar{\mathbf{h}}} + \mathbf{F^{-1}LF}\bar{\mathbf{p}}\cdot\bar{\mathbf{e}} \ + \ \mu_{\mathbf{o}}\mathbf{F^{-1}LF}\bar{\mathbf{m}}\cdot\bar{\mathbf{h}} \ + \ B \ = \ 0. \tag{3.70}$$

The constitutive equations are

$$\begin{aligned}\rho_o\frac{\partial\phi^{P\theta eh}}{\partial\mathbf{P}} &= -\mathbf{F}, \qquad (3.71)\\ \frac{\partial\phi^{P\theta eh}}{\partial\theta} &= -\eta,\\ \rho_o\frac{\partial\phi^{P\theta eh}}{\partial\bar{\mathbf{e}}} &= -\bar{\mathbf{p}},\\ \rho_o\frac{\partial\phi^{P\theta eh}}{\partial\bar{\mathbf{h}}} &= -\mu_o\bar{\mathbf{m}}.\end{aligned}$$

The residual equality is

$$\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{p}}\cdot\bar{\mathbf{e}} \ + \ \mu_o\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{m}}\cdot\bar{\mathbf{h}} \ - \ \frac{Q}{\theta}\frac{\partial\theta}{\partial\mathbf{X}} \ - \ \rho_o\xi\theta \ + \ \bar{\mathbf{j}}\cdot\bar{\mathbf{e}} \ = \ 0. \tag{3.72}$$

### 3.2.8 Family 8

The form of Family 8's Legendre transformation equation is same as Family 4's. The only difference is, the Legendre transformation of Family 8 is substituted in (3.17) where as the one for Family 4 is substituted in (3.15)For example, Consider the case $\theta = \theta(P, \psi, p, h)$ The Legendre transformation for this case is

$$\rho_o\eta\dot{\theta}^{P\phi ph} = \rho_o\eta\dot{\theta} - \mu_o\bar{\mathbf{m}}\cdot\dot{\bar{\mathbf{h}}} - \mu_{\mathbf{o}}\dot{\bar{\mathbf{m}}}\cdot\bar{\mathbf{h}}, \tag{3.73}$$

where as the corresponding legendre transformation for the Family 4 is

$$\rho_o\eta\dot{\theta}^{F\psi ph} = \rho_o\eta\dot{\theta} - \mu_o\bar{\mathbf{m}}\cdot\dot{\bar{\mathbf{h}}} - \mu_{\mathbf{o}}\dot{\bar{\mathbf{m}}}\cdot\bar{\mathbf{h}}. \tag{3.74}$$

Substituting (3.73) into (3.17),

$$-\rho_o\eta\dot{\theta}^{P\phi ph} - \rho_o\dot{\phi} - \mathbf{F}\cdot\dot{\mathbf{P}} + \bar{\mathbf{e}}\cdot\dot{\bar{\mathbf{p}}} - \mu_{\mathbf{o}}\bar{\mathbf{m}}\cdot\dot{\bar{\mathbf{h}}} + \mathbf{F^{-1}LF}\bar{\mathbf{p}}\cdot\bar{\mathbf{e}} \ + \ \mu_{\mathbf{o}}\mathbf{F^{-1}LF}\bar{\mathbf{m}}\cdot\bar{\mathbf{h}} \ + \ B \ = \ 0. \tag{3.75}$$

The constitutive equations are

$$\begin{aligned}\rho_o\eta\frac{\partial\theta^{P\phi ph}}{\partial\mathbf{P}} &= -\mathbf{F}, \\ \frac{\partial\theta^{P\phi ph}}{\partial\psi} &= -\frac{1}{\eta}, \\ \rho_o\eta\frac{\partial\theta^{P\phi ph}}{\partial\bar{\mathbf{p}}} &= \bar{\mathbf{e}}, \\ \rho_o\eta\frac{\partial\theta^{P\phi ph}}{\partial\bar{\mathbf{h}}} &= -\mu_o\bar{\mathbf{m}}.\end{aligned} \tag{3.76}$$

The residual equality is

$$\mathbf{F}^{-1}\mathbf{L}\mathbf{F}\bar{\mathbf{p}}\cdot\bar{\mathbf{e}} \;+\; \mu_o\mathbf{F}^{-1}\mathbf{L}\mathbf{F}\bar{\mathbf{m}}\cdot\bar{\mathbf{h}} \;-\; \frac{Q}{\theta}\frac{\partial\theta}{\partial\mathbf{X}} \;-\; \rho_o\xi\theta \;+\; \bar{\mathbf{j}}\cdot\bar{\mathbf{e}} \;=\; 0. \tag{3.77}$$

Comparing the constitutive equations given in [35], [44], the equations that involve stress as independent variable are same; there is difference in the equations which has strain as independent variable. In the mechanical constitutive equation that are derived in this work, apart from the mechanical Piola-Kirchoff stress component, there is electromagnetic stress term. Moreover, the constitutive equations in [35], [44] are based on linear elasticity where as the most general non-linear form is presented here. The process of obtaining these constitutive equations in [35], [44] started from the differential form of equation for the reversible processes. Here, the integral form of equations are presented and the differential form of equations are deduced from them. Also, the irreversible component of the second law is considered. There is residual inequality obtained in derivation of constitutive equations.

### 3.2.9 Legendre transformations and constitutive equations

The legendre transformation and constitutive equations for all the seventy-two potentials are given in the following tables.

Family1-Transformation

| Potential | Transformation equation | eqn no. | Subs eqn no. |
|---|---|---|---|
| $\varepsilon^{F\eta pm}$ | $\varepsilon^{F\eta pm} = \varepsilon$ | f1a | 3.10 |
| $\varepsilon^{F\eta em}$ | $\rho_o\varepsilon^{F\eta em} = \rho_o\varepsilon - \bar{\mathbf{p}} \cdot \bar{\mathbf{e}}$ | f1b | 3.10 |
| $\varepsilon^{F\eta ph}$ | $\rho_o\varepsilon^{F\eta ph} = \rho_o\varepsilon - \mu_o\bar{\mathbf{m}} \cdot \bar{\mathbf{h}}$ | f1c | 3.10 |
| $\varepsilon^{F\eta eh}$ | $\rho_o\varepsilon^{F\eta eh} = \rho_o\varepsilon - \bar{\mathbf{p}} \cdot \bar{\mathbf{e}} - \mu_o\bar{\mathbf{m}} \cdot \bar{\mathbf{h}}$ | f1d | 3.10 |
| $\varepsilon^{F\eta dm}$ | $\rho_o\dot{\varepsilon}^{F\eta dm} = \rho_o\dot{\varepsilon} + \epsilon_o J\mathbf{F}^{-T}\bar{\mathbf{e}} \cdot (-\mathbf{L}^T\mathbf{F}^{-T}\bar{\mathbf{e}} + \mathbf{F}^{-T}\dot{\bar{\mathbf{e}}}) + \epsilon_{\mathbf{o}}\mathbf{F}^{-\mathbf{T}}\bar{\mathbf{e}} \cdot \mathbf{F}^{-\mathbf{T}}\bar{\mathbf{e}}\ \mathbf{divv}$ | f1e | 3.10 |
| $\varepsilon^{F\eta pb}$ | $\rho_o\dot{\varepsilon}^{F\eta pb} = \rho_o\dot{\varepsilon} + \mu_o J\mathbf{F}^{-T}\bar{\mathbf{h}} \cdot (-\mathbf{L}^T\mathbf{F}^{-T}\bar{\mathbf{h}} + \mathbf{F}^{-T}\dot{\bar{\mathbf{h}}}) + \mu_{\mathbf{o}}\mathbf{F}^{-\mathbf{T}}\bar{\mathbf{h}} \cdot \mathbf{F}^{-\mathbf{T}}\bar{\mathbf{h}}\ \mathbf{divv}$ | f1f | 3.10 |
| $\varepsilon^{F\eta dh}$ | $\rho_o\dot{\varepsilon}^{F\eta dh} = \rho_o\dot{\varepsilon} + \epsilon_o J\mathbf{F}^{-T}\bar{\mathbf{e}} \cdot (-\mathbf{L}^T\mathbf{F}^{-T}\bar{\mathbf{e}} + \mathbf{F}^{-T}\dot{\bar{\mathbf{e}}}) + \epsilon_{\mathbf{o}}\mathbf{F}^{-\mathbf{T}}\bar{\mathbf{e}} \cdot \mathbf{F}^{-\mathbf{T}}\bar{\mathbf{e}}\ \mathbf{divv} - \mu_{\mathbf{o}}\bar{\mathbf{m}} \cdot \dot{\bar{\mathbf{h}}} - \mu_{\mathbf{o}}\dot{\bar{\mathbf{m}}} \cdot \bar{\mathbf{h}}$ | f1g | 3.10 |
| $\varepsilon^{F\eta eb}$ | $\rho_o\dot{\varepsilon}^{F\eta eb} = \rho_o\dot{\varepsilon} + \mu_o J\mathbf{F}^{-T}\bar{\mathbf{h}} \cdot (-\mathbf{L}^T\mathbf{F}^{-T}\bar{\mathbf{h}} + \mathbf{F}^{-T}\dot{\bar{\mathbf{h}}}) + \mu_{\mathbf{o}}\mathbf{F}^{-\mathbf{T}}\bar{\mathbf{h}} \cdot \mathbf{F}^{-\mathbf{T}}\bar{\mathbf{h}}\ \mathbf{divv} - \bar{\mathbf{p}} \cdot \dot{\bar{\mathbf{e}}} - \dot{\bar{\mathbf{p}}} \cdot \bar{\mathbf{e}}$ | f1h | 3.10 |
| $\varepsilon^{F\eta db}$ | $\rho_o\dot{\varepsilon}^{F\eta db} = \rho_o\dot{\varepsilon} + \epsilon_o J\mathbf{F}^{-T}\bar{\mathbf{e}} \cdot (-\mathbf{L}^T\mathbf{F}^{-T}\bar{\mathbf{e}} + \mathbf{F}^{-T}\dot{\bar{\mathbf{e}}}) + \epsilon_{\mathbf{o}}\mathbf{F}^{-\mathbf{T}}\bar{\mathbf{e}} \cdot \mathbf{F}^{-\mathbf{T}}\bar{\mathbf{e}}\ \mathbf{divv}$ $+\mu_o J\mathbf{F}^{-T}\bar{\mathbf{h}} \cdot (-\mathbf{L}^T\mathbf{F}^{-T}\bar{\mathbf{h}} + \mathbf{F}^{-T}\dot{\bar{\mathbf{h}}}) + \mu_{\mathbf{o}}\mathbf{F}^{-\mathbf{T}}\bar{\mathbf{h}} \cdot \mathbf{F}^{-\mathbf{T}}\bar{\mathbf{h}}\ \mathbf{divv}$ | f1i | 3.10 |

Table 3.2: Family 1 - Legendre transformations

Family1- Constitutive Equations

| Residual Equality: $\bar{\mathbf{j}} \cdot \bar{\mathbf{e}} - \frac{\mathbf{Q}}{\theta} \frac{\partial \theta}{\partial \mathbf{X}} - \rho_o \xi \theta = 0$ | | | | |
|---|---|---|---|---|
| potential | Mechanical | Thermal | Electrical | Magnetic |
| $\varepsilon^{F\eta pm}$ | $\rho_o \frac{\partial \varepsilon^{F\eta pm}}{\partial \mathbf{F}} = \mathbf{P} + \mathbf{F}^{-T}(\bar{\mathbf{e}} \otimes \bar{\mathbf{p}} + \mu_o \bar{\mathbf{h}} \otimes \bar{\mathbf{m}})$ | $\frac{\partial \varepsilon^{F\eta pm}}{\partial \eta} = \theta$ | $\rho_o \frac{\partial \varepsilon^{F\eta pm}}{\partial \bar{\mathbf{p}}} = \bar{\mathbf{e}}$ | $\rho_o \frac{\partial \varepsilon^{F\eta pm}}{\partial \bar{\mathbf{m}}} = \mu_o \bar{\mathbf{h}}$ |
| $\varepsilon^{F\eta em}$ | $\rho_o \frac{\partial \varepsilon^{F\eta em}}{\partial \mathbf{F}} = \mathbf{P} + \mathbf{F}^{-T}(\bar{\mathbf{e}} \otimes \bar{\mathbf{p}} + \mu_o \bar{\mathbf{h}} \otimes \bar{\mathbf{m}})$ | $\frac{\partial \varepsilon^{F\eta em}}{\partial \eta} = \theta$ | $\rho_o \frac{\partial \varepsilon^{F\eta em}}{\partial \bar{\mathbf{e}}} = -\bar{\mathbf{p}}$ | $\rho_o \frac{\partial \varepsilon^{F\eta em}}{\partial \bar{\mathbf{m}}} = \mu_o \bar{\mathbf{h}}$ |
| $\varepsilon^{F\eta ph}$ | $\rho_o \frac{\partial \varepsilon^{F\eta ph}}{\partial \mathbf{F}} = \mathbf{P} + \mathbf{F}^{-T}(\bar{\mathbf{e}} \otimes \bar{\mathbf{p}} + \mu_o \bar{\mathbf{h}} \otimes \bar{\mathbf{m}})$ | $\frac{\partial \varepsilon^{F\eta ph}}{\partial \eta} = \theta$ | $\rho_o \frac{\partial \varepsilon^{F\eta ph}}{\partial \bar{\mathbf{p}}} = \bar{\mathbf{e}}$ | $\rho_o \frac{\partial \varepsilon^{F\eta ph}}{\partial \bar{\mathbf{h}}} = -\mu_o \bar{\mathbf{m}}$ |
| $\varepsilon^{F\eta eh}$ | $\rho_o \frac{\partial \varepsilon^{F\eta eh}}{\partial \mathbf{F}} = \mathbf{P} + \mathbf{F}^{-T}(\bar{\mathbf{e}} \otimes \bar{\mathbf{p}} + \mu_o \bar{\mathbf{h}} \otimes \bar{\mathbf{m}})$ | $\frac{\partial \varepsilon^{F\eta eh}}{\partial \eta} = \theta$ | $\rho_o \frac{\partial \varepsilon^{F\eta eh}}{\partial \bar{\mathbf{e}}} = -\bar{\mathbf{p}}$ | $\rho_o \frac{\partial \varepsilon^{F\eta eh}}{\partial \bar{\mathbf{h}}} = -\mu_o \bar{\mathbf{m}}$ |
| $\varepsilon^{F\eta dm}$ | $\rho_o \frac{\partial \varepsilon^{F\eta dm}}{\partial \mathbf{F}} = \mathbf{P} + \mathbf{F}^{-T}(\bar{\mathbf{e}} \otimes \bar{\mathbf{d}} + \mu_o \bar{\mathbf{h}} \otimes \bar{\mathbf{m}})$ | $\frac{\partial \varepsilon^{F\eta dm}}{\partial \eta} = \theta$ | $\rho_o \frac{\partial \varepsilon^{F\eta dm}}{\partial \bar{\mathbf{d}}} = \bar{\mathbf{e}}$ | $\rho_o \frac{\partial \varepsilon^{F\eta dm}}{\partial \bar{\mathbf{m}}} = \mu_o \bar{\mathbf{h}}$ |
| $\varepsilon^{F\eta pb}$ | $\rho_o \frac{\partial \varepsilon^{F\eta pb}}{\partial \mathbf{F}} = \mathbf{P} + \mathbf{F}^{-T}(\bar{\mathbf{e}} \otimes \bar{\mathbf{p}} + \bar{\mathbf{h}} \otimes \bar{\mathbf{b}})$ | $\frac{\partial \varepsilon^{F\eta pb}}{\partial \eta} = \theta$ | $\rho_o \frac{\partial \varepsilon^{F\eta pb}}{\partial \bar{\mathbf{p}}} = \bar{\mathbf{e}}$ | $\rho_o \frac{\partial \varepsilon^{F\eta pb}}{\partial \bar{\mathbf{b}}} = \mu_o \bar{\mathbf{h}}$ |
| $\varepsilon^{F\eta dh}$ | $\rho_o \frac{\partial \varepsilon^{F\eta dh}}{\partial \mathbf{F}} = \mathbf{P} + \mathbf{F}^{-T}(\bar{\mathbf{e}} \otimes \bar{\mathbf{d}} + \mu_o \bar{\mathbf{h}} \otimes \bar{\mathbf{m}})$ | $\frac{\partial \varepsilon^{F\eta dh}}{\partial \eta} = \theta$ | $\rho_o \frac{\partial \varepsilon^{F\eta dh}}{\partial \bar{\mathbf{d}}} = \bar{\mathbf{e}}$ | $\rho_o \frac{\partial \varepsilon^{F\eta dh}}{\partial \bar{\mathbf{h}}} = -\mu_o \bar{\mathbf{m}}$ |
| $\varepsilon^{F\eta eb}$ | $\rho_o \frac{\partial \varepsilon^{F\eta eb}}{\partial \mathbf{F}} = \mathbf{P} + \mathbf{F}^{-T}(\bar{\mathbf{e}} \otimes \bar{\mathbf{p}} + \bar{\mathbf{h}} \otimes \bar{\mathbf{b}})$ | $\frac{\partial \varepsilon^{F\eta eb}}{\partial \eta} = \theta$ | $\rho_o \frac{\partial \varepsilon^{F\eta eb}}{\partial \bar{\mathbf{e}}} = -\bar{\mathbf{p}}$ | $\rho_o \frac{\partial \varepsilon^{F\eta eb}}{\partial \bar{\mathbf{b}}} = \mu_o \bar{\mathbf{h}}$ |
| $\varepsilon^{F\eta db}$ | $\rho_o \frac{\partial \varepsilon^{F\eta db}}{\partial \mathbf{F}} = \mathbf{P} + \mathbf{F}^{-T}(\bar{\mathbf{e}} \otimes \bar{\mathbf{d}} + \mu_o \bar{\mathbf{h}} \otimes \bar{\mathbf{b}})$ | $\frac{\partial \varepsilon^{F\eta db}}{\partial \eta} = \theta$ | $\rho_o \frac{\partial \varepsilon^{F\eta db}}{\partial \bar{\mathbf{d}}} = \bar{\mathbf{e}}$ | $\rho_o \frac{\partial \varepsilon^{F\eta db}}{\partial \bar{\mathbf{b}}} = \mu_o \bar{\mathbf{h}}$ |

Table 3.3: Family 1 - constitutive equations

Family2-Transformation

| Potential | Transformation equation | eqn no. | Subs eqn no. |
|---|---|---|---|
| $\eta^{F\varepsilon pm}$ | $\eta^{F\varepsilon pm} = \eta$ | f2a | 3.10 |
| $\eta^{F\varepsilon em}$ | $\rho_o \dot{\eta}^{F\varepsilon em}\theta = \rho_o \dot{\eta}\theta + \bar{\mathbf{p}} \cdot \dot{\bar{\mathbf{e}}} + \dot{\bar{\mathbf{p}}} \cdot \bar{\mathbf{e}}$ | f2b | 3.10 |
| $\eta^{F\varepsilon ph}$ | $\rho_o \dot{\eta}^{F\varepsilon ph}\theta = \rho_o \dot{\eta}\theta + \mu_o \bar{\mathbf{m}} \cdot \dot{\bar{\mathbf{h}}} + \mu_{\mathbf{o}} \dot{\bar{\mathbf{m}}} \cdot \bar{\mathbf{h}}$ | f2c | 3.10 |
| $\eta^{F\varepsilon eh}$ | $\rho_o \dot{\eta}^{F\varepsilon eh}\theta = \rho_o \dot{\eta}\theta + \bar{\mathbf{p}} \cdot \dot{\bar{\mathbf{e}}} + \dot{\bar{\mathbf{p}}} \cdot \bar{\mathbf{e}} + \mu_{\mathbf{o}} \bar{\mathbf{m}} \cdot \dot{\bar{\mathbf{h}}} + \mu_{\mathbf{o}} \dot{\bar{\mathbf{m}}} \cdot \bar{\mathbf{h}}$ | f2d | 3.10 |
| $\eta^{F\varepsilon dm}$ | $\rho_o \dot{\eta}^{F\varepsilon dm}\theta = \rho_o \dot{\eta}\theta - \epsilon_o J\mathbf{F}^{-T}\bar{\mathbf{e}} \cdot (-\mathbf{L}^T\mathbf{F}^{-T}\bar{\mathbf{e}} + \mathbf{F}^{-T}\dot{\bar{\mathbf{e}}}) - \epsilon_{\mathbf{o}}\mathbf{F}^{-\mathbf{T}}\bar{\mathbf{e}} \cdot \mathbf{F}^{-\mathbf{T}}\bar{\mathbf{e}}\ \mathbf{divv}$ | f2e | 3.10 |
| $\eta^{F\varepsilon pb}$ | $\rho_o \dot{\eta}^{F\varepsilon pb}\theta = \rho_o \dot{\eta}\theta - \mu_o J\mathbf{F}^{-T}\bar{\mathbf{h}} \cdot (-\mathbf{L}^T\mathbf{F}^{-T}\bar{\mathbf{h}} + \mathbf{F}^{-T}\dot{\bar{\mathbf{h}}}) - \mu_{\mathbf{o}}\mathbf{F}^{-\mathbf{T}}\bar{\mathbf{h}} \cdot \mathbf{F}^{-\mathbf{T}}\bar{\mathbf{h}}\ \mathbf{divv}$ | f2f | 3.10 |
| $\eta^{F\varepsilon dh}$ | $\rho_o \dot{\eta}^{F\varepsilon dh}\theta = \rho_o \dot{\eta}\theta - \epsilon_o J\mathbf{F}^{-T}\bar{\mathbf{e}} \cdot (-\mathbf{L}^T\mathbf{F}^{-T}\bar{\mathbf{e}} + \mathbf{F}^{-T}\dot{\bar{\mathbf{e}}}) - \epsilon_{\mathbf{o}}\mathbf{F}^{-\mathbf{T}}\bar{\mathbf{e}} \cdot \mathbf{F}^{-\mathbf{T}}\bar{\mathbf{e}}\ \mathbf{divv} + \mu_{\mathbf{o}}\bar{\mathbf{m}} \cdot \dot{\bar{\mathbf{h}}} + \mu_{\mathbf{o}}\dot{\bar{\mathbf{m}}} \cdot \bar{\mathbf{h}}$ | f2g | 3.10 |
| $\eta^{F\varepsilon eb}$ | $\rho_o \dot{\eta}^{F\varepsilon eb}\theta = \rho_o \dot{\eta}\theta - \mu_o J\mathbf{F}^{-T}\bar{\mathbf{h}} \cdot (-\mathbf{L}^T\mathbf{F}^{-T}\bar{\mathbf{h}} + \mathbf{F}^{-T}\dot{\bar{\mathbf{h}}}) - \mu_{\mathbf{o}}\mathbf{F}^{-\mathbf{T}}\bar{\mathbf{h}} \cdot \mathbf{F}^{-\mathbf{T}}\bar{\mathbf{h}}\ \mathbf{divv} + \bar{\mathbf{p}} \cdot \dot{\bar{\mathbf{e}}} + \dot{\bar{\mathbf{p}}} \cdot \bar{\mathbf{e}}$ | f2h | 3.10 |
| $\eta^{F\varepsilon db}$ | $\rho_o \dot{\eta}^{F\varepsilon db}\theta = \rho_o \dot{\eta}\theta - \epsilon_o J\mathbf{F}^{-T}\bar{\mathbf{e}} \cdot (-\mathbf{L}^T\mathbf{F}^{-T}\bar{\mathbf{e}} + \mathbf{F}^{-T}\dot{\bar{\mathbf{e}}}) - \epsilon_{\mathbf{o}}\mathbf{F}^{-\mathbf{T}}\bar{\mathbf{e}} \cdot \mathbf{F}^{-\mathbf{T}}\bar{\mathbf{e}}\ \mathbf{divv}$<br>$-\mu_o J\mathbf{F}^{-T}\bar{\mathbf{h}} \cdot (-\mathbf{L}^T\mathbf{F}^{-T}\bar{\mathbf{h}} + \mathbf{F}^{-T}\dot{\bar{\mathbf{h}}}) - \mu_{\mathbf{o}}\mathbf{F}^{-\mathbf{T}}\bar{\mathbf{h}} \cdot \mathbf{F}^{-\mathbf{T}}\bar{\mathbf{h}}\ \mathbf{divv}$ | f2i | 3.10 |

Table 3.4: Family 2 - Legendre transformations

Family2- Constitutive Equations

| Residual Equality: $\bar{\mathbf{j}} \cdot \bar{\mathbf{e}} - \frac{\mathbf{Q}}{\theta} \frac{\partial \theta}{\partial \mathbf{X}} - \rho_o \xi \theta = 0$ | | | | |
|---|---|---|---|---|
| potential | Mechanical | Thermal | Electrical | Magnetic |
| $\eta^{F\varepsilon pm}$ | $\rho_o \theta \frac{\partial \eta^{F\varepsilon pm}}{\partial \mathbf{F}} = -(\mathbf{P} + \mathbf{F}^{-T}(\bar{\mathbf{e}} \otimes \bar{\mathbf{p}} + \mu_o \bar{\mathbf{h}} \otimes \bar{\mathbf{m}}))$ | $\frac{\partial \eta^{F\varepsilon pm}}{\partial \varepsilon} = \frac{1}{\theta}$ | $\rho_o \theta \frac{\partial \eta^{F\varepsilon pm}}{\partial \bar{\mathbf{p}}} = -\bar{\mathbf{e}}$ | $\rho_o \theta \frac{\partial \eta^{F\varepsilon pm}}{\partial \bar{\mathbf{m}}} = -\mu_o \bar{\mathbf{h}}$ |
| $\eta^{F\varepsilon em}$ | $\rho_o \theta \frac{\partial \eta^{F\varepsilon em}}{\partial \mathbf{F}} = -(\mathbf{P} + \mathbf{F}^{-T}(\bar{\mathbf{e}} \otimes \bar{\mathbf{p}} + \mu_o \bar{\mathbf{h}} \otimes \bar{\mathbf{m}}))$ | $\frac{\partial \eta^{F\varepsilon em}}{\partial \varepsilon} = \frac{1}{\theta}$ | $\rho_o \theta \frac{\partial \eta^{F\varepsilon em}}{\partial \bar{\mathbf{e}}} = \bar{\mathbf{p}}$ | $\rho_o \theta \frac{\partial \eta^{F\varepsilon em}}{\partial \bar{\mathbf{m}}} = -\mu_o \bar{\mathbf{h}}$ |
| $\eta^{F\varepsilon ph}$ | $\rho_o \theta \frac{\partial \eta^{F\varepsilon ph}}{\partial \mathbf{F}} = -(\mathbf{P} + \mathbf{F}^{-T}(\bar{\mathbf{e}} \otimes \bar{\mathbf{p}} + \mu_o \bar{\mathbf{h}} \otimes \bar{\mathbf{m}}))$ | $\frac{\partial \eta^{F\varepsilon ph}}{\partial \varepsilon} = \frac{1}{\theta}$ | $\rho_o \theta \frac{\partial \eta^{F\varepsilon ph}}{\partial \bar{\mathbf{p}}} = -\bar{\mathbf{e}}$ | $\rho_o \theta \frac{\partial \eta^{F\varepsilon ph}}{\partial \bar{\mathbf{h}}} = \mu_o \bar{\mathbf{m}}$ |
| $\eta^{F\varepsilon eh}$ | $\rho_o \theta \frac{\partial \eta^{F\varepsilon eh}}{\partial \mathbf{F}} = -(\mathbf{P} + \mathbf{F}^{-T}(\bar{\mathbf{e}} \otimes \bar{\mathbf{p}} + \mu_o \bar{\mathbf{h}} \otimes \bar{\mathbf{m}}))$ | $\frac{\partial \eta^{F\varepsilon eh}}{\partial \varepsilon} = \frac{1}{\theta}$ | $\rho_o \theta \frac{\partial \eta^{F\varepsilon eh}}{\partial \bar{\mathbf{e}}} = \bar{\mathbf{p}}$ | $\rho_o \theta \frac{\partial \eta^{F\varepsilon eh}}{\partial \bar{\mathbf{h}}} = \mu_o \bar{\mathbf{m}}$ |
| $\eta^{F\varepsilon dm}$ | $\rho_o \theta \frac{\partial \eta^{F\varepsilon dm}}{\partial \mathbf{F}} = -(\mathbf{P} + \mathbf{F}^{-T}(\bar{\mathbf{e}} \otimes \bar{\mathbf{d}} + \mu_o \bar{\mathbf{h}} \otimes \bar{\mathbf{m}}))$ | $\frac{\partial \eta^{F\varepsilon dm}}{\partial \varepsilon} = \frac{1}{\theta}$ | $\rho_o \theta \frac{\partial \eta^{F\varepsilon dm}}{\partial \bar{\mathbf{d}}} = -\bar{\mathbf{e}}$ | $\rho_o \theta \frac{\partial \eta^{F\varepsilon dm}}{\partial \bar{\mathbf{m}}} = -\mu_o \bar{\mathbf{h}}$ |
| $\eta^{F\varepsilon pb}$ | $\rho_o \theta \frac{\partial \eta^{F\varepsilon pb}}{\partial \mathbf{F}} = -(\mathbf{P} + \mathbf{F}^{-T}(\bar{\mathbf{e}} \otimes \bar{\mathbf{p}} + \bar{\mathbf{h}} \otimes \bar{\mathbf{b}}))$ | $\frac{\partial \eta^{F\varepsilon pb}}{\partial \varepsilon} = \frac{1}{\theta}$ | $\rho_o \theta \frac{\partial \eta^{F\varepsilon pb}}{\partial \bar{\mathbf{p}}} = -\bar{\mathbf{e}}$ | $\rho_o \theta \frac{\partial \eta^{F\varepsilon pb}}{\partial \bar{\mathbf{b}}} = -\mu_o \bar{\mathbf{h}}$ |
| $\eta^{F\varepsilon dh}$ | $\rho_o \theta \frac{\partial \eta^{F\varepsilon dh}}{\partial \mathbf{F}} = -(\mathbf{P} + \mathbf{F}^{-T}(\bar{\mathbf{e}} \otimes \bar{\mathbf{d}} + \mu_o \bar{\mathbf{h}} \otimes \bar{\mathbf{m}}))$ | $\frac{\partial \eta^{F\varepsilon dh}}{\partial \varepsilon} = \frac{1}{\theta}$ | $\rho_o \theta \frac{\partial \eta^{F\varepsilon dh}}{\partial \bar{\mathbf{d}}} = -\bar{\mathbf{e}}$ | $\rho_o \theta \frac{\partial \eta^{F\varepsilon dh}}{\partial \bar{\mathbf{h}}} = \mu_o \bar{\mathbf{m}}$ |
| $\eta^{F\varepsilon eb}$ | $\rho_o \theta \frac{\partial \eta^{F\varepsilon eb}}{\partial \mathbf{F}} = -(\mathbf{P} + \mathbf{F}^{-T}(\bar{\mathbf{e}} \otimes \bar{\mathbf{p}} + \bar{\mathbf{h}} \otimes \bar{\mathbf{b}}))$ | $\frac{\partial \eta^{F\varepsilon eb}}{\partial \varepsilon} = \frac{1}{\theta}$ | $\rho_o \theta \frac{\partial \eta^{F\varepsilon eb}}{\partial \bar{\mathbf{e}}} = \bar{\mathbf{p}}$ | $\rho_o \theta \frac{\partial \eta^{F\varepsilon eb}}{\partial \bar{\mathbf{b}}} = -\mu_o \bar{\mathbf{h}}$ |
| $\eta^{F\varepsilon db}$ | $\rho_o \theta \frac{\partial \eta^{F\varepsilon db}}{\partial \mathbf{F}} = -(\mathbf{P} + \mathbf{F}^{-T}(\bar{\mathbf{e}} \otimes \bar{\mathbf{d}} + \bar{\mathbf{h}} \otimes \bar{\mathbf{b}}))$ | $\frac{\partial \eta^{F\varepsilon db}}{\partial \varepsilon} = \frac{1}{\theta}$ | $\rho_o \theta \frac{\partial \eta^{F\varepsilon db}}{\partial \bar{\mathbf{d}}} = -\bar{\mathbf{e}}$ | $\rho_o \theta \frac{\partial \eta^{F\varepsilon db}}{\partial \bar{\mathbf{b}}} = -\mu_o \bar{\mathbf{h}}$ |

Table 3.5: Family 2 - constitutive equations

Family3-Transformation

| Potential | Transformation equation | eqn no. | Subs eqn no. |
|---|---|---|---|
| $\psi^{F\theta pm}$ | $\psi^{F\theta pm} = \varepsilon^{F\eta pm} - \eta\theta$ | f3a | f1a |
| $\psi^{F\theta em}$ | $\psi^{F\theta em} = \varepsilon^{F\eta em} - \eta\theta$ | f3b | f1b |
| $\psi^{F\theta ph}$ | $\psi^{F\theta ph} = \varepsilon^{F\eta ph} - \eta\theta$ | f3c | f1c |
| $\psi^{F\theta eh}$ | $\psi^{F\theta eh} = \varepsilon^{F\eta eh} - \eta\theta$ | f3d | f1d |
| $\psi^{F\theta dm}$ | $\psi^{F\theta dm} = \varepsilon^{F\eta dm} - \eta\theta$ | f3e | f1e |
| $\psi^{F\theta pb}$ | $\psi^{F\theta pb} = \varepsilon^{F\eta pb} - \eta\theta$ | f3f | f1a |
| $\psi^{F\theta dh}$ | $\psi^{F\theta dh} = \varepsilon^{F\eta dh} - \eta\theta$ | f3g | f1g |
| $\psi^{F\theta eb}$ | $\psi^{F\theta eb} = \varepsilon^{F\eta eb} - \eta\theta$ | f3h | f1h |
| $\psi^{F\theta db}$ | $\psi^{F\theta db} = \varepsilon^{F\eta db} - \eta\theta$ | f3i | f1i |

Table 3.6: Family 3 - Legendre transformations

Family3- Constitutive Equations

| Residual Equality: $\bar{\mathbf{j}} \cdot \bar{\mathbf{e}} - \frac{\mathbf{Q}}{\theta}\frac{\partial \theta}{\partial \mathbf{X}} - \rho_o \xi \theta = 0$ | | | | |
|---|---|---|---|---|
| potential | Mechanical | Thermal | Electrical | Magnetic |
| $\psi^{F\theta pm}$ | $\rho_o \frac{\partial \psi^{F\theta pm}}{\partial \mathbf{F}} = \mathbf{P} + \mathbf{F}^{-T}(\bar{\mathbf{e}} \otimes \bar{\mathbf{p}} + \mu_o \bar{\mathbf{h}} \otimes \bar{\mathbf{m}})$ | $\frac{\partial \psi^{F\theta pm}}{\partial \theta} = -\eta$ | $\rho_o \frac{\partial \psi^{F\theta pm}}{\partial \bar{\mathbf{p}}} = \bar{\mathbf{e}}$ | $\rho_o \frac{\partial \psi^{F\theta pm}}{\partial \bar{\mathbf{m}}} = \mu_o \bar{\mathbf{h}}$ |
| $\psi^{F\theta em}$ | $\rho_o \frac{\partial \psi^{F\theta em}}{\partial \mathbf{F}} = \mathbf{P} + \mathbf{F}^{-T}(\bar{\mathbf{e}} \otimes \bar{\mathbf{p}} + \mu_o \bar{\mathbf{h}} \otimes \bar{\mathbf{m}})$ | $\frac{\partial \psi^{F\theta em}}{\partial \theta} = -\eta$ | $\rho_o \frac{\partial \psi^{F\theta em}}{\partial \bar{\mathbf{e}}} = -\bar{\mathbf{p}}$ | $\rho_o \frac{\partial \psi^{F\theta em}}{\partial \bar{\mathbf{m}}} = \mu_o \bar{\mathbf{h}}$ |
| $\psi^{F\theta ph}$ | $\rho_o \frac{\partial \psi^{F\theta ph}}{\partial \mathbf{F}} = \mathbf{P} + \mathbf{F}^{-T}(\bar{\mathbf{e}} \otimes \bar{\mathbf{p}} + \mu_o \bar{\mathbf{h}} \otimes \bar{\mathbf{m}})$ | $\frac{\partial \psi^{F\theta ph}}{\partial \theta} = -\eta$ | $\rho_o \frac{\partial \psi^{F\theta ph}}{\partial \bar{\mathbf{p}}} = \bar{\mathbf{e}}$ | $\rho_o \frac{\partial \psi^{F\theta ph}}{\partial \bar{\mathbf{h}}} = -\mu_o \bar{\mathbf{m}}$ |
| $\psi^{F\theta eh}$ | $\rho_o \frac{\partial \psi^{F\theta eh}}{\partial \mathbf{F}} = \mathbf{P} + \mathbf{F}^{-T}(\bar{\mathbf{e}} \otimes \bar{\mathbf{p}} + \mu_o \bar{\mathbf{h}} \otimes \bar{\mathbf{m}})$ | $\frac{\partial \psi^{F\theta eh}}{\partial \theta} = -\eta$ | $\rho_o \frac{\partial \psi^{F\theta eh}}{\partial \bar{\mathbf{e}}} = -\bar{\mathbf{p}}$ | $\rho_o \frac{\partial \psi^{F\theta eh}}{\partial \bar{\mathbf{h}}} = -\mu_o \bar{\mathbf{m}}$ |
| $\psi^{F\theta dm}$ | $\rho_o \frac{\partial \psi^{F\theta dm}}{\partial \mathbf{F}} = \mathbf{P} + \mathbf{F}^{-T}(\bar{\mathbf{e}} \otimes \bar{\mathbf{d}} + \mu_o \bar{\mathbf{h}} \otimes \bar{\mathbf{m}})$ | $\frac{\partial \psi^{F\theta dm}}{\partial \theta} = -\eta$ | $\rho_o \frac{\partial \psi^{F\theta dm}}{\partial \bar{\mathbf{d}}} = \bar{\mathbf{e}}$ | $\rho_o \frac{\partial \psi^{F\theta dm}}{\partial \bar{\mathbf{m}}} = \mu_o \bar{\mathbf{h}}$ |
| $\psi^{F\theta pb}$ | $\rho_o \frac{\partial \psi^{F\theta pb}}{\partial \mathbf{F}} = \mathbf{P} + \mathbf{F}^{-T}(\bar{\mathbf{e}} \otimes \bar{\mathbf{p}} + \bar{\mathbf{h}} \otimes \bar{\mathbf{b}})$ | $\frac{\partial \psi^{F\theta pb}}{\partial \theta} = -\eta$ | $\rho_o \frac{\partial \psi^{F\theta pb}}{\partial \bar{\mathbf{p}}} = \bar{\mathbf{e}}$ | $\rho_o \frac{\partial \psi^{F\theta pb}}{\partial \bar{\mathbf{b}}} = \mu_o \bar{\mathbf{h}}$ |
| $\psi^{F\theta dh}$ | $\rho_o \frac{\partial \psi^{F\theta dh}}{\partial \mathbf{F}} = \mathbf{P} + \mathbf{F}^{-T}(\bar{\mathbf{e}} \otimes \bar{\mathbf{d}} + \mu_o \bar{\mathbf{h}} \otimes \bar{\mathbf{m}})$ | $\frac{\partial \psi^{F\theta dh}}{\partial \theta} = -\eta$ | $\rho_o \frac{\partial \psi^{F\theta dh}}{\partial \bar{\mathbf{d}}} = \bar{\mathbf{e}}$ | $\rho_o \frac{\partial \psi^{F\theta dh}}{\partial \bar{\mathbf{h}}} = -\mu_o \bar{\mathbf{m}}$ |
| $\psi^{F\theta eb}$ | $\rho_o \frac{\partial \psi^{F\theta eb}}{\partial \mathbf{F}} = \mathbf{P} + \mathbf{F}^{-T}(\bar{\mathbf{e}} \otimes \bar{\mathbf{p}} + \bar{\mathbf{h}} \otimes \bar{\mathbf{b}})$ | $\frac{\partial \psi^{F\theta eb}}{\partial \theta} = -\eta$ | $\rho_o \frac{\partial \psi^{F\theta eb}}{\partial \bar{\mathbf{e}}} = -\bar{\mathbf{p}}$ | $\rho_o \frac{\partial \psi^{F\theta eb}}{\partial \bar{\mathbf{b}}} = \mu_o \bar{\mathbf{h}}$ |
| $\psi^{F\theta db}$ | $\rho_o \frac{\partial \psi^{F\theta db}}{\partial \mathbf{F}} = \mathbf{P} + \mathbf{F}^{-T}(\bar{\mathbf{e}} \otimes \bar{\mathbf{d}} + \mu_o \bar{\mathbf{h}} \otimes \bar{\mathbf{b}})$ | $\frac{\partial \psi^{F\theta db}}{\partial \theta} = -\eta$ | $\rho_o \frac{\partial \psi^{F\theta db}}{\partial \bar{\mathbf{d}}} = \bar{\mathbf{e}}$ | $\rho_o \frac{\partial \psi^{F\theta db}}{\partial \bar{\mathbf{b}}} = \mu_o \bar{\mathbf{h}}$ |

Table 3.7: Family 3 - constitutive equations

Family4-Transformation

| Potential | Transformation equation | eqn no. | Subs eqn no. |
|---|---|---|---|
| $\theta^{F\psi pm}$ | $\theta^{F\psi pm} = \theta$ | f4a | 3.15 |
| $\theta^{F\psi em}$ | $\rho_o \eta \dot{\theta}^{F\psi em} = \rho_o \eta \dot{\theta} - \bar{\mathbf{p}} \cdot \dot{\bar{\mathbf{e}}} - \dot{\bar{\mathbf{p}}} \cdot \bar{\mathbf{e}}$ | f4b | 3.15 |
| $\theta^{F\psi ph}$ | $\rho_o \eta \dot{\theta}^{F\psi ph} = \rho_o \eta \dot{\theta} - \mu_o \bar{\mathbf{m}} \cdot \dot{\bar{\mathbf{h}}} - \mu_\mathbf{o} \dot{\bar{\mathbf{m}}} \cdot \bar{\mathbf{h}}$ | f4c | 3.15 |
| $\theta^{F\psi eh}$ | $\rho_o \eta \dot{\theta}^{F\psi eh} = \rho_o \eta \dot{\theta} - \bar{\mathbf{p}} \cdot \dot{\bar{\mathbf{e}}} - \dot{\bar{\mathbf{p}}} \cdot \bar{\mathbf{e}} - \mu_\mathbf{o} \bar{\mathbf{m}} \cdot \dot{\bar{\mathbf{h}}} - \mu_\mathbf{o} \dot{\bar{\mathbf{m}}} \cdot \bar{\mathbf{h}}$ | f4d | 3.15 |
| $\theta^{F\psi dm}$ | $\rho_o \eta \dot{\theta}^{F\psi dm} = \rho_o \eta \dot{\theta} + \epsilon_o J \mathbf{F}^{-T} \bar{\mathbf{e}} \cdot (-\mathbf{L}^T \mathbf{F}^{-T} \bar{\mathbf{e}} + \mathbf{F}^{-T} \dot{\bar{\mathbf{e}}}) + \epsilon_\mathbf{o} \mathbf{F}^{-\mathbf{T}} \bar{\mathbf{e}} \cdot \mathbf{F}^{-\mathbf{T}} \bar{\mathbf{e}} \ \mathbf{div v}$ | f4e | 3.15 |
| $\theta^{F\psi pb}$ | $\rho_o \eta \dot{\theta}^{F\psi pb} = \rho_o \eta \dot{\theta} + \mu_o J \mathbf{F}^{-T} \bar{\mathbf{h}} \cdot (-\mathbf{L}^T \mathbf{F}^{-T} \bar{\mathbf{h}} + \mathbf{F}^{-T} \dot{\bar{\mathbf{h}}}) + \mu_\mathbf{o} \mathbf{F}^{-\mathbf{T}} \bar{\mathbf{h}} \cdot \mathbf{F}^{-\mathbf{T}} \bar{\mathbf{h}} \ \mathbf{div v}$ | f4f | 3.15 |
| $\theta^{F\psi dh}$ | $\rho_o \eta \dot{\theta}^{F\psi dh} = \rho_o \eta \dot{\theta} + \epsilon_o J \mathbf{F}^{-T} \bar{\mathbf{e}} \cdot (-\mathbf{L}^T \mathbf{F}^{-T} \bar{\mathbf{e}} + \mathbf{F}^{-T} \dot{\bar{\mathbf{e}}}) + \epsilon_\mathbf{o} \mathbf{F}^{-\mathbf{T}} \bar{\mathbf{e}} \cdot \mathbf{F}^{-\mathbf{T}} \bar{\mathbf{e}} \ \mathbf{div v} - \mu_\mathbf{o} \bar{\mathbf{m}} \cdot \dot{\bar{\mathbf{h}}} - \mu_o \dot{\bar{\mathbf{m}}} \cdot \bar{\mathbf{h}}$ | f4g | 3.15 |
| $\theta^{F\psi eb}$ | $\rho_o \eta \dot{\theta}^{F\psi eb} = \rho_o \eta \dot{\theta} + \mu_o J \mathbf{F}^{-T} \bar{\mathbf{h}} \cdot (-\mathbf{L}^T \mathbf{F}^{-T} \bar{\mathbf{h}} + \mathbf{F}^{-T} \dot{\bar{\mathbf{h}}}) + \mu_\mathbf{o} \mathbf{F}^{-\mathbf{T}} \bar{\mathbf{h}} \cdot \mathbf{F}^{-\mathbf{T}} \bar{\mathbf{h}} \ \mathbf{div v} - \bar{\mathbf{p}} \cdot \dot{\bar{\mathbf{e}}} - \dot{\bar{\mathbf{p}}} \cdot \bar{\mathbf{e}}$ | f4h | 3.15 |
| $\theta^{F\psi db}$ | $\rho_o \eta \dot{\theta}^{F\psi db} = \rho_o \eta \dot{\theta} + \epsilon_o J \mathbf{F}^{-T} \bar{\mathbf{e}} \cdot (-\mathbf{L}^T \mathbf{F}^{-T} \bar{\mathbf{e}} + \mathbf{F}^{-T} \dot{\bar{\mathbf{e}}}) + \epsilon_\mathbf{o} \mathbf{F}^{-\mathbf{T}} \bar{\mathbf{e}} \cdot \mathbf{F}^{-\mathbf{T}} \bar{\mathbf{e}} \ \mathbf{div v}$ $+\mu_o J \mathbf{F}^{-T} \bar{\mathbf{h}} \cdot (-\mathbf{L}^T \mathbf{F}^{-T} \bar{\mathbf{h}} + \mathbf{F}^{-T} \dot{\bar{\mathbf{h}}}) + \mu_\mathbf{o} \mathbf{F}^{-\mathbf{T}} \bar{\mathbf{h}} \cdot \mathbf{F}^{-\mathbf{T}} \bar{\mathbf{h}} \ \mathbf{div v}$ | f4i | 3.15 |

Table 3.8: Family 4 - Legendre transformations

Family4- Constitutive Equations

| Residual Equality: $\bar{\mathbf{j}} \cdot \bar{\mathbf{e}} - \frac{\mathbf{Q}}{\theta} \frac{\partial \theta}{\partial \mathbf{X}} - \rho_o \xi \theta = 0$ | | | | |
|---|---|---|---|---|
| potential | Mechanical | Thermal | Electrical | Magnetic |
| $\theta^{F\psi pm}$ | $\rho_o \eta \frac{\partial \theta^{F\psi pm}}{\partial \mathbf{F}} = (\mathbf{P} + \mathbf{F}^{-T}(\bar{\mathbf{e}} \otimes \bar{\mathbf{p}} + \mu_o \bar{\mathbf{h}} \otimes \bar{\mathbf{m}}))$ | $\frac{\partial \theta^{F\psi pm}}{\partial \psi} = -\frac{1}{\eta}$ | $\rho_o \eta \frac{\partial \theta^{F\psi pm}}{\partial \bar{\mathbf{p}}} = \bar{\mathbf{e}}$ | $\rho_o \eta \frac{\partial \theta^{F\psi pm}}{\partial \bar{\mathbf{m}}} = \mu_o \bar{\mathbf{h}}$ |
| $\theta^{F\psi em}$ | $\rho_o \eta \frac{\partial \theta^{F\psi em}}{\partial \mathbf{F}} = (\mathbf{P} + \mathbf{F}^{-T}(\bar{\mathbf{e}} \otimes \bar{\mathbf{p}} + \mu_o \bar{\mathbf{h}} \otimes \bar{\mathbf{m}}))$ | $\frac{\partial \theta^{F\psi em}}{\partial \psi} = -\frac{1}{\eta}$ | $\rho_o \eta \frac{\partial \theta^{F\psi em}}{\partial \bar{\mathbf{e}}} = -\bar{\mathbf{p}}$ | $\rho_o \eta \frac{\partial \theta^{F\psi em}}{\partial \bar{\mathbf{m}}} = \mu_o \bar{\mathbf{h}}$ |
| $\theta^{F\psi ph}$ | $\rho_o \eta \frac{\partial \theta^{F\psi ph}}{\partial \mathbf{F}} = (\mathbf{P} + \mathbf{F}^{-T}(\bar{\mathbf{e}} \otimes \bar{\mathbf{p}} + \mu_o \bar{\mathbf{h}} \otimes \bar{\mathbf{m}}))$ | $\frac{\partial \theta^{F\psi ph}}{\partial \psi} = -\frac{1}{\eta}$ | $\rho_o \eta \frac{\partial \theta^{F\psi ph}}{\partial \bar{\mathbf{p}}} = \bar{\mathbf{e}}$ | $\rho_o \eta \frac{\partial \theta^{F\psi ph}}{\partial \bar{\mathbf{h}}} = -\mu_o \bar{\mathbf{m}}$ |
| $\theta^{F\psi eh}$ | $\rho_o \eta \frac{\partial \theta^{F\psi eh}}{\partial \mathbf{F}} = (\mathbf{P} + \mathbf{F}^{-T}(\bar{\mathbf{e}} \otimes \bar{\mathbf{p}} + \mu_o \bar{\mathbf{h}} \otimes \bar{\mathbf{m}}))$ | $\frac{\partial \theta^{F\psi eh}}{\partial \psi} = -\frac{1}{\eta}$ | $\rho_o \eta \frac{\partial \theta^{F\psi eh}}{\partial \bar{\mathbf{e}}} = -\bar{\mathbf{p}}$ | $\rho_o \eta \frac{\partial \theta^{F\psi eh}}{\partial \bar{\mathbf{h}}} = -\mu_o \bar{\mathbf{m}}$ |
| $\theta^{F\psi dm}$ | $\rho_o \eta \frac{\partial \theta^{F\psi dm}}{\partial \mathbf{F}} = (\mathbf{P} + \mathbf{F}^{-T}(\bar{\mathbf{e}} \otimes \bar{\mathbf{d}} + \mu_o \bar{\mathbf{h}} \otimes \bar{\mathbf{m}}))$ | $\frac{\partial \theta^{F\psi dm}}{\partial \psi} = -\frac{1}{\eta}$ | $\rho_o \eta \frac{\partial \theta^{F\psi dm}}{\partial \bar{\mathbf{d}}} = \bar{\mathbf{e}}$ | $\rho_o \eta \frac{\partial \theta^{F\psi dm}}{\partial \bar{\mathbf{m}}} = \mu_o \bar{\mathbf{h}}$ |
| $\theta^{F\psi pb}$ | $\rho_o \eta \frac{\partial \theta^{F\psi pb}}{\partial \mathbf{F}} = (\mathbf{P} + \mathbf{F}^{-T}(\bar{\mathbf{e}} \otimes \bar{\mathbf{p}} + \bar{\mathbf{h}} \otimes \bar{\mathbf{b}}))$ | $\frac{\partial \theta^{F\psi pb}}{\partial \psi} = -\frac{1}{\eta}$ | $\rho_o \eta \frac{\partial \theta^{F\psi pb}}{\partial \bar{\mathbf{p}}} = \bar{\mathbf{e}}$ | $\rho_o \eta \frac{\partial \theta^{F\psi pb}}{\partial \bar{\mathbf{b}}} = \mu_o \bar{\mathbf{h}}$ |
| $\theta^{F\psi dh}$ | $\rho_o \eta \frac{\partial \theta^{F\psi dh}}{\partial \mathbf{F}} = (\mathbf{P} + \mathbf{F}^{-T}(\bar{\mathbf{e}} \otimes \bar{\mathbf{d}} + \mu_o \bar{\mathbf{h}} \otimes \bar{\mathbf{m}}))$ | $\frac{\partial \theta^{F\psi dh}}{\partial \psi} = -\frac{1}{\eta}$ | $\rho_o \eta \frac{\partial \theta^{F\psi dh}}{\partial \bar{\mathbf{d}}} = \bar{\mathbf{e}}$ | $\rho_o \eta \frac{\partial \theta^{F\psi dh}}{\partial \bar{\mathbf{h}}} = -\mu_o \bar{\mathbf{m}}$ |
| $\theta^{F\psi eb}$ | $\rho_o \eta \frac{\partial \theta^{F\psi eb}}{\partial \mathbf{F}} = (\mathbf{P} + \mathbf{F}^{-T}(\bar{\mathbf{e}} \otimes \bar{\mathbf{p}} + \bar{\mathbf{h}} \otimes \bar{\mathbf{b}}))$ | $\frac{\partial \theta^{F\psi eb}}{\partial \psi} = -\frac{1}{\eta}$ | $\rho_o \eta \frac{\partial \theta^{F\psi eb}}{\partial \bar{\mathbf{e}}} = -\bar{\mathbf{p}}$ | $\rho_o \eta \frac{\partial \theta^{F\psi eb}}{\partial \bar{\mathbf{b}}} = \mu_o \bar{\mathbf{h}}$ |
| $\theta^{F\psi db}$ | $\rho_o \eta \frac{\partial \theta^{F\psi db}}{\partial \mathbf{F}} = (\mathbf{P} + \mathbf{F}^{-T}(\bar{\mathbf{e}} \otimes \bar{\mathbf{d}} + \bar{\mathbf{h}} \otimes \bar{\mathbf{b}}))$ | $\frac{\partial \theta^{F\psi db}}{\partial \psi} = -\frac{1}{\eta}$ | $\rho_o \eta \frac{\partial \theta^{F\psi db}}{\partial \bar{\mathbf{d}}} = \bar{\mathbf{e}}$ | $\rho_o \eta \frac{\partial \theta^{F\psi db}}{\partial \bar{\mathbf{b}}} = \mu_o \bar{\mathbf{h}}$ |

Table 3.9: Family 4 - constitutive equations

Family5-Transformation

| Potential | Transformation equation | eqn no. | Subs eqn no. |
|---|---|---|---|
| $\chi^{P\eta pm}$ | $\rho_o \chi^{P\eta pm} = \rho_o \varepsilon^{F\eta pm} - \mathbf{P} \cdot \mathbf{F}$ | f5a | f1a |
| $\chi^{P\eta em}$ | $\rho_o \chi^{P\eta em} = \rho_o \varepsilon^{F\eta em} - \mathbf{P} \cdot \mathbf{F}$ | f5b | f1b |
| $\chi^{P\eta ph}$ | $\rho_o \chi^{P\eta ph} = \rho_o \varepsilon^{F\eta ph} - \mathbf{P} \cdot \mathbf{F}$ | f5c | f1c |
| $\chi^{P\eta eh}$ | $\rho_o \chi^{P\eta eh} = \rho_o \varepsilon^{F\eta eh} - \mathbf{P} \cdot \mathbf{F}$ | f5d | f1d |
| $\chi^{P\eta dm}$ | $\rho_o \chi^{P\eta dm} = \rho_o \varepsilon^{F\eta dm} - \mathbf{P} \cdot \mathbf{F}$ | f5e | f1e |
| $\chi^{P\eta pb}$ | $\rho_o \chi^{P\eta pb} = \rho_o \varepsilon^{F\eta pb} - \mathbf{P} \cdot \mathbf{F}$ | f5f | f1a |
| $\chi^{P\eta dh}$ | $\rho_o \chi^{P\eta dh} = \rho_o \varepsilon^{F\eta dh} - \mathbf{P} \cdot \mathbf{F}$ | f5g | f1g |
| $\chi^{P\eta eb}$ | $\rho_o \chi^{P\eta eb} = \rho_o \varepsilon^{F\eta eb} - \mathbf{P} \cdot \mathbf{F}$ | f5h | f1h |
| $\chi^{P\eta db}$ | $\rho_o \chi^{P\eta db} = \rho_o \varepsilon^{F\eta db} - \mathbf{P} \cdot \mathbf{F}$ | f5i | f1i |

Table 3.10: Family 5 - Legendre transformations

Family5- Constitutive Equations

| Residual Equality: $\mathrm{A} + \bar{\mathbf{j}} \cdot \bar{\mathbf{e}} - \frac{\mathbf{Q}}{\theta}\frac{\partial\theta}{\partial\mathbf{X}} - \rho_o\xi\theta = 0$ | | | | | |
|---|---|---|---|---|---|
| potential | Mechanical | Thermal | Electrical | Magnetic | A |
| $\chi^{P\eta pm}$ | $\rho_o\frac{\partial\chi^{P\eta pm}}{\partial\mathbf{P}} = -\mathbf{F}$ | $\frac{\partial\chi^{P\eta pm}}{\partial\eta} = \theta$ | $\rho_o\frac{\partial\chi^{P\eta pm}}{\partial\bar{\mathbf{p}}} = \bar{\mathbf{e}}$ | $\rho_o\frac{\partial\chi^{P\eta pm}}{\partial\bar{\mathbf{m}}} = \mu_o\bar{\mathbf{h}}$ | $\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{p}} \cdot \bar{\mathbf{e}} + \mu_o\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{m}} \cdot \bar{\mathbf{h}}$ |
| $\chi^{P\eta em}$ | $\rho_o\frac{\partial\chi^{P\eta em}}{\partial\mathbf{P}} = -\mathbf{F}$ | $\frac{\partial\chi^{P\eta em}}{\partial\eta} = \theta$ | $\rho_o\frac{\partial\chi^{P\eta em}}{\partial\bar{\mathbf{e}}} = -\bar{\mathbf{p}}$ | $\rho_o\frac{\partial\chi^{P\eta em}}{\partial\bar{\mathbf{m}}} = \mu_o\bar{\mathbf{h}}$ | $\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{p}} \cdot \bar{\mathbf{e}} + \mu_o\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{m}} \cdot \bar{\mathbf{h}}$ |
| $\chi^{P\eta ph}$ | $\rho_o\frac{\partial\chi^{P\eta ph}}{\partial\mathbf{P}} = -\mathbf{F}$ | $\frac{\partial\chi^{P\eta ph}}{\partial\eta} = \theta$ | $\rho_o\frac{\partial\chi^{P\eta ph}}{\partial\bar{\mathbf{p}}} = \bar{\mathbf{e}}$ | $\rho_o\frac{\partial\chi^{P\eta ph}}{\partial\bar{\mathbf{h}}} = -\mu_o\bar{\mathbf{m}}$ | $\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{p}} \cdot \bar{\mathbf{e}} + \mu_o\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{m}} \cdot \bar{\mathbf{h}}$ |
| $\chi^{P\eta eh}$ | $\rho_o\frac{\partial\chi^{P\eta eh}}{\partial\mathbf{P}} = -\mathbf{F}$ | $\frac{\partial\chi^{P\eta eh}}{\partial\eta} = \theta$ | $\rho_o\frac{\partial\chi^{P\eta eh}}{\partial\bar{\mathbf{e}}} = -\bar{\mathbf{p}}$ | $\rho_o\frac{\partial\chi^{P\eta eh}}{\partial\bar{\mathbf{h}}} = -\mu_o\bar{\mathbf{m}}$ | $\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{p}} \cdot \bar{\mathbf{e}} + \mu_o\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{m}} \cdot \bar{\mathbf{h}}$ |
| $\chi^{P\eta dm}$ | $\rho_o\frac{\partial\chi^{P\eta dm}}{\partial\mathbf{P}} = -\mathbf{F}$ | $\frac{\partial\chi^{P\eta dm}}{\partial\eta} = \theta$ | $\rho_o\frac{\partial\chi^{P\eta dm}}{\partial\bar{\mathbf{d}}} = \bar{\mathbf{e}}$ | $\rho_o\frac{\partial\chi^{P\eta dm}}{\partial\bar{\mathbf{m}}} = \mu_o\bar{\mathbf{h}}$ | $\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{d}} \cdot \bar{\mathbf{e}} + \mu_o\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{m}} \cdot \bar{\mathbf{h}}$ |
| $\chi^{P\eta pb}$ | $\rho_o\frac{\partial\chi^{P\eta pb}}{\partial\mathbf{P}} = -\mathbf{F}$ | $\frac{\partial\chi^{P\eta pb}}{\partial\eta} = \theta$ | $\rho_o\frac{\partial\chi^{P\eta pb}}{\partial\bar{\mathbf{p}}} = \bar{\mathbf{e}}$ | $\rho_o\frac{\partial\chi^{P\eta pb}}{\partial\bar{\mathbf{b}}} = \mu_o\bar{\mathbf{h}}$ | $\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{p}} \cdot \bar{\mathbf{e}} + \mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{b}} \cdot \bar{\mathbf{h}}$ |
| $\chi^{P\eta dh}$ | $\rho_o\frac{\partial\chi^{P\eta dh}}{\partial\mathbf{P}} = -\mathbf{F}$ | $\frac{\partial\chi^{P\eta dh}}{\partial\eta} = \theta$ | $\rho_o\frac{\partial\chi^{P\eta dh}}{\partial\bar{\mathbf{d}}} = \bar{\mathbf{e}}$ | $\rho_o\frac{\partial\chi^{P\eta dh}}{\partial\bar{\mathbf{h}}} = -\mu_o\bar{\mathbf{m}}$ | $\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{d}} \cdot \bar{\mathbf{e}} + \mu_o\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{m}} \cdot \bar{\mathbf{h}}$ |
| $\chi^{P\eta eb}$ | $\rho_o\frac{\partial\chi^{P\eta eb}}{\partial\mathbf{P}} = -\mathbf{F}$ | $\frac{\partial\chi^{P\eta eb}}{\partial\eta} = \theta$ | $\rho_o\frac{\partial\chi^{P\eta eb}}{\partial\bar{\mathbf{e}}} = -\bar{\mathbf{p}}$ | $\rho_o\frac{\partial\chi^{P\eta eb}}{\partial\bar{\mathbf{b}}} = \mu_o\bar{\mathbf{h}}$ | $\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{p}} \cdot \bar{\mathbf{e}} + \mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{b}} \cdot \bar{\mathbf{h}}$ |
| $\chi^{P\eta db}$ | $\rho_o\frac{\partial\chi^{P\eta db}}{\partial\mathbf{P}} = -\mathbf{F}$ | $\frac{\partial\chi^{P\eta db}}{\partial\eta} = \theta$ | $\rho_o\frac{\partial\chi^{P\eta db}}{\partial\bar{\mathbf{d}}} = \bar{\mathbf{e}}$ | $\rho_o\frac{\partial\chi^{P\eta db}}{\partial\bar{\mathbf{b}}} = \mu_o\bar{\mathbf{h}}$ | $\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{d}} \cdot \bar{\mathbf{e}} + \mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{b}} \cdot \bar{\mathbf{h}}$ |

Table 3.11: Family 5 - constitutive equations

Family6-Transformation

| Potential | Transformation equation | eqn no. | Subs eqn no. |
|---|---|---|---|
| $\eta^{P\chi pm}$ | $\eta^{P\chi pm} = \eta$ | f6a | 3.16 |
| $\eta^{P\chi em}$ | $\rho_o\dot{\eta}^{P\chi em}\theta = \rho_o\dot{\eta}\theta + \bar{\mathbf{p}} \cdot \dot{\bar{\mathbf{e}}} + \dot{\bar{\mathbf{p}}} \cdot \bar{\mathbf{e}}$ | f6b | 3.16 |
| $\eta^{P\chi ph}$ | $\rho_o\dot{\eta}^{P\chi ph}\theta = \rho_o\dot{\eta}\theta + \mu_o\bar{\mathbf{m}} \cdot \dot{\bar{\mathbf{h}}} + \mu_{\mathbf{o}}\dot{\bar{\mathbf{m}}} \cdot \bar{\mathbf{h}}$ | f6c | 3.16 |
| $\eta^{P\chi eh}$ | $\rho_o\dot{\eta}^{P\chi eh}\theta = \rho_o\dot{\eta}\theta + \bar{\mathbf{p}} \cdot \dot{\bar{\mathbf{e}}} + \dot{\bar{\mathbf{p}}} \cdot \bar{\mathbf{e}} + \mu_{\mathbf{o}}\bar{\mathbf{m}} \cdot \dot{\bar{\mathbf{h}}} + \mu_{\mathbf{o}}\dot{\bar{\mathbf{m}}} \cdot \bar{\mathbf{h}}$ | 6d | 3.16 |
| $\eta^{P\chi dm}$ | $\rho_o\dot{\eta}^{P\chi dm}\theta = \rho_o\dot{\eta}\theta - \epsilon_o J\mathbf{F}^{-T}\bar{\mathbf{e}} \cdot (-\mathbf{L}^T\mathbf{F}^{-T}\bar{\mathbf{e}} + \mathbf{F}^{-T}\dot{\bar{\mathbf{e}}}) - \epsilon_{\mathbf{o}}\mathbf{F}^{-\mathbf{T}}\bar{\mathbf{e}} \cdot \mathbf{F}^{-\mathbf{T}}\bar{\mathbf{e}}\ \mathbf{divv}$ | f6e | 3.16 |
| $\eta^{P\chi pb}$ | $\rho_o\dot{\eta}^{P\chi pb}\theta = \rho_o\dot{\eta}\theta - \mu_o J\mathbf{F}^{-T}\bar{\mathbf{h}} \cdot (-\mathbf{L}^T\mathbf{F}^{-T}\bar{\mathbf{h}} + \mathbf{F}^{-T}\dot{\bar{\mathbf{h}}}) - \mu_{\mathbf{o}}\mathbf{F}^{-\mathbf{T}}\bar{\mathbf{h}} \cdot \mathbf{F}^{-\mathbf{T}}\bar{\mathbf{h}}\ \mathbf{divv}$ | f6f | 3.16 |
| $\eta^{P\chi dh}$ | $\rho_o\dot{\eta}^{P\chi dh}\theta = \rho_o\dot{\eta}\theta - \epsilon_o J\mathbf{F}^{-T}\bar{\mathbf{e}} \cdot (-\mathbf{L}^T\mathbf{F}^{-T}\bar{\mathbf{e}} + \mathbf{F}^{-T}\dot{\bar{\mathbf{e}}}) - \epsilon_{\mathbf{o}}\mathbf{F}^{-\mathbf{T}}\bar{\mathbf{e}} \cdot \mathbf{F}^{-\mathbf{T}}\bar{\mathbf{e}}\ \mathbf{divv} + \mu_{\mathbf{o}}\bar{\mathbf{m}} \cdot \dot{\bar{\mathbf{h}}} + \mu_{\mathbf{o}}\dot{\bar{\mathbf{m}}} \cdot \bar{\mathbf{h}}$ | f6g | 3.16 |
| $\eta^{P\chi eb}$ | $\rho_o\dot{\eta}^{P\chi eb}\theta = \rho_o\dot{\eta}\theta - \mu_o J\mathbf{F}^{-T}\bar{\mathbf{h}} \cdot (-\mathbf{L}^T\mathbf{F}^{-T}\bar{\mathbf{h}} + \mathbf{F}^{-T}\dot{\bar{\mathbf{h}}}) - \mu_{\mathbf{o}}\mathbf{F}^{-\mathbf{T}}\bar{\mathbf{h}} \cdot \mathbf{F}^{-\mathbf{T}}\bar{\mathbf{h}}\ \mathbf{divv} + \bar{\mathbf{p}} \cdot \dot{\bar{\mathbf{e}}} + \dot{\bar{\mathbf{p}}} \cdot \bar{\mathbf{e}}$ | f6h | 3.16 |
| $\eta^{P\chi db}$ | $\rho_o\dot{\eta}^{P\chi db}\theta = \rho_o\dot{\eta}\theta - \epsilon_o J\mathbf{F}^{-T}\bar{\mathbf{e}} \cdot (-\mathbf{L}^T\mathbf{F}^{-T}\bar{\mathbf{e}} + \mathbf{F}^{-T}\dot{\bar{\mathbf{e}}}) - \epsilon_{\mathbf{o}}\mathbf{F}^{-\mathbf{T}}\bar{\mathbf{e}} \cdot \mathbf{F}^{-\mathbf{T}}\bar{\mathbf{e}}\ \mathbf{divv}$ | f6i | 3.16 |
| | $-\mu_o J\mathbf{F}^{-T}\bar{\mathbf{h}} \cdot (-\mathbf{L}^T\mathbf{F}^{-T}\bar{\mathbf{h}} + \mathbf{F}^{-T}\dot{\bar{\mathbf{h}}}) - \mu_{\mathbf{o}}\mathbf{F}^{-\mathbf{T}}\bar{\mathbf{h}} \cdot \mathbf{F}^{-\mathbf{T}}\bar{\mathbf{h}}\ \mathbf{divv}$ | | |

Table 3.12: Family 6 - Legendre transformations

Family6- Constitutive Equations

| Residual Equality: $\mathrm{A} + \bar{\mathbf{j}} \cdot \bar{\mathbf{e}} - \frac{\mathbf{Q}}{\theta} \frac{\partial \theta}{\partial \mathbf{X}} - \rho_o \xi \theta = 0$ | | | | | |
|---|---|---|---|---|---|
| potential | Mechanical | Thermal | Electrical | Magnetic | A |
| $\eta^{P\chi pm}$ | $\rho_o \theta \frac{\partial \eta^{P\chi pm}}{\partial \mathbf{P}} = \mathbf{F}$ | $\frac{\partial \eta^{P\chi pm}}{\partial \chi} = \frac{1}{\theta}$ | $\rho_o \theta \frac{\partial \eta^{P\chi pm}}{\partial \bar{\mathbf{p}}} = -\bar{\mathbf{e}}$ | $\rho_o \theta \frac{\partial \eta^{P\chi pm}}{\partial \bar{\mathbf{m}}} = -\mu_o \bar{\mathbf{h}}$ | $\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{p}} \cdot \bar{\mathbf{e}} + \mu_o \mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{m}} \cdot \bar{\mathbf{h}}$ |
| $\eta^{P\chi em}$ | $\rho_o \theta \frac{\partial \eta^{P\chi em}}{\partial \mathbf{P}} = \mathbf{F}$ | $\frac{\partial \eta^{P\chi em}}{\partial \chi} = \frac{1}{\theta}$ | $\rho_o \theta \frac{\partial \eta^{P\chi em}}{\partial \bar{\mathbf{e}}} = \bar{\mathbf{p}}$ | $\rho_o \theta \frac{\partial \eta^{P\chi em}}{\partial \bar{\mathbf{m}}} = -\mu_o \bar{\mathbf{h}}$ | $\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{p}} \cdot \bar{\mathbf{e}} + \mu_o \mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{m}} \cdot \bar{\mathbf{h}}$ |
| $\eta^{P\chi ph}$ | $\rho_o \theta \frac{\partial \eta^{P\chi ph}}{\partial \mathbf{P}} = \mathbf{F}$ | $\frac{\partial \eta^{P\chi ph}}{\partial \chi} = \frac{1}{\theta}$ | $\rho_o \theta \frac{\partial \eta^{P\chi ph}}{\partial \bar{\mathbf{p}}} = -\bar{\mathbf{e}}$ | $\rho_o \theta \frac{\partial \eta^{P\chi ph}}{\partial \bar{\mathbf{h}}} = \mu_o \bar{\mathbf{m}}$ | $\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{p}} \cdot \bar{\mathbf{e}} + \mu_o \mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{m}} \cdot \bar{\mathbf{h}}$ |
| $\eta^{P\chi eh}$ | $\rho_o \theta \frac{\partial \eta^{P\chi eh}}{\partial \mathbf{P}} = \mathbf{F}$ | $\frac{\partial \eta^{P\chi eh}}{\partial \chi} = \frac{1}{\theta}$ | $\rho_o \theta \frac{\partial \eta^{P\chi eh}}{\partial \bar{\mathbf{e}}} = \bar{\mathbf{p}}$ | $\rho_o \theta \frac{\partial \eta^{P\chi eh}}{\partial \bar{\mathbf{h}}} = \mu_o \bar{\mathbf{m}}$ | $\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{p}} \cdot \bar{\mathbf{e}} + \mu_o \mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{m}} \cdot \bar{\mathbf{h}}$ |
| $\eta^{P\chi dm}$ | $\rho_o \theta \frac{\partial \eta^{P\chi dm}}{\partial \mathbf{P}} = \mathbf{F}$ | $\frac{\partial \eta^{P\chi dm}}{\partial \chi} = \frac{1}{\theta}$ | $\rho_o \theta \frac{\partial \eta^{P\chi dm}}{\partial \bar{\mathbf{d}}} = -\bar{\mathbf{e}}$ | $\rho_o \theta \frac{\partial \eta^{P\chi dm}}{\partial \bar{\mathbf{m}}} = -\mu_o \bar{\mathbf{h}}$ | $\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{d}} \cdot \bar{\mathbf{e}} + \mu_o \mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{m}} \cdot \bar{\mathbf{h}}$ |
| $\eta^{P\chi pb}$ | $\rho_o \theta \frac{\partial \eta^{P\chi pb}}{\partial \mathbf{P}} = \mathbf{F}$ | $\frac{\partial \eta^{P\chi pb}}{\partial \chi} = \frac{1}{\theta}$ | $\rho_o \theta \frac{\partial \eta^{P\chi pb}}{\partial \bar{\mathbf{p}}} = -\bar{\mathbf{e}}$ | $\rho_o \theta \frac{\partial \eta^{P\chi pb}}{\partial \bar{\mathbf{b}}} = -\mu_o \bar{\mathbf{h}}$ | $\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{p}} \cdot \bar{\mathbf{e}} + \mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{b}} \cdot \bar{\mathbf{h}}$ |
| $\eta^{P\chi dh}$ | $\rho_o \theta \frac{\partial \eta^{P\chi dh}}{\partial \mathbf{P}} = \mathbf{F}$ | $\frac{\partial \eta^{P\chi dh}}{\partial \chi} = \frac{1}{\theta}$ | $\rho_o \theta \frac{\partial \eta^{P\chi dh}}{\partial \bar{\mathbf{d}}} = -\bar{\mathbf{e}}$ | $\rho_o \theta \frac{\partial \eta^{P\chi dh}}{\partial \bar{\mathbf{h}}} = \mu_o \bar{\mathbf{m}}$ | $\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{d}} \cdot \bar{\mathbf{e}} + \mu_o \mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{m}} \cdot \bar{\mathbf{h}}$ |
| $\eta^{P\chi eb}$ | $\rho_o \theta \frac{\partial \eta^{P\chi eb}}{\partial \mathbf{P}} = \mathbf{F}$ | $\frac{\partial \eta^{P\chi eb}}{\partial \chi} = \frac{1}{\theta}$ | $\rho_o \theta \frac{\partial \eta^{P\chi eb}}{\partial \bar{\mathbf{e}}} = \bar{\mathbf{p}}$ | $\rho_o \theta \frac{\partial \eta^{P\chi eb}}{\partial \bar{\mathbf{b}}} = -\mu_o \bar{\mathbf{h}}$ | $\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{p}} \cdot \bar{\mathbf{e}} + \mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{b}} \cdot \bar{\mathbf{h}}$ |
| $\eta^{P\chi db}$ | $\rho_o \theta \frac{\partial \eta^{P\chi db}}{\partial \mathbf{P}} = \mathbf{F}$ | $\frac{\partial \eta^{P\chi db}}{\partial \chi} = \frac{1}{\theta}$ | $\rho_o \theta \frac{\partial \eta^{P\chi db}}{\partial \bar{\mathbf{d}}} = -\bar{\mathbf{e}}$ | $\rho_o \theta \frac{\partial \eta^{P\chi db}}{\partial \bar{\mathbf{b}}} = -\mu_o \bar{\mathbf{h}}$ | $\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{d}} \cdot \bar{\mathbf{e}} + \mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{b}} \cdot \bar{\mathbf{h}}$ |

Table 3.13: Family 6 - constitutive equations

Family7-Transformation

| Potential | Transformation equation | eqn no. | Subs eqn no. |
|---|---|---|---|
| $\phi^{P\theta pm}$ | $\rho_o \phi^{P\theta pm} = \rho_o \varepsilon^{F\eta pm} - \mathbf{P} \cdot \mathbf{F} - \rho_o \eta \theta$ | f7a | f1a |
| $\phi^{P\theta em}$ | $\rho_o \phi^{P\theta em} = \rho_o \varepsilon^{F\eta em} - \mathbf{P} \cdot \mathbf{F} - \rho_o \eta \theta$ | f7b | f1b |
| $\phi^{P\theta ph}$ | $\rho_o \phi^{P\theta ph} = \rho_o \varepsilon^{F\eta ph} - \mathbf{P} \cdot \mathbf{F} - \rho_o \eta \theta$ | f7c | f1c |
| $\phi^{P\theta eh}$ | $\rho_o \phi^{P\theta eh} = \rho_o \varepsilon^{F\eta eh} - \mathbf{P} \cdot \mathbf{F} - \rho_o \eta \theta$ | f7d | f1d |
| $\phi^{P\theta dm}$ | $\rho_o \phi^{P\theta dm} = \rho_o \varepsilon^{F\eta dm} - \mathbf{P} \cdot \mathbf{F} - \rho_o \eta \theta$ | f7e | f1e |
| $\phi^{P\theta pb}$ | $\rho_o \phi^{P\theta pb} = \rho_o \varepsilon^{F\eta pb} - \mathbf{P} \cdot \mathbf{F} - \rho_o \eta \theta$ | f7f | f1a |
| $\phi^{P\theta dh}$ | $\rho_o \phi^{P\theta dh} = \rho_o \varepsilon^{F\eta dh} - \mathbf{P} \cdot \mathbf{F} - \rho_o \eta \theta$ | f7g | f1g |
| $\phi^{P\theta eb}$ | $\rho_o \phi^{P\theta eb} = \rho_o \varepsilon^{F\eta eb} - \mathbf{P} \cdot \mathbf{F} - \rho_o \eta \theta$ | f7h | f1h |
| $\phi^{P\theta db}$ | $\rho_o \phi^{P\theta db} = \rho_o \varepsilon^{F\eta db} - \mathbf{P} \cdot \mathbf{F} - \rho_o \eta \theta$ | f7i | f1i |

Table 3.14: Family 7 - Legendre transformations

Family7- Constitutive Equations

| Residual Equality: A+ $\bar{\mathbf{j}} \cdot \bar{\mathbf{e}} - \frac{\mathbf{Q}}{\theta}\frac{\partial \theta}{\partial \mathbf{X}} - \rho_o \xi \theta = 0$ | | | | | |
|---|---|---|---|---|---|
| potential | Mechanical | Thermal | Electrical | Magnetic | A |
| $\phi^{P\theta pm}$ | $\rho_o \frac{\partial \phi^{P\theta pm}}{\partial \mathbf{P}} = -\mathbf{F}$ | $\frac{\partial \phi^{P\theta pm}}{\partial \theta} = -\eta$ | $\rho_o \frac{\partial \phi^{P\theta pm}}{\partial \bar{\mathbf{p}}} = \bar{\mathbf{e}}$ | $\rho_o \frac{\partial \phi^{P\theta pm}}{\partial \bar{\mathbf{m}}} = \mu_o \bar{\mathbf{h}}$ | $\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{p}} \cdot \bar{\mathbf{e}} + \mu_o \mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{m}} \cdot \bar{\mathbf{h}}$ |
| $\phi^{P\theta em}$ | $\rho_o \frac{\partial \phi^{P\theta em}}{\partial \mathbf{P}} = -\mathbf{F}$ | $\frac{\partial \phi^{P\theta em}}{\partial \theta} = -\eta$ | $\rho_o \frac{\partial \phi^{P\theta em}}{\partial \bar{\mathbf{e}}} = -\bar{\mathbf{p}}$ | $\rho_o \frac{\partial \phi^{P\theta em}}{\partial \bar{\mathbf{m}}} = \mu_o \bar{\mathbf{h}}$ | $\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{p}} \cdot \bar{\mathbf{e}} + \mu_o \mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{m}} \cdot \bar{\mathbf{h}}$ |
| $\phi^{P\theta ph}$ | $\rho_o \frac{\partial \phi^{P\theta ph}}{\partial \mathbf{P}} = -\mathbf{F}$ | $\frac{\partial \phi^{P\theta ph}}{\partial \theta} = -\eta$ | $\rho_o \frac{\partial \phi^{P\theta ph}}{\partial \bar{\mathbf{p}}} = \bar{\mathbf{e}}$ | $\rho_o \frac{\partial \phi^{P\theta ph}}{\partial \bar{\mathbf{h}}} = -\mu_o \bar{\mathbf{m}}$ | $\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{p}} \cdot \bar{\mathbf{e}} + \mu_o \mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{m}} \cdot \bar{\mathbf{h}}$ |
| $\phi^{P\theta eh}$ | $\rho_o \frac{\partial \phi^{P\theta eh}}{\partial \mathbf{P}} = -\mathbf{F}$ | $\frac{\partial \phi^{P\theta eh}}{\partial \theta} = -\eta$ | $\rho_o \frac{\partial \phi^{P\theta eh}}{\partial \bar{\mathbf{e}}} = -\bar{\mathbf{p}}$ | $\rho_o \frac{\partial \phi^{P\theta eh}}{\partial \bar{\mathbf{h}}} = -\mu_o \bar{\mathbf{m}}$ | $\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{p}} \cdot \bar{\mathbf{e}} + \mu_o \mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{m}} \cdot \bar{\mathbf{h}}$ |
| $\phi^{P\theta dm}$ | $\rho_o \frac{\partial \phi^{P\theta dm}}{\partial \mathbf{P}} = -\mathbf{F}$ | $\frac{\partial \phi^{P\theta dm}}{\partial \theta} = -\eta$ | $\rho_o \frac{\partial \phi^{P\theta dm}}{\partial \bar{\mathbf{d}}} = \bar{\mathbf{e}}$ | $\rho_o \frac{\partial \phi^{P\theta dm}}{\partial \bar{\mathbf{m}}} = \mu_o \bar{\mathbf{h}}$ | $\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{d}} \cdot \bar{\mathbf{e}} + \mu_o \mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{m}} \cdot \bar{\mathbf{h}}$ |
| $\phi^{P\theta pb}$ | $\rho_o \frac{\partial \phi^{P\theta pb}}{\partial \mathbf{P}} = -\mathbf{F}$ | $\frac{\partial \phi^{P\theta pb}}{\partial \theta} = -\eta$ | $\rho_o \frac{\partial \phi^{P\theta pb}}{\partial \bar{\mathbf{p}}} = \bar{\mathbf{e}}$ | $\rho_o \frac{\partial \phi^{P\theta pb}}{\partial \bar{\mathbf{b}}} = \mu_o \bar{\mathbf{h}}$ | $\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{p}} \cdot \bar{\mathbf{e}} + \mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{b}} \cdot \bar{\mathbf{h}}$ |
| $\phi^{P\theta dh}$ | $\rho_o \frac{\partial \phi^{P\theta dh}}{\partial \mathbf{P}} = -\mathbf{F}$ | $\frac{\partial \phi^{P\theta dh}}{\partial \theta} = -\eta$ | $\rho_o \frac{\partial \phi^{P\theta dh}}{\partial \bar{\mathbf{d}}} = \bar{\mathbf{e}}$ | $\rho_o \frac{\partial \phi^{P\theta dh}}{\partial \bar{\mathbf{h}}} = -\mu_o \bar{\mathbf{m}}$ | $\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{d}} \cdot \bar{\mathbf{e}} + \mu_o \mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{m}} \cdot \bar{\mathbf{h}}$ |
| $\phi^{P\theta eb}$ | $\rho_o \frac{\partial \phi^{P\theta eb}}{\partial \mathbf{P}} = -\mathbf{F}$ | $\frac{\partial \phi^{P\theta eb}}{\partial \theta} = -\eta$ | $\rho_o \frac{\partial \phi^{P\theta eb}}{\partial \bar{\mathbf{e}}} = -\bar{\mathbf{p}}$ | $\rho_o \frac{\partial \phi^{P\theta eb}}{\partial \bar{\mathbf{b}}} = \mu_o \bar{\mathbf{h}}$ | $\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{p}} \cdot \bar{\mathbf{e}} + \mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{b}} \cdot \bar{\mathbf{h}}$ |
| $\phi^{P\theta db}$ | $\rho_o \frac{\partial \phi^{P\theta db}}{\partial \mathbf{P}} = -\mathbf{F}$ | $\frac{\partial \phi^{P\theta db}}{\partial \theta} = -\eta$ | $\rho_o \frac{\partial \phi^{P\theta db}}{\partial \bar{\mathbf{d}}} = \bar{\mathbf{e}}$ | $\rho_o \frac{\partial \phi^{P\theta db}}{\partial \bar{\mathbf{b}}} = \mu_o \bar{\mathbf{h}}$ | $\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{d}} \cdot \bar{\mathbf{e}} + \mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{b}} \cdot \bar{\mathbf{h}}$ |

Table 3.15: Family 7 - constitutive equations

Family8-Transformation

| Potential | Transformation equation | eqn no. | Subs eqn no. |
|---|---|---|---|
| $\theta^{P\phi pm}$ | $\theta^{P\phi pm} = \theta$ | f8a | 3.17 |
| $\theta^{P\phi em}$ | $\rho_o \eta \dot{\theta}^{P\phi em} = \rho_o \eta \dot{\theta} - \bar{\mathbf{p}} \cdot \dot{\bar{\mathbf{e}}} - \dot{\bar{\mathbf{p}}} \cdot \bar{\mathbf{e}}$ | f8b | 3.17 |
| $\theta^{P\phi ph}$ | $\rho_o \eta \dot{\theta}^{P\phi ph} = \rho_o \eta \dot{\theta} - \mu_o \bar{\mathbf{m}} \cdot \dot{\bar{\mathbf{h}}} - \mu_\mathbf{o} \dot{\bar{\mathbf{m}}} \cdot \bar{\mathbf{h}}$ | f8c | 3.17 |
| $\theta^{P\phi eh}$ | $\rho_o \eta \dot{\theta}^{P\phi eh} = \rho_o \eta \dot{\theta} - \bar{\mathbf{p}} \cdot \dot{\bar{\mathbf{e}}} - \dot{\bar{\mathbf{p}}} \cdot \bar{\mathbf{e}} - \mu_\mathbf{o} \bar{\mathbf{m}} \cdot \dot{\bar{\mathbf{h}}} - \mu_\mathbf{o} \dot{\bar{\mathbf{m}}} \cdot \bar{\mathbf{h}}$ | f8d | 3.17 |
| $\theta^{P\phi dm}$ | $\rho_o \eta \dot{\theta}^{P\phi dm} = \rho_o \eta \dot{\theta} + \epsilon_o J \mathbf{F}^{-T} \bar{\mathbf{e}} \cdot (-\mathbf{L}^T \mathbf{F}^{-T} \bar{\mathbf{e}} + \mathbf{F}^{-T} \dot{\bar{\mathbf{e}}}) + \epsilon_\mathbf{o} \mathbf{F}^{-\mathbf{T}} \bar{\mathbf{e}} \cdot \mathbf{F}^{-\mathbf{T}} \bar{\mathbf{e}} \; \mathbf{divv}$ | f8e | 3.17 |
| $\theta^{P\phi pb}$ | $\rho_o \eta \dot{\theta}^{P\phi pb} = \rho_o \eta \dot{\theta} + \mu_o J \mathbf{F}^{-T} \bar{\mathbf{h}} \cdot (-\mathbf{L}^T \mathbf{F}^{-T} \bar{\mathbf{h}} + \mathbf{F}^{-T} \dot{\bar{\mathbf{h}}}) + \mu_\mathbf{o} \mathbf{F}^{-\mathbf{T}} \bar{\mathbf{h}} \cdot \mathbf{F}^{-\mathbf{T}} \bar{\mathbf{h}} \; \mathbf{divv}$ | f8f | 3.17 |
| $\theta^{P\phi dh}$ | $\rho_o \eta \dot{\theta}^{P\phi dh} = \rho_o \eta \dot{\theta} + \epsilon_o J \mathbf{F}^{-T} \bar{\mathbf{e}} \cdot (-\mathbf{L}^T \mathbf{F}^{-T} \bar{\mathbf{e}} + \mathbf{F}^{-T} \dot{\bar{\mathbf{e}}}) + \epsilon_\mathbf{o} \mathbf{F}^{-\mathbf{T}} \bar{\mathbf{e}} \cdot \mathbf{F}^{-\mathbf{T}} \bar{\mathbf{e}} \; \mathbf{divv} - \mu_\mathbf{o} \bar{\mathbf{m}} \cdot \dot{\bar{\mathbf{h}}} - \mu_\mathbf{o} \dot{\bar{\mathbf{m}}} \cdot \bar{\mathbf{h}}$ | f8g | 3.17 |
| $\theta^{P\phi eb}$ | $\rho_o \eta \dot{\theta}^{P\phi eb} = \rho_o \eta \dot{\theta} + \mu_o J \mathbf{F}^{-T} \bar{\mathbf{h}} \cdot (-\mathbf{L}^T \mathbf{F}^{-T} \bar{\mathbf{h}} + \mathbf{F}^{-T} \dot{\bar{\mathbf{h}}}) + \mu_\mathbf{o} \mathbf{F}^{-\mathbf{T}} \bar{\mathbf{h}} \cdot \mathbf{F}^{-\mathbf{T}} \bar{\mathbf{h}} \; \mathbf{divv} - \bar{\mathbf{p}} \cdot \dot{\bar{\mathbf{e}}} - \dot{\bar{\mathbf{p}}} \cdot \bar{\mathbf{e}}$ | f8h | 3.17 |
| $\theta^{P\phi db}$ | $\rho_o \eta \dot{\theta}^{P\phi db} = \rho_o \eta \dot{\theta} + \epsilon_o J \mathbf{F}^{-T} \bar{\mathbf{e}} \cdot (-\mathbf{L}^T \mathbf{F}^{-T} \bar{\mathbf{e}} + \mathbf{F}^{-T} \dot{\bar{\mathbf{e}}}) + \epsilon_\mathbf{o} \mathbf{F}^{-\mathbf{T}} \bar{\mathbf{e}} \cdot \mathbf{F}^{-\mathbf{T}} \bar{\mathbf{e}} \; \mathbf{divv}$ <br> $+\mu_o J \mathbf{F}^{-T} \bar{\mathbf{h}} \cdot (-\mathbf{L}^T \mathbf{F}^{-T} \bar{\mathbf{h}} + \mathbf{F}^{-T} \dot{\bar{\mathbf{h}}}) + \mu_\mathbf{o} \mathbf{F}^{-\mathbf{T}} \bar{\mathbf{h}} \cdot \mathbf{F}^{-\mathbf{T}} \bar{\mathbf{h}} \; \mathbf{divv}$ | f8i | 3.17 |

Table 3.16: Family 8 - Legendre transformations

Family8- Constitutive Equations

| Residual Equality: A$+\bar{\mathbf{j}}\cdot\bar{\mathbf{e}}-\frac{\mathbf{Q}}{\theta}\frac{\partial\theta}{\partial\mathbf{X}}-\rho_o\xi\theta=0$ | | | | | |
|---|---|---|---|---|---|
| potential | Mechanical | Thermal | Electrical | Magnetic | A |
| $\theta^{P\phi pm}$ | $\rho_o\eta\frac{\partial\theta^{P\phi pm}}{\partial\mathbf{P}}=-\mathbf{F}$ | $\frac{\partial\theta^{P\phi pm}}{\partial\psi}=-\frac{1}{\eta}$ | $\rho_o\eta\frac{\partial\theta^{P\phi pm}}{\partial\bar{\mathbf{p}}}=\bar{\mathbf{e}}$ | $\rho_o\eta\frac{\partial\theta^{P\phi pm}}{\partial\bar{\mathbf{m}}}=\mu_o\bar{\mathbf{h}}$ | $\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{p}}\cdot\bar{\mathbf{e}}+\mu_o\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{m}}\cdot\bar{\mathbf{h}}$ |
| $\theta^{P\phi em}$ | $\rho_o\eta\frac{\partial\theta^{P\phi em}}{\partial\mathbf{P}}=-\mathbf{F}$ | $\frac{\partial\theta^{P\phi em}}{\partial\psi}=-\frac{1}{\eta}$ | $\rho_o\eta\frac{\partial\theta^{P\phi em}}{\partial\bar{\mathbf{e}}}=-\bar{\mathbf{p}}$ | $\rho_o\eta\frac{\partial\theta^{P\phi em}}{\partial\bar{\mathbf{m}}}=\mu_o\bar{\mathbf{h}}$ | $\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{p}}\cdot\bar{\mathbf{e}}+\mu_o\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{m}}\cdot\bar{\mathbf{h}}$ |
| $\theta^{P\phi ph}$ | $\rho_o\eta\frac{\partial\theta^{P\phi ph}}{\partial\mathbf{P}}=-\mathbf{F}$ | $\frac{\partial\theta^{P\phi ph}}{\partial\psi}=-\frac{1}{\eta}$ | $\rho_o\eta\frac{\partial\theta^{P\phi ph}}{\partial\bar{\mathbf{p}}}=\bar{\mathbf{e}}$ | $\rho_o\eta\frac{\partial\theta^{P\phi ph}}{\partial\bar{\mathbf{h}}}=-\mu_o\bar{\mathbf{m}}$ | $\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{p}}\cdot\bar{\mathbf{e}}+\mu_o\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{m}}\cdot\bar{\mathbf{h}}$ |
| $\theta^{P\phi eh}$ | $\rho_o\eta\frac{\partial\theta^{P\phi eh}}{\partial\mathbf{P}}=-\mathbf{F}$ | $\frac{\partial\theta^{P\phi eh}}{\partial\psi}=-\frac{1}{\eta}$ | $\rho_o\eta\frac{\partial\theta^{P\phi eh}}{\partial\bar{\mathbf{e}}}=-\bar{\mathbf{p}}$ | $\rho_o\eta\frac{\partial\theta^{P\phi eh}}{\partial\bar{\mathbf{h}}}=-\mu_o\bar{\mathbf{m}}$ | $\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{p}}\cdot\bar{\mathbf{e}}+\mu_o\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{m}}\cdot\bar{\mathbf{h}}$ |
| $\theta^{P\phi dm}$ | $\rho_o\eta\frac{\partial\theta^{P\phi dm}}{\partial\mathbf{P}}=-\mathbf{F}$ | $\frac{\partial\theta^{P\phi dm}}{\partial\psi}=-\frac{1}{\eta}$ | $\rho_o\eta\frac{\partial\theta^{P\phi dm}}{\partial\bar{\mathbf{d}}}=\bar{\mathbf{e}}$ | $\rho_o\eta\frac{\partial\theta^{P\phi dm}}{\partial\bar{\mathbf{m}}}=\mu_o\bar{\mathbf{h}}$ | $\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{d}}\cdot\bar{\mathbf{e}}+\mu_o\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{m}}\cdot\bar{\mathbf{h}}$ |
| $\theta^{P\phi pb}$ | $\rho_o\eta\frac{\partial\theta^{P\phi pb}}{\partial\mathbf{P}}=-\mathbf{F}$ | $\frac{\partial\theta^{P\phi pb}}{\partial\psi}=-\frac{1}{\eta}$ | $\rho_o\eta\frac{\partial\theta^{P\phi pb}}{\partial\bar{\mathbf{p}}}=\bar{\mathbf{e}}$ | $\rho_o\eta\frac{\partial\theta^{P\phi pb}}{\partial\bar{\mathbf{b}}}=\mu_o\bar{\mathbf{h}}$ | $\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{p}}\cdot\bar{\mathbf{e}}+\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{b}}\cdot\bar{\mathbf{h}}$ |
| $\theta^{P\phi dh}$ | $\rho_o\eta\frac{\partial\theta^{P\phi dh}}{\partial\mathbf{P}}=-\mathbf{F}$ | $\frac{\partial\theta^{P\phi dh}}{\partial\psi}=-\frac{1}{\eta}$ | $\rho_o\eta\frac{\partial\theta^{P\phi dh}}{\partial\bar{\mathbf{d}}}=\bar{\mathbf{e}}$ | $\rho_o\eta\frac{\partial\theta^{P\phi dh}}{\partial\bar{\mathbf{h}}}=-\mu_o\bar{\mathbf{m}}$ | $\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{d}}\cdot\bar{\mathbf{e}}+\mu_o\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{m}}\cdot\bar{\mathbf{h}}$ |
| $\theta^{P\phi eb}$ | $\rho_o\eta\frac{\partial\theta^{P\phi eb}}{\partial\mathbf{P}}=-\mathbf{F}$ | $\frac{\partial\theta^{P\phi eb}}{\partial\psi}=-\frac{1}{\eta}$ | $\rho_o\eta\frac{\partial\theta^{P\phi eb}}{\partial\bar{\mathbf{e}}}=-\bar{\mathbf{p}}$ | $\rho_o\eta\frac{\partial\theta^{P\phi eb}}{\partial\bar{\mathbf{b}}}=\mu_o\bar{\mathbf{h}}$ | $\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{p}}\cdot\bar{\mathbf{e}}+\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{b}}\cdot\bar{\mathbf{h}}$ |
| $\theta^{P\phi db}$ | $\rho_o\eta\frac{\partial\theta^{P\phi db}}{\partial\mathbf{P}}=-\mathbf{F}$ | $\frac{\partial\theta^{P\phi db}}{\partial\psi}=-\frac{1}{\eta}$ | $\rho_o\eta\frac{\partial\theta^{P\phi db}}{\partial\bar{\mathbf{d}}}=\bar{\mathbf{e}}$ | $\rho_o\eta\frac{\partial\theta^{P\phi db}}{\partial\bar{\mathbf{b}}}=\mu_o\bar{\mathbf{h}}$ | $\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{d}}\cdot\bar{\mathbf{e}}+\mathbf{F}^{-1}\mathbf{LF}\bar{\mathbf{b}}\cdot\bar{\mathbf{h}}$ |

Table 3.17: Family 8 - constitutive equations

Next, for a case of constant pressure as a restricted form of internal energy, an attempt is made to obtain the expression for the potential.

Consider the potential $\varepsilon^{F\eta eh}$. The constitutive equations for this potential are

$$\begin{aligned}
\rho_o\frac{\partial\varepsilon^{F\eta eh}}{\partial\mathbf{F}} &= \mathbf{P}+\mathbf{F}^{-T}(\bar{\mathbf{e}}\otimes\bar{\mathbf{p}}+\mu_o\bar{\mathbf{h}}\otimes\bar{\mathbf{m}}),\\
\rho_o\frac{\partial\varepsilon^{F\eta eh}}{\partial\eta} &= \theta,\\
\rho_o\frac{\partial\varepsilon^{F\eta eh}}{\partial\bar{\mathbf{e}}} &= -\bar{\mathbf{p}},\\
\rho_o\frac{\partial\varepsilon^{F\eta eh}}{\partial\bar{\mathbf{h}}} &= -\mu_o\bar{\mathbf{m}}.
\end{aligned} \tag{3.78}$$

Let $\bar{\mathbf{P}}$ be constant Piola-Kirchoff stress tensor. Since Pressure is constant,

$$\begin{aligned}
\frac{\partial\mathbf{P}}{\partial\mathbf{F}} &\to 0,\\
\frac{\partial\mathbf{P}}{\partial\eta} &\to 0,\\
\frac{\partial\mathbf{P}}{\partial\bar{\mathbf{e}}} &\to 0,\\
\frac{\partial\mathbf{P}}{\partial\bar{\mathbf{h}}} &\to 0.
\end{aligned} \tag{3.79}$$

Combining (3.78) and (3.79), we obtain the following conditions:

$$\rho_o \frac{\partial^2 \varepsilon^{F\eta eh}}{\partial \mathbf{F}^2} = \frac{\partial}{\partial \mathbf{F}}(\bar{\mathbf{e}} \otimes \bar{\mathbf{p}} + \mu_o \bar{\mathbf{h}} \otimes \bar{\mathbf{m}}), \tag{3.80}$$

$$\rho_o \frac{\partial^2 \varepsilon^{F\eta eh}}{\partial \mathbf{F} \partial \eta} = \frac{\partial}{\partial \eta}(\bar{\mathbf{e}} \otimes \bar{\mathbf{p}} + \mu_o \bar{\mathbf{h}} \otimes \bar{\mathbf{m}}), \tag{3.81}$$

$$\rho_o \frac{\partial^2 \varepsilon^{F\eta eh}}{\partial \mathbf{F} \partial \bar{\mathbf{e}}} = \frac{\partial}{\partial \bar{\mathbf{e}}}(\bar{\mathbf{e}} \otimes \bar{\mathbf{p}} + \mu_o \bar{\mathbf{h}} \otimes \bar{\mathbf{m}}), \tag{3.82}$$

$$\rho_o \frac{\partial^2 \varepsilon^{F\eta eh}}{\partial \mathbf{F} \partial \bar{\mathbf{h}}} = \frac{\partial}{\partial \bar{\mathbf{h}}}(\bar{\mathbf{e}} \otimes \bar{\mathbf{p}} + \mu_o \bar{\mathbf{h}} \otimes \bar{\mathbf{m}}). \tag{3.83}$$

On performing integration, one obtains

$$\rho_o \frac{\partial \varepsilon^{F\eta eh}}{\partial \mathbf{F}} = \bar{\mathbf{P}} + \mathbf{F}^{-T}(\bar{\mathbf{e}} \otimes \bar{\mathbf{p}} + \mu_o \bar{\mathbf{h}} \otimes \bar{\mathbf{m}}). \tag{3.84}$$

Integrating (3.84) again, one gets the expression to evaluate the internal energy for a case of constant pressure,

$$\rho_o \varepsilon^{F\eta eh} = \bar{\mathbf{P}} \cdot \mathbf{F} + \int \mathbf{F}^{-T}(\bar{\mathbf{e}} \otimes \bar{\mathbf{p}} + \mu_o \bar{\mathbf{h}} \otimes \bar{\mathbf{m}}) \cdot d\mathbf{F} + c(\eta, \bar{\mathbf{e}}, \bar{\mathbf{h}}). \tag{3.85}$$

## 3.3 Determination of potential from Galfenol data

Magnetization vs field, magnetization vs stress, strain vs field, strain vs stress plots of magnetostrictive Galfenol obtained by [12] are shown in Figures (3.1), (3.2). The 3D versions of these graphs are shown in (3.3), (3.4). Considering two independent state variables $(\mathbf{T}, \mathbf{h})$, the corresponding constitutive equations are

$$\frac{\partial \phi}{\partial \mathbf{h}} = -\mathbf{m}, \qquad \frac{\partial \phi}{\partial T} = \epsilon. \tag{3.86}$$

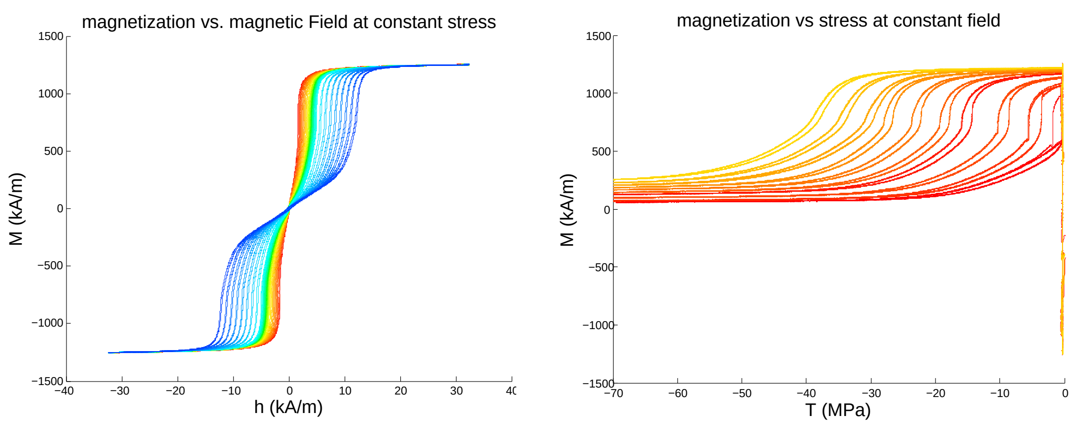
Considering unidirectional magnetic field and stress, integration of (3.86) gives

$$\phi(h) = -\int \frac{\mu_o}{\rho_o} m(h) dh + c_1 \tag{3.87}$$

$$\phi(T) = \int \frac{1}{\rho_o}\epsilon(T)dT + c_2 \tag{3.88}$$

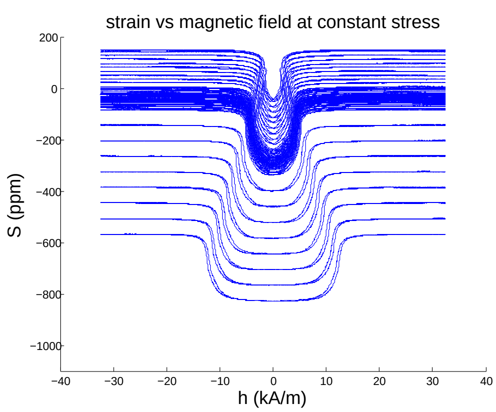
The plots of potential as a funtion of field and potential as a function of stress are shown in Figures (3.5)

### 3.3.1 Results

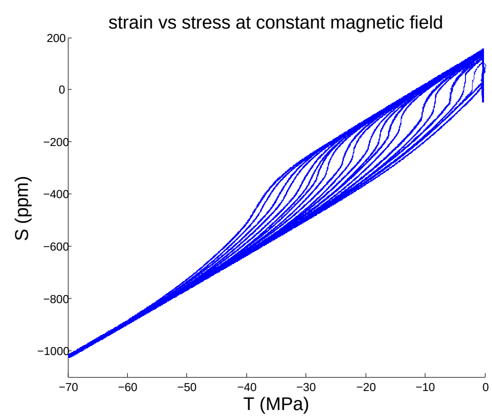



(a) (b)

Figure 3.1: Magnetization as a function of stress and potential [12] (a) magnetization as a function of magnetic field at constant stress and (b) magnetizaton as a function of stress at constant field

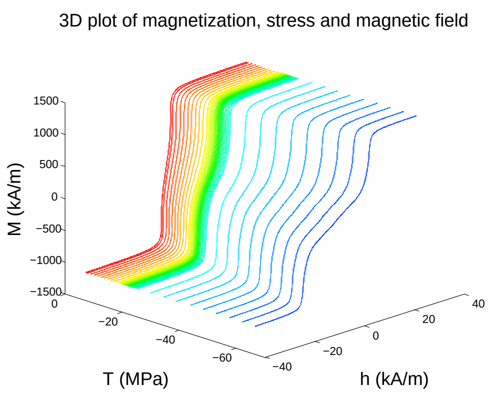



(a)

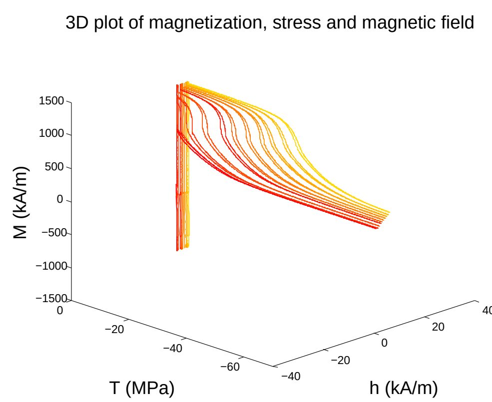



(b)

Figure 3.2: Strain as a function of stress and potential [12] (a) strain as a function of magnetic field at constant stress and (b) stress as a function of stress at constant field

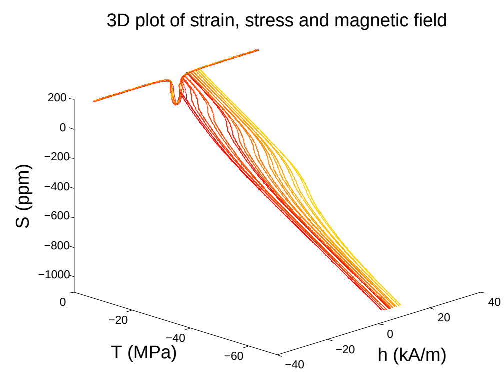



(a)

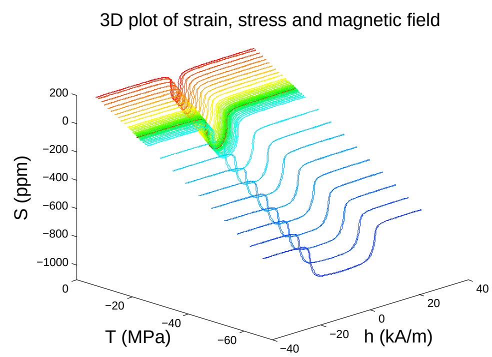



(b)

Figure 3.3: 3D plot of magnetization as a function of stress and potential (a) at constant stress and (b) at constant magnetic field

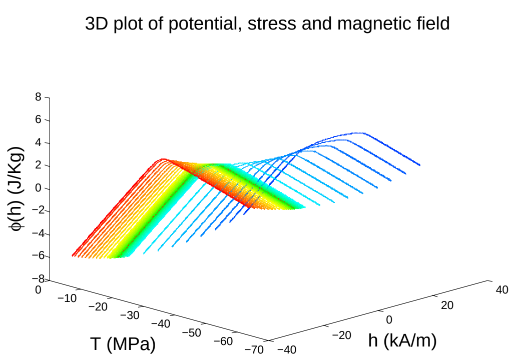



(a)

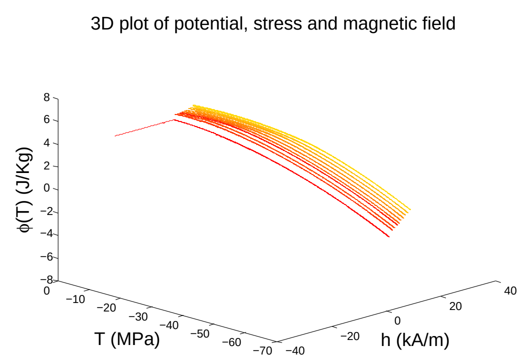



(b)

Figure 3.4: 3D plot of strain as a function of stress and potential (a) at constant magnetic field and (b) at constant stress




(a)




(b)

Figure 3.5: 3D plots of potential, stress and magnetic field (a) potential as a function of magnetic field and (b) Potential as a function of stress

# CHAPTER 4

# USE OF SMART MATERIALS IN FUEL INJECTORS

A fuel injection system consists of a pump that brings the fuel up to a high pressure level of up to 2000 bar and a nozzle that injects finely dosed quantities of fuel into the engine cylinder with the aid of a valve. There are two kinds of injection designs that are dominant today: common rail injector and pump injector. In the common rail fuel injector, the fuel is fed by a separate pump to the injection nozzle via a common rail and injected via a valve. In the pump fuel injector, the injection pump and the nozzle are integrated in a single module. The camshaft of the engine drives the pump cylinder of the injection element via a roller rocker arm. A needle in the valve doses the necessary quantity for injection. The valve needles in both systems are activated by either electromagnetic or smart materials.

The characteristics of fuel injectors are [20]:

1. Injection quantity
2. Injection pressure
3. Direction of control valve

The performance evaluation parameters are [20]:

1. Injection control

2. Injection mounting interference

3. Actuator capability in engine environment

Improving fuel injection strategies is the key to optimized engine performance. Considerable research has been conducted to control the injection quantity, pressure and profile of the fuel injection in order to reduce harmful emissions. In general, the higher the pressure and the more accurate the dosing and time of injection, the more efficient and less polluting the combustion [43].

## 4.1 Conventional Fuel Injectors

Conventionally, a fuel injector is activated through an electromagnetic solenoid arrangement. A solenoid is an insulating conducting wire wound to form a tight helical coil. When current passes through the wire, a magnetic field is generated within the coil in a direction parallel to the axis of the coil. When the coil is energized, the resulting magnetic field exerts a force on a moveable ferromagnetic armature located within the coil. This causes the armature to move a needle valve into an open position in opposition to force generated by a return spring. The force exerted on the armature is proportional to the strength of the magnetic field; the strength of the magnetic field depends on the number of turns of the coil and the amount of current passing through the coil [8].

The needle movement in conventional fuel injectors depends on various factors like spring pre-load holding the injector closed, the friction and the inertia of the needle, fuel pressure, eddy currents in the magnetic materials, and the magnetic characteristics of the design. The armature will not move until the magnetic force builds to a level high enough to overcome the opposing forces. In the same way, the needle will not return to a closed position until the magnetic force decays to a low enough level for the closing spring to overcome the fuel flow pressure and needle inertia. Once the needle begins opening or closing, it may continue to accelerate until it impacts with its respective end-stop, creating wear in the needle valve seat, needle bounce, unwanted vibrations, and noise problems. Electromagnets waste a large amount of input power through resistive heating losses. Electronically optimized control of the opening and closing of the valve reduces operating switching time by 30% and makes multiple injections possible. However, two-way solenoid valves are typically limited to digital operation: they are either fully open or fully closed. This characteristic is beneficial for controlling fuel quantity, and in some cases the injection timing, but is generally poor for shaping the flowrate profile. Thus there is a need for an improved fuel injector actuation method that will provide reduced noise, longer seat life, elimination of bounce, and full actuator force applied during the entire armature stroke, where the force is large as compared with the force resulting from fuel pressure effects. An electronic, high speed, proportional control valve could add rate-shaping capability to simple injection timing and fuel quantity control. This should also reduce particulate matter and nitrogen oxides in diesel engine emissions [8].

## 4.2 Piezoelectric based fuel injectors

Some of the smart materials that are in use today are piezoelectric materials, shape memory alloys, electrostrictive materials, magnetostrictive materials, electrorheological fluids, magneto-rheological fluids. Three types of proportional actuators that can be considered for the ACC fuel modulation system (piezoelectric stacks, electromagnetic shakers, and magnetostrictive actuators) are currently the most promising technologies available for proportional, high-bandwidth, linear actuation. Piezoelectric and magnetostrictive materials are capable of driving proportional actuators to frequencies exceeding 1kHz. Other types of materials such as shape memory alloys and electrostrictive materials were not appropriate for this application. Commercially available electrostrictive actuators offer no significant advantages to piezoelectric ceramics and the response time for shape memory alloy actuators is far too slow for active combustion control [43].

Use of active materials over conventional electromagnetics is preferred for a number of reasons. Piezoelectric actuation enables proportional authority over the injector's control valve, as opposed to traditional digital (on/off) operation. Piezoelectric actuators have the bandwidth needed for extremely fast switching. Typical switching times are less than 100 $\mu$s with no delays, while solenoid valves are up to ten times slower and have substantial lag due to magnetic reluctance. Active materials are capable of delivering much higher actuation forces (50,000 N). This characteristic lends itself to opening larger valve flow sections than comparable solenoid actuators to enable faster needle velocities. The actuation delay is a order of magnitude shorter and the control valve rise time is three times faster for a piezo. The spray cone angle for the piezo driven injector is about ten degrees larger than that of the solenoid injector system. The piezo-driven injector also reaches maximum injection rate quickly because the overall system losses are low, hence more pressure energy is converted into fuel kinetic energy leading to higher liquid velocities. This leads to better atomization. Another advantage of piezoelectric actuation is that the piezostack applies full force during the armature travel, allowing for controlled trajectory operation. Hysteresis of a piezoelectric stack does not play a significant role in a pulsed application such as a fuel injection system. Electrical energy conservation is also an advantage, since energy can be regained from a piezoelectric load due to its capacitive nature. Its wear proof and can be used in low temperature environment. On the whole, the piezoelectric fuel injection system has a simple structure, compact size, good displacement accuracy, low power consumption, high reliability, fuel consumption reduction to 20%, reduced carbondioxide emissions, exact control of injection discharge rate and injection fuel quantity, faster fuel intake, better air entrainment and faster spray vaporization [34].

The most common form of piezoceramic used today is based on lead zirconate titanate (PZT). Very high voltages must be applied to ensure that larger piezoelectric crystals or ceramic blocks expand significantly. In contrast, no more than 160 V is needed to trigger the piezo-effect in a single ceramic layer about 80 $\mu$m thick. But the layer thickness then changes by only $1/10^{th}$ of a micron. So, the key is to stack many of these layers together, sinter them monolitically and connect them mechanically in series but electrically in parallel with several hundred piezo layers stacked upon each other so that each piezoelectric ceramic plate has the same voltage. Since the direction of polarization is along the axial direction of piezoelectric multilayer actuator, its displacement is equal to the sum of the displacements of all the ceramic plates and thus effective elongation becomes 80 $\mu$m. Such a stack can generate a force of about 2500 N.

There are two types of piezoelectric stack actuators; plate through and co-fired. The plate through actuators are manufactured by stacking a large number of very thin

(0.2 mm) piezoelectric discs, with copper shims in between each disc to act as ground and positive electrodes to opposite sides of the actuator stack. External connectors enabling the activation of all the actuator elements are then used to connect all of the ground and positive electrodes. Although these actuators provide good piezoelectric actuator properties, they require high operating voltages (500 to 1000 V) and are very expensive to manufacture.

The co-fired multilayer technology offers the major advantage of edibility of component design (enabling much thinner ceramic layers, a customized internal electrode structure, and processing of a variety of forms and shapes). In this type, the electrodes are incorporated while growing the crystal. These inner electrodes which are connected in series do not completely cover the piezoceramic layer. If the ceramic layers are 10 times thinner, then the operating voltage needed to obtain the equivalent electric field strength and strain in the actuator can be reduced by a factor of 10. These kind of actuators are mass produced and hence cheaper compared to plate through actuators.

The piezostack may be attached to a mechanical member or needle performing a similar function as the needle in the conventional injector. When the piezostack has a high voltage potential applied across the wafers, the piezoelectric effect causes the stack to change dimension, thereby opening the fuel injector.

### 4.2.1 Failure of Piezoelectric stack actuators

Under the extreme dynamic, large signal driving conditions and hostile environment, that fuel injectors endure, life times of more than $10^9$ cycles and long term failure rates lower than $10^{-5}$ must be guaranteed. By examining the S-N diagram of a piezo stack, it can be seen that they fail at approximately 1000 cycles, which is far too low for the desired application. This failure can be attributed to two factors. PZT ceramics are brittle and hence cannot withstand tensile or shear stresses. If the co-fired piezoelectric stack actuator is put to use for fuel injector application, then this design could potentially fail due to tensile stresses. Also, with commercial PZT composition in the operating temperature and pressure range, a tetragonal to monoclinic phase transition can occur. In monoclinic phase, shear stresses are stable in the presence of hydrostatic pressure and due to this reason, the fuel injector employing the piezoelectric stack actuator can undergo failure.

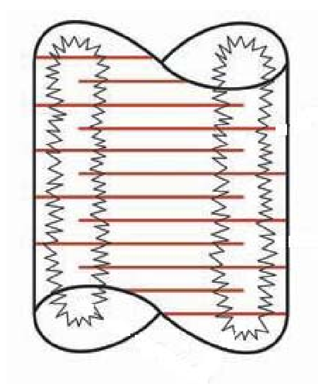


Figure 4.1: Cracks due to tensile stresses [1]

In co-fired stack actuators, since the inner electrodes do not completely cover the piezoceramic layer, there are inactive insulation volumes (gaps) which are not field accessed. In every layer of inner electrode, there is the active region and inactive region (gap). When the whole stack starts expanding or contracting, the active regions expand along the actuator axis where as the inactive regions do not. Thus, tensile stresses are induced in these inactive regions. Due to repeated cycling, in the presence of these tensile stresses, cracks start forming in the gaps as shown in Figure 4.1. Once the conducting electrode material enters into the crack, short-circuiting takes place and the whole stack fails. The higher the tensile stresses, the higher the probability of formation of uncontrolled cracks in axial direction. This can cause either a dielectric breakdown or just separate an internal electrode from the termination. It can be shown that the tensile stresses increase when the actuator height increases and the inactive volume increases as well. The magnitude of the tensile stress increases with increase in number of ceramic layers. The magnitude of the displacement decreases as the length of the inactive part increases. One possibility to limit this stress accumulation could be the segmentation of the actuator by gluing chips together. Then the stress would be released partially by the glued layers. But this solution has principal drawbacks. Chips, as well as actuators, do not have precise mechanical dimensions when they are fired. So either all the single chips have to be grounded before gluing, which is expensive and needs thick passive layers in the design (i.e., stress concentration and loss of strain), or they are glued "as fired", which results in non-homogeneous gluing and termination layers.

At high temperatures PZT is cubic with a pervoksite structure. When the temperature is lowered, the material becomes ferroelectric, with symmetry being tetragonal for Ti rich compositions and rhombohedral for Zr rich compositions. The morphotropic phase boundary (MPB) is at x=0.48 Ti. The maximum values of dielectric permittivity, electromechanical coupling factor, and piezoelectric coefficients of

$$
\begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \\ \hline D_1 \\ D_2 \\ D_3 \end{bmatrix}
=
\left[\begin{array}{cccccc|ccc}
s_{11}^E & s_{12}^E & s_{13}^E & 0 & 0 & 0 & 0 & 0 & d_{31} \\
s_{12}^E & s_{11}^E & s_{13}^E & 0 & 0 & 0 & 0 & 0 & d_{31} \\
s_{13}^E & s_{13}^E & s_{33}^E & 0 & 0 & 0 & 0 & 0 & d_{33} \\
0 & 0 & 0 & s_{44}^E & 0 & 0 & 0 & d_{15} & 0 \\
0 & 0 & 0 & 0 & s_{44}^E & 0 & d_{15} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & s_{66}^E & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & d_{15} & 0 & \varepsilon_{11}^T & 0 & 0 \\
0 & 0 & 0 & d_{15} & 0 & 0 & 0 & \varepsilon_{11}^T & 0 \\
d_{31} & d_{31} & d_{33} & 0 & 0 & 0 & 0 & 0 & \varepsilon_{33}^T
\end{array}\right]
\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ \hline E_1 \\ E_2 \\ E_3 \end{bmatrix}
$$

or, equivalently, $S = s^E T + d^* E$

$$D = d\,T + \varepsilon^T E$$

where $d^*$ denotes the transpose of $d$.

Figure 4.2: Condensed matrix notation of linear constitutive equations [9]

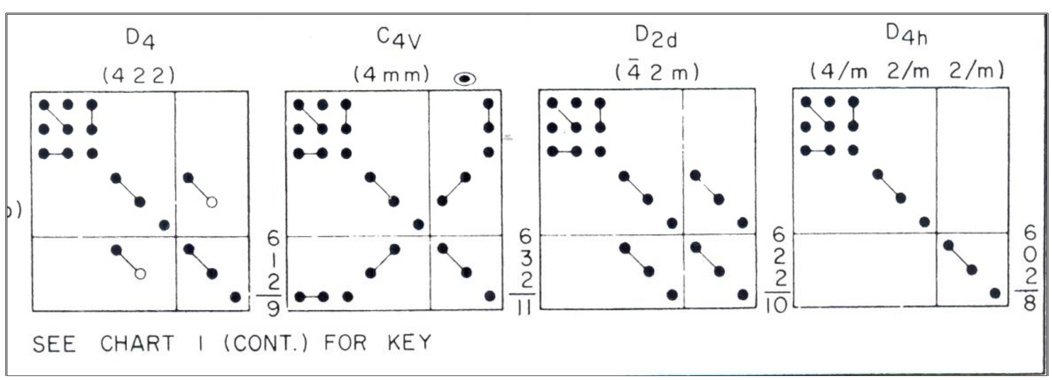



Figure 4.3: Elastoelectric matrices for tetragonal symmetry [35]

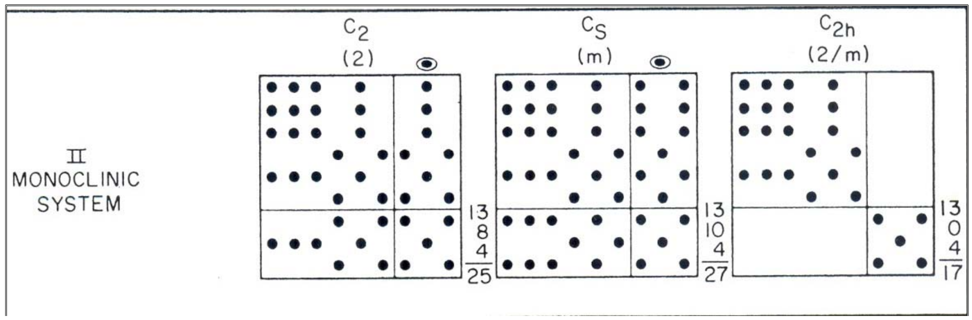



Figure 4.4: Elastoelectric matrices for monoclinic symmetry [35]

PZT occur at room temperature on the MPB boundary. Hence, this composition $Pb(Zr_{0.52}Ti_{0.48})O_3$ is used in commercial PZT materials. A PZT sample with x=0.48 is tetragonal just below Curie temperature and rhombohedral below room temperature. From cubic phase, when the temperature is lowered, the tetragonal phase is observed around 300 K. Below this temperature (at around 200 K), new features appeared in diffractograms. They are not comparable with rhombohedral phase or with a mixture of both tetragonal or rhombohedral phases. They correspond to monoclinic symmetry. Thus, the so called morphotrophic phase boundary is not a boundary, but rather a phase with monoclinic symmetry. This can be seen in Figure 4.5. At ambient pressure, below 210 K, low temperature monoclinic phase exists. From 210 to 305 K, high temperature monoclinic phase exists; and at 305 K, transition to tetragonal phase results. The monoclinic phase is found to be particularly stable with respect to hydrostatic pressure.

The elastoelectric matrices of the tetragonal symmetry as shown in Figure 4.3 clearly indicates that there is no possibility of shear strain in the presence of hydrostatic stress. On the other hand the elastoelectric matrices of the monoclinic symmetry in Figure 4.4 indicates that there is non-zero shear strain possible in the presence of hydrostatic stress. As PZT ceramics are brittle and cannot withstand shear mode, failure of PZT stack actuators occurs due to monoclinic phase transition.

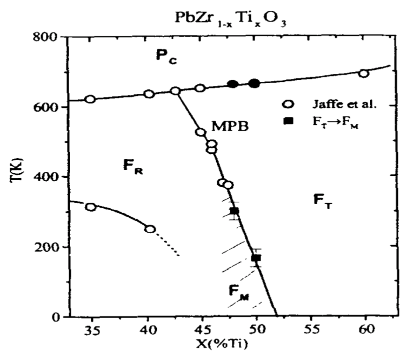



Figure 4.5: Monoclinic phase at MPB [36]

## 4.3 Magnetostrictive based fuel injectors

Magnetostrictive actuators have advantages in terms of durability and supply voltage. These actuators also excel in resonant applications because of large dynamic strains and high electromechanical efficiencies. While piezoelectric materials require very large electric fields ( 5kV/cm) and may suffer from self-heating problems, magnetostrictive materials require current carrying coils to produce varying magnetic fields, which can make the actuator bulky.

### 4.3.1 Principle of operation

A coil is provided for generating a magnetic field. The coil is arranged in the proximity of the magnetostrictive element. When the coil is energized, electrical current flowing through the coil induces a magnetic field which acts to realign the magnetic domains in the magnetostrictive material. As the domains rotate, they distort the atomic structure causing the material to grow and contract as the current is applied and removed. This efficiently converts the electricity into motion thereby actuating the valve. The result is proportional, positive and gives repeatable expansion in microseconds. Since the relationship between magnetic field and mechanical strain is quadratic, positive expansion results regardless of the direction of the magnetic field. Therefore, a preload mechanism similar to the type used with a piezoelectric stack is employed to compress the materials. Some prestress opposing the direction of the desired displacement of the magnetostrictive member is preferred. This is because a slight compression by a disk or coil spring results in greater needle displacement when the magnetizing force is applied. The prestress should not be so great as to prevent the displacement of the magnetostrictive material. Magnetostrictive strain and prestress should be considered in selecting the geometry of the magnetostrictive member so as to avoid surpassing the yield stress of the materials [8].

### 4.3.2 Terfenol-D based fuel injectors

Terfenol-D, an alloy of Terbium, Dysposium, and Iron exhibits giant magnetostrictive properties. It exhibits large magnetostriction (∼2000ppm) at room temperatures. An applied magnetic field is not the only factor that controls the magnetostrictive properties of a Terfenol-D actuator. Terfenol-D has a Curie temperature of 380$^o$C which lets it provide this magnetostrictive performance from room temperature to around 200$^o$C (Automotive applications operating temperatures are in the range of -40 to 150$^o$C). There is also a lower operating temperature limit of 15$^o$C. Lower temperatures can be reached by adjusting the stoichiometry of the alloy. This can be done to enable fuel-injector applications which require operation down to $-40^o$C. At lower frequencies, say 10 to 100 Hz, Terfenol-D actuators can provide repeatable displacements in the range of hundreds of micrometers or even greater. This makes them candidates for the high precision motion necessary to realize various state of the art manufacturing processes. The material can also respond at very high frequencies, in excess of 20 kHz, while still producing a large amount of force. Under a pre-stress and bias condition, it is equivalent to piezoceramics low stiffness value and can operatre under high pre-stress.

The ratio of reaction time of the Terfenol-D to the current pulses is very fast, allowing for fuel injection on the order of 6,000 cycles per second. The high injection speed allow for the injector to deliver precise combustion control, maximizing the power of each piston stroke. The control leads to significant gains in fuel economy, lowered emissions, and uniform power output from all cylinders.

One of the drawbacks of Terfenol-D is that it is brittle which limits its ability to withstand shock loads or operate in tension.

### 4.3.3 Galfenol based fuel injectors

Galfenol is an Iron-Gallium alloy, $Fe_{1-x}Ga_x$ $(0.13 < x < 0.3)$ developed at the Naval Surface Warfare Center [5, 6, 7]. It demonstrates moderate magnetostriction (∼350ppm) under low magnetic field (∼100 Oe) and has very low hysteresis, while demonstrating high tensile strength (∼500 MPa). It has high curie temperature ($675^oC$), and in general machinable, ductile and corrosion resistant [26, 27, 28, 29]. Magnetostriction peaks in FeGa alloys at a volume fraction of 17 percent Gallium.

Magnetostrictive FeGa alloys have certain unique properties, which may make them better suited than either piezoelectrics or Terfenol-D in certain actuation and sensing applications. For example, the high tensile strength (20 times that of typical piezoelectric and Terfenol-D) may enable the use of these alloys as actuators and sensors in harsh and shock prone environments. The bias field required for FeGa alloy is ten times smaller than that for Terfenol-D. Its material cost is low when compared to Terfenol-D. Galfenol is tough and can be machined, while Terfenol-D is a brittle material. Its mangetostriction is only a third to a quarter that of Terfenol-D, but can operate at significantly lower drive fields. The magnetostriction can be increased by applying preload to the material. Recent research has also shown that Galfenol can operate in tension and compression, something no other high frequency smart material can do [11].

Galfenol could fundamentally change the manner in which fuel injectors are made. Recent advances suggest that these new alloys are rapidly approaching the energy density requirements of common-rail fuel injectors, while their structural grade mechanical properties promise improved durability, design flexibility, ease of control, and lower costs over the most advanced designs (piezoelectric injectors).

# CHAPTER 5

# APPLICATION OF THE FRAMEWORK TO GALFENOL

Chapter 2 presents the governing equations of balance laws of mechanics and Maxwell equations describing a TEMM material. Chapter 3 develops a framework that results in the constitutive equations necessary to evaluate thermodynamic potentials. This is part of the inverse problem. As the prospects of Galfenol in fuel injector applications is explored in Chapter 4, Chapter 5 focuses on creating a framework that can be used to solve a boundary value problem consisting of magnetostrictive Galfenol through finite-element method in the spirit of [40]. In this respect, the constitutive equations that will be used here are obtained from a potential developed from a phenomenological model [12]. This is part of the direct problem. For a magnetostatic problem, the Maxwell equations $(2.17)_1$, (2.23) reduce to

$$\nabla \cdot \mathbf{b} = 0, \tag{5.1}$$

$$\nabla \times \mathbf{h} = \mathbf{j}_f, \tag{5.2}$$

where $\mathbf{j}_f$ is the prescribed current density. The equation for conservation of linear momentum $(2.19)_2$ can be rewritten by writing the electromagnetic force in terms of

Maxwell stress tensor. The density of the material is assumed to be constant $\rho_o$.

$$\nabla \cdot (\mathbf{T} + \mathbf{T}_M) + \rho_o \mathbf{f}^{ext} = \rho_o \ddot{u}, \tag{5.3}$$

where $\mathbf{f}^{ext}$ is the specified force, $\mathbf{u}$ is the displacement, $\mathbf{T}_M$ is the Maxwell stress tensor which for magnetostrictive materials reduces to

$$\mathbf{T}_M = \mathbf{h} \otimes \mathbf{b} - \frac{1}{2}\mu_o(\mathbf{h} \cdot \mathbf{h})\mathbf{I}. \tag{5.4}$$

The constitutive equations that are presented here are adopted from [13]. The free energy has terms for magnetic anisotropy, magnetomechanical coupling, zeeman or field energy and elastic strain energy. These energies will be expressed while idealizing the complex domain structure of ferromagnetic materials as a system of non-interacting, single-domain, Stoner-Wohlfarth (S-W) particles. The total free energy of the material is

$$G = \sum_{k=1}^{r} \xi^k G^k + \mathbf{T}.\mathbf{s}\mathbf{T}, \tag{5.5}$$

where $G^k$ is the energy of the S-W particle in easy direction $\mathbf{c}^k$ given by

$$G^k = \frac{1}{2}\mathbf{K}^k|\mathbf{m}^k - c^k|^2 - \lambda^k \cdot \mathbf{T} - \mu_o M_s \mathbf{m}^k \cdot \mathbf{h}. \tag{5.6}$$

For cubic materials, the $\langle 100 \rangle$ or $\langle 111 \rangle$ tend to be the easy directions. The anisotropy coefficient $K^k$ in each direction family is the same, thus $K^k = K_{100}$ for all six $\langle 100 \rangle$ directions and $K^k = K_{111}$ for all eight $\langle 111 \rangle$ directions. The magnetostriction $\lambda^k$ is only a function of stress and field through its dependence on $\mathbf{m}$ which for the longitudinal components is

$$\lambda_i^k = \frac{3}{2}\lambda_{100} m_i^{k^2}, i = 1, 2, 3 \tag{5.7}$$

and shear components

$$\lambda_4^k = 3\lambda_{111} m_1^k m_2^k, \tag{5.8}$$

$$\lambda_5^k = 3\lambda_{111} m_2^k m_3^k, \tag{5.9}$$

$$\lambda_6^k = 3\lambda_{111} m_3^k m_1^k. \tag{5.10}$$

The magnetic orientation $\mathbf{m}^k$ is given by

$$\mathbf{m}^k = (\mathbf{K}^k)^{-1}(\mathbf{B}^k + \frac{1 - c^k \cdot (\mathbf{K}^k)^{-1}\mathbf{B}^k}{c^k \cdot (\mathbf{K}^k)^{-1} c^k} c^k), \tag{5.11}$$

where the magnetic stiffness matrix $\mathbf{K}^k$ and force vector $\mathbf{B}^k$ are

$$\mathbf{K^k} = \begin{pmatrix} K^k - 3\lambda_{100}T_1 & -3\lambda_{111}T_4 & -3\lambda_{111}T_6 \\ -3\lambda_{111}T_4 & K^k - 3\lambda_{100}T_2 & -3\lambda_{111}T_5 \\ -3\lambda_{111}T_6 & -3\lambda_{111}T_5 & K^k - 3\lambda_{100}T_3 \end{pmatrix},$$

$$\mathbf{B^k} = \begin{pmatrix} c_1^k K^k + \mu_o M_s h_1 & c_2^k K^k + \mu_o M_s h_2 & c_3^k K^k + \mu_o M_s h_3 \end{pmatrix}^T.$$

The constitutive equations are

$$\mathbf{b} = \mu_o(\mathbf{h} + \mathbf{m}(\mathbf{h}, \mathbf{T})), \tag{5.12}$$

$$\mathbf{S} = \mathbf{S}_e + \mathbf{S}_{me}(\mathbf{h}, \mathbf{T}), \tag{5.13}$$

where $\mathbf{S}_e$ is the elastic strain given by

$$\mathbf{S}_e = s\mathbf{T}, \tag{5.14}$$

s is the compliance coefficient, $\mathbf{S}_{me}$ is the magnetoelastic strain. The macroscopic magnetization and magnetostriction are the sum of the contributions of the six domain families.

$$\mathbf{S}_{me} = \sum_{k=1}^{r} \xi^k \lambda^k, \tag{5.15}$$

$$\mathbf{m} = M_s \sum_{k=1}^{r} \xi^k \mathbf{m}^k. \tag{5.16}$$

The equilibrium volume fraction for each of six domains is given by

$$\xi^k = \frac{e^{-\frac{G^k}{\omega}}}{\sum_{j=1}^{r} e^{-\frac{G^j}{\omega}}}. \tag{5.17}$$

The constants have the values of

$$\omega = \frac{k_B \theta}{V} = 500 - 1200 J/m^3, \tag{5.18}$$

$$\mu_o M_s = 1.6. \tag{5.19}$$

The compatibility equations are

$$\mathbf{S} = \nabla^s \mathbf{u}, \tag{5.20}$$

$$\mathbf{b} = \nabla \times \mathbf{A}. \tag{5.21}$$

In order the magnetic potential $\mathbf{A}$ is unique, Coulomb guage condition is established

$$\nabla \cdot \mathbf{A} = 0. \tag{5.22}$$

The strain operator $\nabla^s$ is defined as

$$\nabla^{\mathbf{s}} = \left( \begin{array}{ccc|ccc|ccc} \frac{\partial}{\partial x_1} & 0 & 0 & 0 & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} & 0 & \frac{\partial}{\partial x_3} & -\frac{\partial}{\partial x_2} \\ 0 & \frac{\partial}{\partial x_2} & 0 & \frac{\partial}{\partial x_3} & 0 & \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_3} & 0 & -\frac{\partial}{\partial x_1} \\ 0 & 0 & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 & \frac{\partial}{\partial x_2} & -\frac{\partial}{\partial x_1} & 0 \end{array} \right)^T.$$

The divergence operator $\nabla \cdot$ is

$$\nabla \cdot = \left( \begin{array}{ccc} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \end{array} \right)^T.$$

The curl operator $\nabla \times$ is

$$\nabla \times = \left( \begin{array}{ccc} 0 & -\frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} & 0 & -\frac{\partial}{\partial x_1} \\ -\frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 \end{array} \right)^T.$$

The operator $\nabla^s$ in the above equation relates to a tensor component order

$$T = (T_{11}, T_{22}, T_{33}, T_{23}, T_{13}, T_{12}, T_{32}, T_{31}, T_{21})^T \tag{5.23}$$

$$= (T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8, T_9)n \tag{5.24}$$

## 5.1 FEM formulation

The nodal unknowns of this problem are

$$\{\mathbf{u}, \mathbf{A}\}^t = \{u_1, u_2, u_3, A_1, A_2, A_3\}^T. \tag{5.25}$$

**Weak form**:

The Maxwell equations and linear momentum equation are weighted by the auxiliary functions (variations)

$$\{\omega_u, \omega_A\}^T = \{\omega_{u_1}, \omega_{u_2}, \omega_{u_3}, \omega_{A_1}, \omega_{A_2}, \omega_{A_3}\}^T. \tag{5.26}$$

These functions satisfy homogeneous form of essential boundary condition

$$\omega_u = 0, \omega_A = 0 \qquad \text{on } \partial\Omega_u, \partial\Omega_A. \tag{5.27}$$

After some transformations based on integration by parts, vector analysis and the divergence and stokes theorems, one gets

$$\begin{aligned} -\int_\Omega (\nabla^s \omega_u)^T \sigma d\Omega + \int_{\partial\Omega_f} \omega_u^T \bar{\mathbf{t}} d\Gamma + \int_\Omega \omega_u^T \rho_o \mathbf{F} d\Omega - \int_\Omega \omega_u^T \rho_o \ddot{\mathbf{u}} d\Omega = 0, \\ -\int_\Omega (\nabla \times \omega_A)^T \mathbf{h} d\Omega + \int_{\partial\Omega_h} \omega_A^T \bar{\mathbf{h}} d\Gamma + \int_\Omega \omega_A^T \mathbf{j}_f d\Omega = 0. \end{aligned} \tag{5.28}$$

**Galerkin form**:

The zero-derivative variables $\{\mathbf{u}, \mathbf{A}\}^\mathbf{T}$, $\{\omega_u, \omega_A\}^T$ are discretized through simple polynomial expansions. These spatial polynomials, called shape functions, $\mathbf{N}(x) = \sum_B N^B(x)$ are predetermined and each of them is related to a node, globally numbered with the integers A, B from 1 to $n_{np}$ (number of nodal points). The coefficients of these polynomial expansions are the nodal unknowns, $u_i, A_i$ in each spatial direction i:

$$\{u_i^q, A_i^q\}^T = \sum_{B=1}^{n_{np}} \{u_i^B, A_i^B\}^T, \tag{5.29}$$

$$\{\omega^q_{u_i}, \omega^q_{A_i}\}^T = \sum_{B=1}^{n_{np}} N^B \{c^B_{u_i}, c^B_{A_i}\}^T, \tag{5.30}$$

where superscript (q) denotes approximation for the spatial discretization and the parameters $c^B$ are auxiliary, without clear physical meaning. Since $N^B$ are spatial functions, the previous variables are also functions of the position: $u^h_i(x)$ etc. In what follows the summation over B will be implicit from the index repitition with the range B= 1,....,$n_{np}$, e.g. for the first of equations, $u^h_i = N^B u^B_i$. Also, since the problem is non-linear, further variables need to be discretized. For i=1,2,3 and j= 1,....,6,

$$h^q_i = N^B h^B_i, \qquad \sigma^q_j = N^B \sigma^B_j. \tag{5.31}$$

Linearity of $\nabla^s$, $\nabla\cdot$, $\nabla\times$ operators ensures

$$(\nabla^s \omega^q_u)^t = (c^B_u)^t (B^B_{\nabla_s})^t, \qquad (\nabla \times \omega^h_A)^t = (c^B_A)^t (B^B_\times)^t. \tag{5.32}$$

The FE compatibility matrices $B^B_{\nabla_s}$, $B^B_\times$ are again spatial functions, with different values for different integration points. At each point B we have

$$B^B_{\nabla^s} = \nabla^s N^B(x), \qquad B^B_\times = \nabla \times N^B(x). \tag{5.33}$$

To denote the columns of the compatibility matrices, we will use the subscripts i=1, 2, 3 as in $B^B_{\nabla_s}$, $B^B_\times$. Using (5.29) – (5.33), the weak form equation (5.28) becomes

$$c^B_{u_i}[\int_{\partial\Omega_f} N^B \bar{\mathbf{t}} d\Gamma + \int_\Omega \rho_o N^B F_i d\Omega - \int_\Omega (B^B_{\nabla^s_i})^t \sigma^q d\Omega - \int_\Omega N^B \rho_o \ddot{u}^h_i d\Omega] = 0, \tag{5.34}$$

$$c^B_{A_i}[\int_{\partial\Omega_h} N^B \bar{h}_i d\Gamma - \int_\Omega (B^B_{\times_i})^t h^q d\Omega + \int_\Omega N^B j_i d\Omega] = 0. \tag{5.35}$$

As a next step, the residual vector is formed:

The expressions inside parenthesis in (5.34), (5.35) must vanish independently of the coefficients $c^B$, although in the context of a non-linear solver a set of six residuals for each node B is defined as

$$R^B = \{R_u^B, R_A^B\}^T, \tag{5.36}$$

with values (from now on the superscript q is removed for the sake of clarity)

$$R_u^B = \int_{\partial\Omega_f} (N^B\mathbf{I})\bar{\mathbf{t}}d\Gamma + \int_{\Omega} (N^B\mathbf{I})\rho_o F^B d\Omega - \int_{\Omega} (B_{\nabla^s}^B)^t \sigma^B d\Omega - \int_{\Omega} (N^B\mathbf{I})\rho_o \ddot{\mathbf{u}} d\Gamma, \tag{5.37}$$

$$R_A^B = \int_{\partial\Omega_h} (N^B\mathbf{I})\bar{\mathbf{h}}d\Gamma + \int_{\Omega} (N^B\mathbf{I}) j^B d\Omega - \int_{\Omega} (B_{\times}^B)^t h^B d\Omega. \tag{5.38}$$

Starting from an initial guess, for each step it will be necessary to iterate the above non-linear equations until the norm of $\mathbf{R}$ (vector valued in all nodes) is set to zero, up to a fraction of the precision machine for instance, with a good enough set of basic variables. Note that in the above equations, several values: $\mathbf{T}^B, \ddot{\mathbf{u}}^B, \mathbf{h}^B$ are not known at the beginning of a step, but the values from the previous one can be used, resulting in a good convergence rate. The next stage is to build a consistent tangent matrix. Each entry of this matrix is obtained as the negative of the partial node A with respect to the corresponding field variable at a generic node B. In these residuals there are tensors defined in previous equations, for which the chain rule will have to be used several times. From the compatibility equations (5.20), (5.21) and the expansion equations (5.29),

$$\frac{\partial \mathbf{S}}{\partial u_j^B} = B_{\nabla_j^s}^B, \qquad \frac{\partial \mathbf{b}}{\partial A_j^B} = B_{\times j}^B. \tag{5.39}$$

From (5.39) and the constitutive equations (5.12), (5.13) the derivatives of the stress tensor and magnetic field are obtained.

$$\sigma = \mathbf{T} + \mathbf{T}_M, \tag{5.40}$$

$$\frac{\partial \sigma}{\partial u_j^B} = \frac{\partial \mathbf{T}}{\partial u_j^B} + \frac{\partial \mathbf{T}_M}{\partial u_j^B}, \tag{5.41}$$

$$\frac{\partial \sigma}{\partial A_j^B} = \frac{\partial \mathbf{T}}{\partial A_j^B} + \frac{\partial \mathbf{T}_M}{\partial A_j^B}, \tag{5.42}$$

$$\mathbf{h} = \frac{1}{\mu_o}\mathbf{b} - \mathbf{m}(\mathbf{h}, \mathbf{T}), \tag{5.43}$$

$$\frac{\partial \mathbf{h}}{\partial u_j^B} = \frac{1}{\mu_o}\frac{\partial \mathbf{b}}{\partial u_j^B} - \frac{\partial \mathbf{m}(\mathbf{h}, \mathbf{T})}{\partial u_j^B}, \tag{5.44}$$

$$\frac{\partial \mathbf{h}}{\partial A_j^B} = \frac{1}{\mu_o}\frac{\partial \mathbf{b}}{\partial A_j^B} - \frac{\partial \mathbf{m}(\mathbf{h}, \mathbf{T})}{\partial A_j^B}, \tag{5.45}$$

One can evaluate the expressions $\frac{\partial \mathbf{T}}{\partial u_j^B}$, $\frac{\partial \mathbf{T}}{\partial A_j^B}$, $\frac{\partial \mathbf{h}}{\partial u_j^B}$, $\frac{\partial \mathbf{h}}{\partial A_j^B}$ from the constitutive equations (5.12), (5.13). These expressions are not presented here.

$$\frac{\partial \mathbf{T}_M}{\partial u_j^B} = \frac{\partial \mathbf{h}}{\partial u_j^B} \otimes \mathbf{b} - \mu_o(\frac{\partial \mathbf{h}}{\partial u_j^B})^T \mathbf{h} \otimes \mathbf{I}, \tag{5.46}$$

$$\frac{\partial \mathbf{T}_M}{\partial A_j^B} = \frac{\partial \mathbf{h}}{\partial A_j^B} \otimes \mathbf{b} - \mu_o(\frac{\partial \mathbf{h}}{\partial A_j^B})^T \mathbf{h} \otimes \mathbf{I}, \tag{5.47}$$

Each tangent nodal matrix (6×6) represents the interaction between the degrees of freedom of node A (rows) and those of B (columns). Upon derivation of the residuals in equations,

$$\mathbf{K}_{uu_{ij}} = -\frac{\partial R_{u_i}^A}{\partial u_j^B} = \int_\Omega (B_{\nabla_i^s}^A)^t \frac{\partial \sigma}{\partial u_j^B} d\Omega, \qquad i, j = 1, 2, 3 \tag{5.48}$$

$$\mathbf{K}_{uA_{ij}} = -\frac{\partial R_{u_i}^A}{\partial A_j^B} = \int_\Omega (B_{\nabla_i^s}^A)^t \frac{\partial \sigma}{\partial A_j^B} d\Omega, \qquad i = 1, 2, 3, j = 4, 5, 6 \tag{5.49}$$

$$\mathbf{K}_{Au_{ij}} = -\frac{\partial R_{A_i}^A}{\partial u_j^B} = \int_\Omega (B_{\nabla_{\times i}^s}^A)^t \frac{\partial \mathbf{h}}{\partial u_j^B} d\Omega, \qquad i = 4, 5, 6, j = 1, 2, 3 \tag{5.50}$$

$$\mathbf{K}_{AA_{ij}} = -\frac{\partial R_{A_i}^A}{\partial A_j^B} = \int_\Omega (B_{\nabla_{\times i}^s}^A)^t \frac{\partial \mathbf{h}}{\partial A_j^B} d\Omega. \qquad i, j = 4, 5, 6 \tag{5.51}$$

Equation(5.3) contains a dynamic term $\ddot{\mathbf{u}}$ wiht a second time derivative; it is a hyperbolic equation. If the corresponding mass matrix is lumped (diagonal), a central difference integrator can be used. Assuming $t_0$=0 and $t_{n+1} = t_n + \Delta t$

$$\ddot{\mathbf{u}} = \frac{\mathbf{u}(t_{n+1}) - 2\mathbf{u}(t_n) + \mathbf{u}(t_{n-1})}{\Delta t^2}. \tag{5.52}$$

All unknowns are defined at time step $t_{n+1}$, while the rest of the variables at time $t_n$ or $t_{n-1}$ are known from the previous steps. From these results and equations (5.29) evaluated at time $t_{n+1}$,

$$\frac{\partial \ddot{\mathbf{u}}}{\partial u_j^B} = \frac{(N^B \mathbf{I})_j}{\Delta t^2}, \tag{5.53}$$

where $(N^B\mathbf{I})_j$ is the jth column of this diagonal matrix. Derivation of the residuals in equations (5.37), (5.38) with respect to the dynamic unknowns $\ddot{\mathbf{u}}$ yields the non-zero terms of the dynamic matrices:

$$\mathbf{M}_{uu_{ij}} = -\frac{\partial R_{u_i}^A}{\partial u_j^B} = \frac{1}{\Delta t^2} \int_\Omega (N^A \mathbf{I})_i \rho_o (N^B \mathbf{I})_j. \tag{5.54}$$

The above submatrix equation is diagonal, but that does not mean that the global dynamic matrices including all $M_{uu}$ are also, unless a diagonalization technique is applied. Thic could be especially useful for the inertia matrix, since it allows us to apply a simple central difference integrator and to directly compute eigen solutions. Once the dynamic nodal values have been computed, the spatial dynamic variables can be approximated over the domain with polynomial expansions as in equations

$$\dot{u}_i^q = N^B \dot{u}_i^B, \qquad \ddot{u}_i^q = N^B \ddot{u}_i^B. \tag{5.55}$$

From equations (5.48) – (5.51), (5.54) the profiles of the complete matrices are

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_{uu} & \mathbf{K}_{uA} \\ \mathbf{K}_{Au} & \mathbf{K}_{AA} \end{bmatrix},$$

$$\mathbf{M}^{Mech} = \begin{bmatrix} \mathbf{M}_{uu} & 0 \\ 0 & 0 \end{bmatrix}.$$

The total tangent matrix is automatically assembled as

$$\mathbf{K} \leftarrow c_1 \mathbf{K} + c_2 \mathbf{M}^{Mech}, \tag{5.56}$$

where $c_k$, k = 1, 2, 3, are scalars that include the values of time steps, integration weights etc. The complete non-linear system can then be represented as

$$\begin{bmatrix} c_1\mathbf{K}_{uu} + c_2\mathbf{M}_{uu} & c_1\mathbf{K}_u A \\ -c_1\mathbf{K}_{Au} & c1\mathbf{K}_{AA} \end{bmatrix} \begin{bmatrix} d\mathbf{u} \\ d\mathbf{A} \end{bmatrix} = \begin{bmatrix} R_u \\ R_A \end{bmatrix}.$$

# CHAPTER 6

# CONCLUSION

## 6.1 Fully-coupled characterization of TEMM materials

For a rate independent process, the complete combinatorial analysis of possible potentials and state variables that describe 3D, nonlinear, coupled, material behavior of TEMM materials is created. In all, the combination of mechanical and thermal state variables correspond to four basic potentials (Gibbs free fnergy, Helmholtz free energy, enthalpy, internal energy). With the inclusion of electrical and magnetic state variables, the combinatorial analysis results in seventy-two possible potential functions. In order to obtain the constitutive equations for each of the seventy-two potential functions, a detailed analysis of the possible governing equations that are to be considered is presented. The Maxwell equations and balance laws of mechanics are presented in both integral and pointwise form. These equations and the jump conditions are rewritten in material description as it is suitable for solids. For second law, balance law of entropy [1][4] is used instead of Classius-Duhem inequality. For an unconstrained material, with Maxwell equations, balance laws of mechanics and balance law of entropy, the constitutive equations for each of the seventy-two potentials are derived in a systematic way and tabulated.

A case of constant pressure process is taken and the expression to evaluate the potential is derived. From the strain-stress, strain-field, magnetization-stress, and magnetization-field plots obtained from the experimental data, numerical integration is carried out to obtain the plots of potential as a function of field and potential as a function of stress.

## 6.2 Nonlinear 3D FEM formulation

The 3D tensorial frame work of the basic equations for magnetostrictive materials is combined with constitutive equations that are obtained from a nonlinear phenomenological model [12]. A finite element formulation for a boundary value problem set up using these equations is given in detailed manner.

## 6.3 Future Work

- For the case of the constant pressure process shown in this research work, novel approximation methods can be explored to perform integration and evaluate the potential.

- Similarly, an approximation method can be explored to evaluate numerically the potential as a function of stress and magnetic field from the plots of potential vs field and potential vs stress

- The finite element formulation laid out can be implemented using COSMOL or FEAP or self written code. More complicated geometries close to experimental set up of fuel injectors with Galfenol actuators can be considered as a next step.

# APPENDIX A

# STATISTICAL MODEL FOR MAXWELL EQUATIONS

Magnetic field originates from current which in turn is the product of charge density and velocity. The magnetic interactions between parallel currents depend only on the product of the currents, neither on the charge densities nor on velocities separately. Electrons in vacuum can travel in the order of velocity of light. Electrons in conductors on average have a drift velocity in the order of $10^{-4}$ m/s. Positive and negative ions travel in fluid at a relatively slower speed. Mesoscale electromagnetic behavior depends only on the average current of a mesoscale collection of electrons. The mechanism of charge transport i.e., many charges moving slowly or few charges moving at greater speeds do not matter. If there is a current of $3.3 \times 10^{-3}$ A, it does not matter whether this current is composed of high-energy electrons moving with 99 percent of the speed of light, of electrons in a metal executing nearly random thermal motions with a slight drift in one direction, or of charged ions in solution with positive ions moving one way, negative the other. All these charge carriers can contribute to same amount of current. The Lorentz force equation which is mentioned later in this section is no way restricted to small velocities, either for the charge carriers in the wire or for a moving charge q [41].

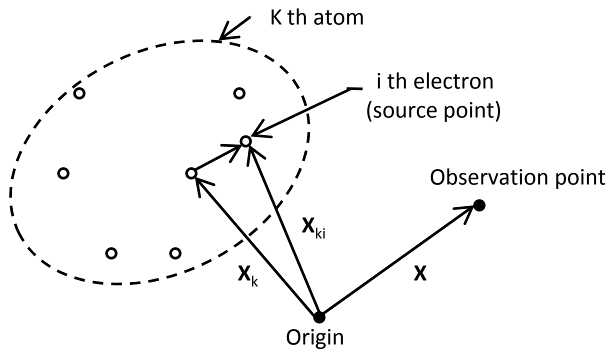



Figure A.1: Electrons (charged particles) and atoms (stable groups) [37]

According to statistical formulation, charged particles in the material medium are gathered into stable groups which are termed as atoms. The location of the $i$th electron in the $k$th atom is denoted by (Fig.1)

$$\mathbf{x}_{ki} = \mathbf{x}_k + \zeta_{ki}, \tag{A.1}$$

where $\mathbf{x}_k$ is the position vector of the center of mass of the stable group and $\zeta_{ki}$, the internal coordinate within the atom. At an observation point $\mathbf{x}$, the microscopic fields $\tilde{\mathbf{e}}(\mathbf{x})$ and $\tilde{\mathbf{b}}(\mathbf{x})$ are generated by large number of atoms, each containing many electrons (the electrons here are different from the electrons in the modern atomic theory by J.J. Thompson). These fields are governed by Maxwell equations in aether [37].

$$\begin{aligned}
&\nabla \cdot \tilde{\mathbf{b}} = 0, \\
&\nabla \times \tilde{\mathbf{e}} + \frac{\partial \tilde{\mathbf{b}}}{\partial t} = 0, \\
&\epsilon_o \nabla \cdot \tilde{\mathbf{e}} = \sum_k \sum_i q_{ki} \delta(x_{ki} - x), \\
&\mu_o^{-1} \nabla \times \tilde{\mathbf{b}} - \epsilon_o \frac{\partial \tilde{\mathbf{e}}}{\partial t} = \sum_k \sum_i q_{ki} \dot{x}_{ki} \delta(x_{ki} - x),
\end{aligned} \tag{A.2}$$

where $q_{ki}$ is the charge of the $i$th electron in the $k$th atom, $q_{ki}\dot{x}_{ki}$ is the convective current.

The fields at $\mathbf{x}$ generated by a single electron $q_{lj}$ at $\mathbf{x}_{lj}$ can be calculated by solving $(8)_3$ and $(8)_4$ with [37]

$$\begin{aligned}\tilde{\mathbf{e}}_{lj}(x) &= -\nabla_x[q_{lj}/(4\pi\epsilon_o|\mathbf{x}-\mathbf{x}_{lj}|)],\\ \tilde{\mathbf{b}}_{lj}(x) &= \nabla_x \times [q_{lj}\dot{x}_{lj}/(4\pi\epsilon_o c^2|\mathbf{x}-\mathbf{x}_{lj}|)].\end{aligned} \tag{A.3}$$

The Lorentz force represents force on a charge particle in the presence of electric and magnetic fields. For an $i$th electron in $k$th atom, this force is defined as [37]

$$F = q_{ki}(\tilde{\mathbf{e}}_t + \dot{\mathbf{x}}_{ki} \times \tilde{\mathbf{b}}_t), \tag{A.4}$$

where $\tilde{\mathbf{e}}_t$, $\tilde{\mathbf{b}}_t$ are the total electric field and magnetic induction at the location $\mathbf{x}_{ki}$.The total field represents sum of the intra-atomic field, inter-atomic field and external field.

While the Maxwell equations describe how electrically charged particles and objects give rise to electric and magnetic fields, the Lorentz force law completes the picture by describing the force acting on a moving point charge in the presence of electromagnetic fields. We define [37]

$$\begin{gathered}q_k = \sum_i q_{ki}, \qquad \mu_k = \sum_i q_{ki}\zeta_{ki}, \qquad \nu_k = \frac{1}{2}\sum_i q_{ki}\zeta_{ki} \times \dot{\zeta}_{ki}, \qquad \sigma_q = \sum_k q_k\delta(\mathbf{x}_k - \mathbf{x}),\\ \tilde{\mathbf{p}} = \sum_k \mu_k\delta(\mathbf{x}_k - \mathbf{x}), \qquad \tilde{\mathbf{j}} = \sum_k q_k\dot{\mathbf{x}}_k\delta(\mathbf{x}_k - \mathbf{x}), \qquad \tilde{\mathbf{m}} = \sum_k(\mu_k \times \dot{\mathbf{x}}_k + \nu_k)\delta(\mathbf{x}_k - \mathbf{x}).\end{gathered} \tag{A.5}$$

Because of the presence of the delta functions, these microscopic fields fluctuate rapidly in space. However, the physical dimension of phenomenological laws are much larger than the size of each atom. The macroscopic field quantities are defined in terms of statistical average of the microscopic fields over regions that contain a large number of atoms. In statistical mechanics, the averaging formula used [3] is

$$\mathbf{h}(\mathbf{x}, t) = \langle \tilde{\mathbf{h}} \rangle = \int \tilde{\mathbf{h}}(\mathbf{x}; \mathbf{r}) f(t; \mathbf{r}) d\mathbf{r}, \tag{A.6}$$

where $\mathbf{r}$ represents the ensemble $(\mathbf{x}_k, \mathbf{x}_k, \zeta_{ki}, \dot{\zeta}_{ki})$, and $d\mathbf{r} = d\mathbf{x}_k d\mathbf{x}_k \Pi(d\zeta_{ki}, d\dot{\zeta}_{ki})$ is an element of fluxion space. The product $fd\mathbf{r}$ represents the probability to find $\mathbf{h}$ in the fluxion space element $d\mathbf{r}$.

According to this definition, the macroscopic field variables are represented [37] as

$$\langle \tilde{\mathbf{e}} \rangle = \mathbf{e}, \qquad \langle \tilde{\mathbf{b}} \rangle = \mathbf{b}, \qquad \langle \tilde{\mathbf{p}} \rangle = \mathbf{p}, \qquad \langle \tilde{\mathbf{m}} \rangle = \mathbf{m}, \qquad \langle \sigma_q \rangle = \sigma, \qquad \langle \tilde{\mathbf{j}} \rangle = \mathbf{j}. \tag{A.7}$$

The Maxwell equations expressed in terms of these macroscopic field quantities [37] are

$$\begin{aligned}
&\nabla \cdot \mathbf{b} = 0, \\
&\nabla \times \mathbf{e} + \frac{\partial \mathbf{b}}{\partial t} = 0, \\
&\epsilon_o \nabla \cdot \mathbf{e} = \sigma - \nabla \cdot \mathbf{p}, \\
&\mu_o^{-1} \nabla \times \mathbf{b} - \epsilon_o \frac{\partial \mathbf{e}}{\partial t} = \mathbf{j} + \frac{\partial \mathbf{p}}{\partial t} + \nabla \times \mathbf{m}.
\end{aligned} \tag{A.8}$$

The Lorentz force on a single charge particle is written. These forces add up, to obtain the net force on the atom and then statistical averaging is done to obtain the Lorentz force on the whole body. The balance laws of mechanics are first written for atoms and statistical averaging is done in a similar way to get the macroscopic equations.

# APPENDIX B

# DENSITY OF GALFENOL

Density of Gallium is 5.91e3 $kg/m^3$, Density of Iron is 7.874e3 $kg/m^3$. Galfenol used in the experiments is 18.5% Gallium and the remaining percent is Iron.

$$\rho_{\text{Galfenol}} = 0.185 * \rho_{\text{Gallium}} + 0.815 * \rho_{\text{Iron}} = 7.51066e3 \tag{B.1}$$

# BIBLIOGRAPHY

[1] "Low votage co-fired multilayer stacks, rings and chips for actuation". www.piezomechanik.com, 2006.

[2] A.N. Abd-alla. "Nonlinear Constitutive Equations for Thermo-electroelastic Materials". *Mechanics Research Communications*, 26(3):335–346, 1999.

[3] W. Armstrong. "Magnetization and magnetostriction processes in $Tb_{(0.27-0.30)}Dy_{(0.73-0.70)}Fe_{(1.9-2.0)}$". *Journal of Applied Physics*, 81(5), 1997.

[4] G.P. Carman and M. Mitrovic. "Nonlinear Constitutive Relations for Magnetostrictive Materials with Applications to 1-D Problems". *Journal of Intelligent Material Systems and Structures*, 6:673, 1995.

[5] A.E. Clark, M. Wun-Fogle, J.B. Restorff, and T.A. Lograsso. "Magnetic and Magnetostrictive properties of Galfenol Alloys Under Large Compressive Stresses". In *International Symposium on Smart Materials-Fundamentals and System Applications, Pacific Rim Conference on Advanced Materials and Processing (PRICM-4)*, 2001.

[6] A.E. Clark, M. Wun-Fogle, J.B. Restorff, T.A. Lograsso, and J.R. Cullen. "Effect of Quenching on the Magnetostriction of $Fe_{1-x}Ga_x, (0.13 < x < 0.21)$". In $8^{th}$ *Joint MMM-Intermag Conference*, 2001.

[7] J.R. Cullen, A.E. Clark, M. Wun-Fogle, J.B. Restorff, and T.A. Lograsso. "Magnetoelasticity of Fe-Ga and Fe-Al Alloys". In *ICM*, 2002.

[8] Czimmek and P. Robert. "Electronic fuel injector actuated by mangetostrictive transduction". www.freepatentsonline.com, 2002.

[9] M.J. Dapino. " Smart Materials Course Notes". The Ohio State University, 2007.

[10] M.J. Dapino, R.C. Smith, F.T. Calkins, and A.B. Flatau. "A Coupled Magnetomechanical Model for Magnetostrictive Transducers and its Application to Villari-Effect Sensors". *Journal of Intelligent Material Systems and Structures*, 13(3):493–502, 2002.

[11] Bryon Dudley. "Precision moves with Magnetostriction". www.machinedesign.com, 2004.

[12] P.G. Evans and M.J. Dapino. "Measurement and Modeling of Magnetomechanical Coupling in Magnetostrictive Iron-Gallium Alloys".

[13] P.G. Evans and M.J. Dapino. "Fully-coupled magnetoelastic model for Galfenol alloys incorporating eddy current losses and thermal relaxation". In *Proceedings of SPIE Smart Structures and Materials*, volume 6929, page 69291W, San Diego, CA, March 2008.

[14] P.G. Evans and M.J. Dapino. "State-Space Constitutive Model for Magnetization and Magnetostriction of Galfenol Alloys". *IEEE Transactions on Magnetics*, 44(7):1711–1720, 2008.

[15] D. Fang, Y. Wan, X. Feng, and A.K. Soh. "Deformation and Fracture of Functional Ferromagnetics". *Applied Mechanics Reviews*, 61:020803–23, 2008.

[16] R. M. Fano, L.C. chu, and R.B. Adler. *"Electromagnetic Fields, Energy and Forces"*. John Wiley & Sons, Inc.,, New york, NY, 1960.

[17] A.E. Green and P.M. Naghdi. "Aspects of the Second Law of Thermodynamics in the presences of Electromagnetic Effects". *Q.Jl Mech.appl.Math*, 37(2), 1984.

[18] A.E. Green and P.M. Naghdi. "A unified procedure for construction of theories of deformable media. I. Classical continuum physics". *Proc.R.Soc.Lond.A*, 448:335–356, 1995.

[19] S.R. De Groot and L.G. Suttorp. *"Foundations of Electrodynamics"*. North-Holland Publishing Co.,, Amsterdam, 1972.

[20] N. Hakim, Y. Kalish, S. Li, and H. Jiang. "Smart Materials for Fuel Injection Actuation". Detroit Diesel Corporation, 2000.

[21] C.L. Hom and N. Shankar. "A Fully Coupled Constitutive Model for Electrostrictive Ceramic Materials". *Journal of Intelligent Material Systems and Structures*, 5:795, 1994.

[22] K. Hutter. " On Thermodynamics and Thermostatics of Viscous Thermoelastic Solids in the Electromagnetic Fields: A Lagrangian Formulation". *Arch.Rat. Mech. Anal.*, 58:339–368, 1975.

[23] K. Hutter. "A thermodynamic Theory of Fluids and Solids in the Electromagnetic Fields". *Arch.Rat. Mech. Anal.*, 64:269–298, 1977.

[24] K. Hutter, A.A.F. van de Ven, and A. Ursescu. *"Electromagnetic Field Matter Interactions in Thermoelastic Solids and Viscous Fluids"*. Springer, New York, NY, 2006.

[25] D.C. Jiles and J.B. Thoelke. "Theoretical Modelling of the Effects of Anisotropy and Stress on the Magnetization and Magnetostriction of $Tb_{0.3}Dy_{0.7}Fe_2$". *Journal of magnetism and magnetic materials*, 134:143–160, 1994.

[26] R.A. Kellogg. "The Delta-E effect in terfenol-D and its applications in a tunable mechanical resonator". Master's thesis, Iowa State University, 2000.

[27] R.A. Kellogg, A.B.Flatau, A.E. Clark, M. Wun-Fogle, and T.A. Lograsso. "Temperature and stress dependencies of the magnetic and magnetostrictive properties of $Fe_{0.81}Ga_{0.19}$". *Journal of Applied Physics*, 91(10):7821, 2002.

[28] R.A. Kellogg, A.B.Flatau, A.E. Clark, M. Wun-Fogle, and T.A. Lograsso. "Texture and grain morphology dependencies of saturation magnetostriction in rolled polycrystalline $Fe_{83}Ga_{17}$". *Journal of Applied Physics*, 93(10):8495–8497, 2003.

[29] R.A. Kellogg, A.M.Russell, T.A. Lograsso, A.B.Flatau, A.E. Clark, and M. Wun-Fogle. "Tensile properties of magnetostrictive iron-gallium alloys". *Acta Materialia*, 52(17):5043–5050, 2004.

[30] B. Kiefer, D. Rosato, and C. Miehe. "Finite Element Analysis of General Magnetomechanical Coupling Phenomena". *Proceedings in Applied Mathematics and Mechanics*, 8:10505–10506, 2008.

[31] B. Kiefer, D. Rosato, and C. Miehe. "Modeling and Computational Analysis of Materials Exhibiting Intrinsic Magnetomechanical Coupling". In *Proceedings of SPIE Smart Structures and Materials*, volume 6929, page 692920, San Diego, CA, March 2008.

[32] S. Klinkel. "A phenomenological constitutive model for ferroelastic and ferroelectric hysteresis effects in ferroelectric ceramics". *International Journal of Solids and Structures*, 43:7197–7222, 2006.

[33] K. Linnemann and S. Klinkel. "A Constitutive Model for Magnetostrictive Materials - Theory and Finite Element Implementation". *Proceedings in Applied Mathematics and Mechanics*, 6:393–394, 2006.

[34] B.J. MacLachlan, N. Elvin, C. Blaurock, and N.J. Keegan. "Piezoelectric valve actuator for flexible diesel operation". In *Smart Structures and Materials 2004:*

*Industrial & Commercial Applications of Smart Structures Technologies*, volume 5388, pages 167–178, San Diego, CA, July 2004.

[35] W.P. Mason. *"Physical Acoustics, principles and methods"*, volume 1. Academic Press, 1964.

[36] B. Noheda, J.A. Gonzalo, R. Guo, S.-E. Park, L.E. Cross, D.E. Cox, and G. Shirane. "The monoclinic phase in PZT: new light on morphotropic phase boundaries". In *Proceedings of the workshop on Fundamental Physics of ferroelectrics*, Aspen, February 2000.

[37] Y.-H. Pao. *"Electromagnetic forces in deformable continua"*. Pergamon Press, Inc.,, 1978.

[38] Y.H. Pao and K. Hutter. "Electrodynamics of Moving Elastic Solids and Viscous Fluids". *Proc. I.E.E.E*, 63:1011–1021, 1975.

[39] P. Pennfield and H.A. Haus. *"Electrodynamics of Moving Media"*. M.I.T Press, Cambridge, MA, 1967.

[40] J.L. Perez-aparicio and H. Sosa. "A continuum three-dimensional, fully coupled, dynamics, non-linear finite element formulation for magnetostrictive materials". *Smart Materials and Structures*, 13(3):493–502, 2004.

[41] E.M. Purcell. *"Electricity and Magnetism, Berkeley Physics Course, Vol 2"*. McGraw-Hill Higher Education, 1984.

[42] F.J. Rooney and S.E. Bechtel. "Constraints, Constitutive Limits and Instability in Finite Thermoelasticity". *Journal of Elasticity*, 74(2):109–133, 2004.

[43] A. Scharf. "Piezo-actuators in fuel injection systems". www.epcos.com, 2006.

[44] R.C. Smith. *"Smart Materials Systems: Model Development"*. Society of Industrial and Applied Mathematics, Philadelphia, PA, 2005.

[45] A.K. Soh and J.X. Liu. "On the Constitutive Equations of Magnetoelectroelastic Solids". *Journal of Intelligent Material Systems and Structures*, 16:597, 2005.

[46] R.A. Toupin. "Int. J. Eng. Sc.". *Arch.Rat. Mech. Anal.*, 1:101–126, 1963.

[47] Y. Wan, D. Fang, and K-C. Hwang. "Non-linear constitutive relations for magnetostrictive materials". *International Journal of Non-Linear Mechanics*, 38:1053–1065, 2003.

[48] X. Zheng and L. Sun. "A one-dimension coupled hysteresis model for giant magnetostrictive materials". *Journal of magnetism and magnetic materials*, 309(2):263–271, 2007.

[49] X.J. Zheng and X.E. Liu. "A nonlinear constitutive model for Terfenol-D rods". *Journal of applied physics*, 97(5):053901, 2005.