Quantum measurement as transformation

Quantum measurement (often expressed as the third axiom) can be seen as a unitary transformation (often expressed as the second axiom). We illustrate the ideas with a generic state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$.

Suppose $|\psi\rangle$ is measured in the computation basis. Then, with probability α^2 the eigenvalue associated with post-measurement state $|0\rangle$ is realized and with probability β^2 the eigenvalues associated with post-measurement state $|1\rangle$ is realized. Of course, quantum measurement can be depicted via projection into the post-measurement state. However, we can also think of it simply as a unitary transformation.

Let $U_0 = \beta X + \alpha Z$ and $U_1 = \alpha X - \beta Z$ where $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ are Pauli operators (both unitary and Hermitian). Then, $U_0|\psi\rangle = |0\rangle$ and $U_1|\psi\rangle = |1\rangle$.

Suppose we measure in the $|+\rangle; |-\rangle$ basis. Then, the unitary operation can be written $HU_0|\psi\rangle = |+\rangle$ and $HU_1|-\rangle$.

More generally, suppose we measure in the Hadamard basis, $|h_0\rangle = \begin{bmatrix} 0.92388 \\ 0.38268 \end{bmatrix}$; $|h_1\rangle = \begin{bmatrix} 0.38268 \\ -0.92388 \end{bmatrix}$. Let $U_H = 0.38268X + 0.92388Z$, then $U_H U_0|\psi\rangle = |h_0\rangle$ and $U_H U_1|\psi\rangle = |h_1\rangle$.

This latter result suggests a fully general transformation. Suppose the measurement basis is $|z_0\rangle = \begin{bmatrix} c \\ d \end{bmatrix}$;$|z_1\rangle = \begin{bmatrix} d \\ -c \end{bmatrix}$. Let $U = dX + cZ$, then $UU_0|\psi\rangle = |z_0\rangle$ and $UU_1|\psi\rangle = |z_1\rangle$.