Ralph's quantum measurement

Ralph knows the state of the system is either $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ or $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, where $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and no measurement can perfectly discover the state. Rather, measurement produces a mix of outcomes (eigenvalues) with certain probabilities. Since probabilities are non-negative, quantum measurement formalism employs positive operators. **Positive operators** are Hermitian and have non-negative, real eigenvalues.

POVM

In general, positive operator valued measurement (POVM) simply requires a complete set of positive operators which sum to the identity matrix.

$$E_1 + E_2 + \cdots + E_n = I$$

For instance, let $E_1 = \frac{\sqrt{2}}{1+\sqrt{2}} |1\rangle \langle 1|$, $E_2 = \frac{\sqrt{2}}{1+\sqrt{2}} |-\rangle \langle -|$, and $E_3 = I - E_1 - E_2$ where $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. Let the measurement device be

$$E = \{E_1, E_2, E_3\}$$

If $\lambda_E = 1$ is realized then the state is not $|0\rangle$ and must be $|+\rangle$, if $\lambda_E = 2$ is realized then the state is not $|+\rangle$ and must be $|0\rangle$. In other words, we are able to tease out the hidden nature of the state with certainty part of the time. However, if $\lambda_E = 3$ we learn nothing about the state.

Projective measurement

Projective measurement is a special case of POVM where the eigenvalues are zero and one (for a single qubit system). For instance, suppose the measurement device is

$$M = 1 |0\rangle \langle 0| + 2 |1\rangle \langle 1|$$

That is, projective measurement has the additional property that the elements are orthogonal to one another. For measurement M, if $\lambda_M = 2$ then the state must be $|+\rangle$. On the other hand, if $\lambda_M = 1$ then the state is ambiguous. Again, no measurement system can perfectly reveal a quantum state.

Suggested:

1. Verify the above measurement claims. Suppose the state is $|0\rangle$, for measurements E and M find the probability of each possible realization λ_E and λ_M. (hint: $\Pr(\lambda_E = 1 \mid |0\rangle) = \langle 0| E_1 |0\rangle$)

2. Repeat 1 supposing the state is $|+\rangle$.