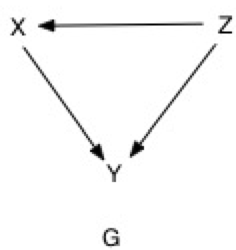
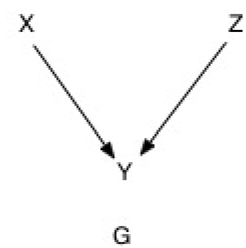
Reconciliation of back-door and front-door adjustments

First, consider the case comparing the back-door adjustment for a typical back-door (below left) with a classic randomized design (below right), the back-door adjustment applies consistently to each.





The back-door adjustment is

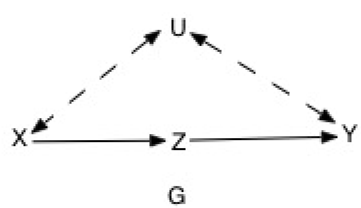
$$\Pr(y \mid do(x)) = \sum_z \Pr(y \mid x, z) \Pr(z) \qquad \text{(back-door adj)}$$

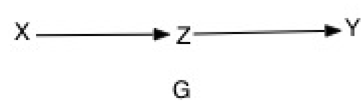
The DAG on the right implies action and observation of X are equivalent

$$\Pr(y \mid do(x)) = \Pr(y \mid x) = \sum_z \Pr(y \mid x, z) \Pr(z \mid x) = \sum_z \Pr(y \mid x, z) \Pr(z)$$

The latter follows since X and Z are independent in the DAG.

Next, consider two front-door DAGs. The one on the left is the prototype but both satisfy the definition of a front-door.





[Definition] A set of variables Z is defined a **front-door** for the ordered pair (X, Y) if
(i) Z intercepts all directed paths from X to Y,
(ii) there is no unblocked back-door path from X to Z, and
(iii) all back-door paths from Z to Y are blocked by X.

The front-door adjustment is

$$\Pr(y \mid do(x)) = \sum_z \Pr(z \mid x) \sum_{x'} \Pr\left(y \mid x', z\right) \Pr\left(x'\right) \qquad \text{(front-door adj)}$$

As is the case with back-door adjustment, front-door adjustment reconciliation relies entirely on conditional independence rather than further appeal to do-calculus. For the front-door DAG on the right action is equivalent to observation $\Pr(y \mid do(x)) = \Pr(y \mid x)$. The key resides with the second summation and conditional independence of X and Y given Z.

$$\sum_{x'} \Pr\left(y \mid x^{'}, z\right) \Pr\left(x^{'}\right) = \sum_{x'} \Pr(y \mid z) \Pr\left(x^{'}\right) = \Pr(y \mid z)$$

Then, combine with the first term and insert X back into the conditioning and by Bayes' chain rule the reconciliation is complete.

$$\sum_{z} \Pr(z \mid x) \Pr(y \mid z) = \sum_{z} \Pr(z \mid x) \Pr(y \mid x, z) = \Pr(y \mid x)$$

The final reconciliation involves the prototypical front-door DAG on the left. Suppose U is measured. Then, the back-door adjustment identifies the causal effect of X on Y.

$$\Pr(y \mid do(x)) = \sum_{u} \Pr(y \mid x, u) \Pr(u)$$

There are two conditional independence relations to reconcile the right-hand side of the back-door adjustment with the right-hand side of the front-door adjustment (again, without further reference to do-calculus).

$$X \perp Y \mid Z, U$$
$$U \perp Z \mid X$$

First, rewrite the back-door adjustment to include Z with Bayes chain rule.

$$\sum_{u} \Pr(y \mid x, u) \Pr(u) = \sum_{u} \sum_{z} \Pr(y \mid x, u, z) \Pr(z \mid x, u) \Pr(u)$$

Now utilize the conditional independence conditions and rearrange.

$$\sum_{u} \sum_{z} \Pr(y \mid x, u, z) \Pr(z \mid x, u) \Pr(u) = \sum_{u} \sum_{z} \Pr(z \mid x) \Pr(y \mid u, z) \Pr(u)$$

Then, use Bayes chain rule along with conditional independence to insert X'in the last two terms.

$$\sum_{u} \sum_{z} \Pr(z \mid x) \Pr(y \mid u, z) \Pr(u)$$
$$= \sum_{u} \sum_{z} \Pr(z \mid x) \sum_{x'} \Pr(y \mid x', u, z) \Pr(u \mid x', z) \Pr(x')$$

Finally, summation over U produces the front-door adjustment.

$$\sum_{z} \Pr(z \mid x) \sum_{x'} \Pr\left(y \mid x^{'}, z\right) \Pr\left(x^{'}\right)$$