We are interested in designing the fast algorithm to solve the general linear programming (LP) problem of the form,
\[
\min_{x \in \mathbb{R}^n} \quad c^T x
\]
\[
st. \quad Ax = b, \quad x \geq 0, \quad i \in [n].
\]

Applications of LP in machine learning:
- \(l_1\)-regularized support vector machine (SVM) problem.
- Nonnegative matrix factorization problem.
- Sparse inverse covariance matrix estimation problem.
- Markov decision process (MDP) problem.
- Maximum a posterior estimation problem.

We separate the equality and inequality constraints by adding another group of variables \(y\).

The Augmented Lagrangian function of the primal problem is
\[
L(x, y, z) = c^T x + \frac{1}{2}\|x - A y - b\|_2^2 + \sum_{i \in [n]} \frac{d_i}{2}(z_i - y_i)^2.
\]

Observations:
- It converges fast in the initial phase, but exhibits a slow and fluctuating tail convergence.
- Theoretically, it can be recovered by an exact Uzawa method (local second-order approximates ADMM).

**New Variable Splitting Method**

We separate the equality and inequality constraints by adding another group of variables \(y\).

**Primal**

\[
\begin{align*}
\min_{x \in \mathbb{R}^n} & \quad c^T x + y^T (A x + A y - b) + \frac{1}{2} \|A x + A y - b\|_2^2 \\
\text{s.t.} & \quad y \geq 0, \quad y_i \in [n].
\end{align*}
\]

**Dual**

\[
\begin{align*}
\min_{b, x, z} & \quad \frac{1}{2}\|A^T y + z - b\|_2^2 \\
\text{s.t.} & \quad \|x - A y - b\|_2^2 \\
& \quad A^T z - y \leq 0, \quad y_i \in [n], \\
& \quad x_i = 0, \quad z_i = 0, \quad i \in [n].
\end{align*}
\]

**Related Works**

First-order algorithm requires a matrix vector multiplication \(A x\) in each iteration with complexity \(O(nz(A)) \ll \min\{mn, n^2\}\).
- Subgradient descent method
- Augmented Lagrangian Method (ALM) [2]
- Alternating Direction Method of Multiplier (ADMM) [1]

**Tail Convergence of the Existing ADMM Method [1]**

**Global Linear Convergence of New Splitting Method**

**Lemma 1 (Convergence [3])**

\[
\begin{align*}
\|p^{(k+1)} - p^{(k)}\|_2 & \leq \|p^{(k)} - p^{(k-1)}\|_2 \\
& \leq 2^{-k} \|p^{(1)} - p^{(0)}\|_2.
\end{align*}
\]

**Lemma 2 (Geometry of the optimal solution set of LP)**

- Feasibility: \(Ax = b, x \succeq y^2 - A z - z = z\) with \(y \geq 0, z_i \leq 0, i \in [n] \land z_i = 0, i \in [n]\).
- Optimal strong duality: \(c^T x + b^T z = 0\)

**Lemma 3 (Hoffman bound [4])**

\[
\|x - y\|_2 \leq \frac{\theta_n}{\|A\|_2} \|A x - b\|_2.
\]

**Lemma 4 (Estimation of residuals)**

\[
\begin{align*}
A_1 x^{k+1} + A_2 y^k - b = (p^{k+1} - p^k)/\rho, \\
c + A_1 z^k = A_1 (p^k - p^{k-1}), \\
c + A_1 z^{k+1} + b^T y^k = (A_1 x^{k+1} - z^k)/\rho - (p^k - p^{k-1}), \\
y \geq 0, z_i \leq 0, i \in [n]; z_i = 0, i \in [n].
\end{align*}
\]

**Theorem 1 (Global linear convergence)**

To guarantee that \(\|x^{k+1} - x^*\|_2 \leq \varepsilon\), it suffices to run \(K = 2\rho \log(2\|D_{b0}\|/\varepsilon)\) ADMM iterations with solving accuracy \(\varepsilon^2 / \rho^2 / \|K\|_2^2\).

**Theorem 2 (Overall Convergence)**

If we use the ACDM to solve the inner linear system, the overall complexity of algorithm 1 is
\[
O\left(\rho n z(A) \|A\| \log(1/\varepsilon)\right).
\]

The complexity of existing ADMM [1] is
\[
O(n z(A) \rho (n z + d_{max}) \log(1/\varepsilon)).
\]

**Algorithm 1: Alternating Direction Method of Multiplier with Inexact Subproblem Solver**

- 1. Primal update: \(x^{k+1}\) by solving the linear system with accuracy \(\varepsilon^2 \|D_{b0}\| / \rho^2 / \|K\|_2^2\).
- 2. Dual update: \(y^{k+1}\) by solving the linear system with accuracy \(\varepsilon^2 \|D_{b0}\| / \rho^2 / \|K\|_2^2\).
- 3. Update of \(z^{k+1}\).

**Simulation Results**

- 2X-40X speedup compared with state-of-arts.
- Significantly faster than commercial software CPLEX.
- Flexibility to tackle various problems.