Ralph's quantum coordination

Ralph owns an organization with two business units and realizes the key to success is building synergy. That is, the two units must work together but in an uncertain environment. Formally, this is most simply and elegantly represented by quantum entanglement. Two **entangled** qubits can be represented by a Bell state.

$$|\beta_{00}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

The first qubit lies in the control of a business unit managed by Alice and the second qubit is controlled by Bob, the manager of the other business unit.

Alice's business unit acquires information $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. Effective coordination requires this information by communicated to Bob. Alice and Bob recognize this is delicate but can be accomplished by quantum teleportation (see Ralph's teleportion).

Communication via teleportation

Briefly, teleporation involves

1. combine $|\psi\rangle$ with $|\beta_{00}\rangle$ to form $|\psi\beta_{00}\rangle$.

2. Alice applies $C_1 - X_2$ (if the first qubit is one, bit flip the second qubit) followed by H_1 to the first two qubits she controls.

3. Alice measures her first two transformed qubits with measurement device

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$
$$= 1|00\rangle\langle 00| + 2|01\rangle\langle 01| + 3|10\rangle\langle 10| + 4|11\rangle\langle 11|$$

4. If $\lambda = 1$ is realized, Alice communicates Bob's (third) qubit is $|\psi\rangle$ and no transformation is needed.

If $\lambda = 2$ is realized, Alice communicates Bob's qubit is $X|\psi\rangle$ and $|\psi\rangle$ is recoved if Bob applies X to his qubit.

If $\lambda = 3$ is realized, Alice communicates Bob's qubit is $Z|\psi\rangle$ and $|\psi\rangle$ is recoved if Bob applies Z to his qubit.

If $\lambda = 4$ is realized, Alice communicates Bob's qubit is $XZ|\psi\rangle$ and $|\psi\rangle$ is recoved if Bob applies ZX to his qubit.

Teleportation illustrates the principle of *implicit measurement*. Qubits in the chain that are not directly measured are implicitly measured (the teleportation algorithm directly measures the first and second qubits with the third qubit implicitly measured). Implicit measurement along with no cloning of an unknown state ensure the impossibility of perfect identification of an unknown state or even perfectly distinguishing between non-orthogonal states (such as $|0\rangle$ and $|+\rangle$).

Centralization vs decentralization

Suppose the organization is managed such that $|\psi\beta_{00}\rangle$ is centrally controlled. Then, there is an alternative mechanism to deliver the information $|\psi\rangle$ acquired at the front end (say, marketing) to the final destination (say, production) that eliminates the messy communication-transformation step described in the teleportation algorithm above. The alternative algorithm is

1. combine $|\psi\rangle$ with $|\beta_{00}\rangle$ to form $|\psi\beta_{00}\rangle$.

2. apply $C_1 - X_2$ (if the first qubit is one, bit flip the second qubit) followed by H_1.

3.apply $C_2 - X_3$ followed by $C_1 - Z_3$ (this step is outside the control of Alice under decentralization).

4. measure the first two qubits with M from above. For any realization, $\lambda = 1, 2, 3,$ or 4, the third qubit is $|\psi\rangle$.

Of course, this begs the question could the organization effectively acquire $|\psi\rangle$ without specialization? The remaining discussion proceeds with the decentralized organization.

Synergy and entropy

Prior to teleportation the state of the organization is a pure entangled state $|\psi\beta_{00}\rangle$ or in canonical density operator form

$$\rho = |\psi\beta_{00}\rangle \langle\psi\beta_{00}|$$
$$= \frac{1}{2}\begin{pmatrix} \alpha^2 |000\rangle\langle 000| + \alpha^{\mathbf{2}} |\mathbf{000}\rangle\langle\mathbf{011}| + \alpha\beta |000\rangle\langle 100| + \alpha\beta |\mathbf{000}\rangle\langle\mathbf{111}| \\ +\alpha^{\mathbf{2}} |\mathbf{011}\rangle\langle\mathbf{000}| + \alpha^2 |011\rangle\langle 011| + \alpha\beta |\mathbf{011}\rangle\langle\mathbf{100}| + \alpha\beta |011\rangle\langle 111| \\ +\alpha\beta |100\rangle\langle 000| + \alpha\beta |\mathbf{100}\rangle\langle\mathbf{011}| + \beta^2 |100\rangle\langle 100| + \beta^{\mathbf{2}} |\mathbf{100}\rangle\langle\mathbf{111}| \\ +\alpha\beta |\mathbf{111}\rangle\langle\mathbf{000}| + \alpha\beta |111\rangle\langle 011| + \beta^{\mathbf{2}} |\mathbf{111}\rangle\langle\mathbf{100}| + \beta^2 |111\rangle\langle 111| \end{pmatrix}$$

As a pure state, the quantum entropy or uncertainty is zero. Unfortunately, without Alice's specialization (local attention to detail) the organization may not effectively acquire $|\psi\rangle$. Alice and Bob only have local control (Alice the first two qubits and Bob the third). In other words, Alice has direct influence over her reduced density operator (tracing out the third qubit).

$$\rho_A = \frac{1}{2}\begin{pmatrix} \alpha^2 |00\rangle\langle 00| + \alpha\beta |00\rangle\langle 10| \\ +\alpha^2 |01\rangle\langle 01| + \alpha\beta |01\rangle\langle 11| \\ +\alpha\beta |10\rangle\langle 00| + \beta^2 |10\rangle\langle 10| \\ +\alpha\beta |11\rangle\langle 01| + \beta^2 |11\rangle\langle 11| \end{pmatrix}$$

The eigenvalues of ρ_A are $\frac{1}{2}, \frac{1}{2}, 0, 0$ (a mixed state), therefore Alice's quantum entropy is $\log 2$. Similarly, Bob's reduced density operator (following tracing out the first two qubits to produce a mixed state) is

$$\rho_B = \frac{1}{2}\left(\left[\alpha^2 + \beta^2\right] |0\rangle\langle 0| + \left[\alpha^2 + \beta^2\right] |1\rangle\langle 1|\right) = \frac{1}{2}I$$

Bob faces quantum entropy equal to $\log 2$ (maximum uncertainty for a single qubit state) while the organization in total faces zero entropy. Hence, while potentially beneficial, decentralization also involves cost. This suggests coordination is important to mitigate the impact of uncertainty.

Further, it's not surprising that coordination or cooperation between business units is challenging for a decentralized organization. Alice has access only to ρ_A and Bob has access only to ρ_B but the eight components in bold from ρ are not within purview of either but essential for effective communication of $|\psi\rangle$ from Alice to Bob and coordination of productive activities. Remarkably, communication/coordination via teleportation accomplishes the task just as effectively in a decentralized organization as a centralized organization. Either organizational form faces imperfect information as $|\psi\rangle$ cannot be measured perfectly.

Imperfect information

While $|\psi\rangle$ is important to Bob (and the organization) it is still highly uncertain. However, through her actions associated with teleportation Alice is able to reduce the uncertainty considerably by learning the state is either $|0\rangle$ or $|+\rangle = H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and advises Bob to employ $POVM$ (positive operator valued measurement; see Ralph's quantum measurement).

That is, Bob either employs

$$E = \{E_1, E_2, E_3\}$$

where $\lambda_E = 1, 2$, or 3, $E_1 = \frac{\sqrt{2}}{1+\sqrt{2}} |1\rangle \langle 1|$, $E_2 = \frac{\sqrt{2}}{1+\sqrt{2}} |-\rangle \langle -|$, $E_3 = I - E_1 - E_2$ and $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$, or projective measurement

$$M = 1|0\rangle\langle 0| + 2|1\rangle\langle 1|$$

where $\lambda_M = 1$ or 2. Either measurement system imperfectly reveals the state but some times identifies the state exactly. In other words, the information transferred is imperfectly revealed (no surprise!).

State-act-outcome

Bob faces two alternative actions, A and B, with the following state-contingent payoffs.

	$\|0\rangle$	$\|+\rangle$
A	10	-10
B	-10	10

Bob believes the states are equally likely without evidence λ_E or λ_M.

Suggested:

1. Verify decentralized teleportation of $|\psi\rangle$ from Alice to Bob and centralized transmission of $|\psi\rangle$ from the first to third qubit. Notice the effect is the same except for the locus of control.

2. Repeat Ralph's quantum measurement. That is, assume Bob receives $|\psi\rangle$ by faithful communication with Alice and determine the probability of each possible realization λ_E and λ_M given each possible state $|0\rangle$ and $|+\rangle$. When is the information $|\psi\rangle$ exactly identified?

3. Compare the results in 1 with those realized if communication fails such that Bob is working with $X|\psi\rangle$, $Z|\psi\rangle$, or $XZ|\psi\rangle$ rather than $|\psi\rangle$. Again, when is the information $|\psi\rangle$ exactly identified?

4. Verify Alice and Bob's reduced density operators and associated quantum entropy.

5. Compare the expected payoffs under the various scenarios described in 1 and 2. Assume Bob acts as if $|\psi\rangle$ is effectively communicated even when this is erroneous in 2.

6. Determine the value of effective coordination/communication. That is, compare the expected payoff in 1 for the most efficient measurement with the expected payoff in 2 (again, for the same measurement system as used for 1) where $|\psi\rangle$ is equally likely to be transformed by I, X, Z, XZ.