The **pseudoinverse** or Moore-Penrose inverse is an operator X with the following properties.

$$AXA = A \tag{1}$$

$$XAX = X \tag{2}$$

$$AX = (AX)^T = X^T A^T \tag{3}$$

$$XA = (XA)^T = A^T X^T \tag{4}$$

Combining (2) and (3) yields

$$XAX = XX^T A^T = X \tag{5}$$

Combining (1) and (4) along with transposition produces

$$(AXA)^T = A^T X^T A^T = XAA^T = A^T \tag{6}$$

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Uniqueness

There is exactly one operator with properties (1) − (4) for any matrix A. Uniqueness can be demonstrated by considering another operator Y with the following properties.

$$Y = A^T Y^T Y \tag{7}$$

$$A^T = A^T AY \tag{8}$$

(7) is derived by substituting (4) into (2) after replacing X with Y and (8) substitutes (3) into (1). Utilizing the equations above demonstrates X and Y are the same operators.

$$X = XX^T A^T = XX^T A^T AY = XAY = XAA^T Y^T Y = A^T Y^T Y = Y \tag{9}$$

The pseudoinverse of A is usually denoted $A^\dagger$. Of course, the left-, right- inverse of A, A^{-1}, is a special case of $A^\dagger$ when A is nonsingular.

SVD construction of $\mathbf{A}^\dagger$

Singular value decomposition (SVD) applies to any matrix $A = U\Sigma V^T$ where U and V have orthonormal columns and Σ is a diagonal matrix. Since the columns of U and V are orthonormal, $U^\dagger = U^T$ and $V^\dagger = V^T$. Hence,

$$A^\dagger = \left(U\Sigma V^T\right)^\dagger = V\Sigma^\dagger U^T \tag{10}$$

Properties (1) − (4) along with uniqueness of $A^\dagger$ assures $\left(A^\dagger\right)^\dagger = A$.

LU construction of $\mathbf{A}^\dagger$

First, factor A as PLU where U is fully row reduced to produce rows of zero if A is less than full row rank. Next, construct $\overline{U}$ by dropping the rows of zero from U so that $\overline{U}$ has full row rank. Similarly, construct $\overline{L}$ by dropping columns from the right of L so that the remaining, L_r, is full column rank. Multiply the result by P to form $\overline{L} = PL_r$. Now,

$$A^\dagger = \overline{U}^T \left(\overline{U}\overline{U}^T\right)^{-1} \left(\overline{L}^T \overline{L}\right)^{-1} \overline{L}^T \tag{11}$$