Ralph's Holevo bound

Quantum information is enormously more powerful than classical information as is clear from results like teleportation, superdense coding, quantum factoring and quantum discrete log. The challenge involves accessing/exploiting this information. The Holevo bound provides a limit on accessible quantum information.

$$I\left(X;Y\right)=H\left(X\right)-H\left(X\mid Y\right)\leq S\left(\rho\right)-\sum_{x}p_{x}S\left(\rho_{x}\right)\equiv\chi$$

where $I\left(X;Y\right)$ is classical (Shannon) mutual information, ensemble X is the prepared message, Y is the measured outcome, $H(\cdot)$ is classical (Shannon) entropy, $S(\cdot)$ is quantum (von Neumann) entropy, ρ is a quantum density operator, p_x is the probability of event x, and $\rho=\sum_x p_x\rho_x$.

In classical terms, $I\left(X;Y\right)=H\left(X\right)$ indicates information is fully accessible. This is intuitive as it implies $H\left(X\mid Y\right)=0$, that is, given information Y no uncertainty remains regarding X. The right hand side (RHS) χ indicates a bound on the amount of accessible classical information recoverable from quantum information. χ roughly mirrors quantum mutual information as it indicates the amount of uncertainty remaining in the system on knowing the state of preparation ρ_x from the ensemble of preparations $\rho_x, x=1,2,\ldots,n$.

Holevo information is denoted χ and represents a bound on accessible information. If the alphabet is prepared from pure states, then $\chi=S\left(\rho\right)$ as $S\left(\rho_x\right)=0$ for pure states ρ_x. If the alphabet involves orthogonal states, then $\chi=H\left(X\right)=S\left(\rho\right)$ as measurement perfectly distinguishes orthogonal states of a known ensemble. If the alphabet involves mutually orthogonal mixed states, then $Tr\left(\rho_1\rho_2\right)=0$ and $\chi=H\left(X\right)$.

Suggested:

1. Suppose $\rho=0.3\left|0\right\rangle\left\langle 0\right|+0.7\left|1\right\rangle\left\langle 1\right|$. Find the Holevo bound χ.

2. Suppose $\rho=0.3\left|0\right\rangle\left\langle 0\right|+0.7|+\rangle\left\langle +\right|$ where $\left|+\right\rangle=\frac{1}{\sqrt{2}}\left(\left|0\right\rangle+\left|1\right\rangle\right)$. Find the Holevo bound χ.

3. Suppose $\rho=0.5\rho_1+0.5\rho_2$ where $\rho_1=0.3\left|0\right\rangle\left\langle 0\right|+0.7|+\rangle\langle+|$. and $\rho_2=0.8\left|1\right\rangle\left\langle 1\right|+0.2|-\rangle\left\langle -\right|$ where $\left|-\right\rangle=\frac{1}{\sqrt{2}}\left(\left|0\right\rangle-\left|1\right\rangle\right)$. Find the Holevo bound χ.

4. Suppose $\rho=0.5\rho_1+0.5\rho_2$ where $\rho_1=\begin{bmatrix}0.25 & -0.1 & 0.25 & -0.1\\ -0.1 & 0.25 & -0.1 & 0.25\\ 0.25 & -0.1 & 0.25 & -0.1\\ -0.1 & 0.25 & -0.1 & 0.25\end{bmatrix}$ and $\rho_2=\begin{bmatrix}0.25 & 0.15 & -0.25 & -0.15\\ 0.15 & 0.25 & -0.15 & -0.25\\ -0.25 & -0.15 & 0.25 & 0.15\\ -0.15 & -0.25 & 0.15 & 0.25\end{bmatrix}$. Compute $Tr\left(\rho_1\rho_2\right)$ (if mixed density operators are mutually orthogonal $Tr\left(\rho_1\rho_2\right)=0$). Find the Holevo bound χ and compare it with $H\left(X\right)$.