The row component

Identifying transactions from financial statements is an important task for accounting experts.

$$Ay = x$$

where A is the $m \times n$ accounting matrix with (simple) journal entries in the columns and T accounts in the rows, x is an m-element vector of changes in account balances, and y is an n-element vector of transactions amounts. Since there is one redundant row in A and typically $m < n$, there are usually many solutions for the transactions amounts y. Let any consistent solution be y^p then

$$Ay^p = x$$

and dropping any redundant row leads to

$$A_0 y^p = x_0$$

where A_0 and x_0 drop the same row from their respective components A and x.

Usually, y^p has a row component (a consistent solution that is entirely comprised of linear combinations of the rows of A) and a null component (a component comprised of linear combinations of the nullspace of A, N, an $(n-m+1) \times n$ matrix).

$$y^p = y^{\text{row}} + y^{null}$$

$$Ay^{\text{row}} = x$$

$$A_0 y^{\text{row}} = x_0$$

and

$$AN^T = 0$$

$$Ay^{null} = 0$$

$$y^{null} = N^T k$$

where k is an $(n-m+1)$-element vector of weights.

The key to decomposing y or y^p is uncovering the row component. There are many approaches. We'll pursue a few of them in this note.

Gaussian elimination with concatenated matrix

Concatenate N with A_0, this is a full rank n matrix. On the right hand side, concatenate $n-m+1$ zeroes ($Ny^{\text{row}} = 0$) with x_0 to form an n-element vector. By Gaussian elimination and back-substitution solve for y^{row}. The solution is unique.

$$\begin{bmatrix} A_0 \\ N \end{bmatrix} y^{\text{row}} = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}$$

Row component as linear combination of rows of A

Let b be an $(m-1)$-element vector of weights on the rows of A_0. Then, $b^T A_0 = (y^{\mathrm{row}})^T$ or $y^{\mathrm{row}} = A_0^T b$ provided $A_0 A_0^T b = x_0$. These conditions allow us to identify b.

$$b = \left(A_0 A_0^T\right)^{-1} x_0$$

$$y^{\mathrm{row}} = A_0^T \left(A_0 A_0^T\right)^{-1} x_0 = A_0^T \left(A_0 A_0^T\right)^{-1} A_0 y^p$$

Projecting into rows of A

Since $y^p = y^{\text{row}} + y^{null} = A_0^T b + N^T k$ and $A_0 N^T = 0$, we can write $y^{null} = y^p - y^{\text{row}} = y^p - A_0^T b$. Now, exploit orthogonality.

$$A_0 y^{null} = 0$$

$$A_0 \left(y^p - A_0^T b \right) = 0$$

This leads to the normal equations.

$$A_0 A_0^T b = A_0 y^p$$

Solving for b

$$b = \left(A_0 A_0^T \right)^{-1} A_0 y^p$$

leads to the same result as above ($A_0 y^p = x_0$; hence, $b = \left(A_0 A_0^T \right)^{-1} x_0$). Further,

$$y^{\text{row}} = A_0^T b = A_0^T \left(A_0 A_0^T \right)^{-1} A_0 y^p = P_{R(A)} y^p$$

where $P_{R(A)}$ refers to the (symmetric and idempotent) projection matrix into the rows of A.

Projecting into nullspace of A

Following the arguments above, we can solve for $y^{\text{row}} = y^p - y^{null} = y^p - N^T k$. Again, exploit orthogonality.

$$Ny^{\text{row}} = 0$$

$$N\left(y^p - N^T k\right) = 0$$

This leads to the normal equations in terms of the nullspace.

$$NN^T k = Ny^p$$

Solving for k

$$k = \left(NN^T\right)^{-1} Ny^p$$

$$y^{null} = N^T k = N^T \left(NN^T\right)^{-1} Ny^p$$

and

$$y^{\text{row}} = y^p - y^{null} = \left(I - N^T \left(NN^T\right)^{-1} N\right) y^p = \left(I - P_{N(A)}\right) y^p = P_{R(A)} y^p$$

where $P_{N(A)}$ is the (symmetric and idempotent) projection matrix into the nullspace of A. Also, $P_{R(A)} + P_{N(A)} = I$.

QR

QR_1: Let $A_0^T = QR$ where Q is an $n \times (m-1)$ matrix of orthonormal columns and R is a right or upper triangular, $(m-1) \times (m-1)$ full rank matrix ($r = m-1$). Q is formed by applying Gram-Schmidt orthonormalization to the rows of A_0 (columns of A_0^T). Then, $Q^T A_0^T = Q^T Q R = R$.

$$P_{R(A)} = A_0^T \left(A_0 A_0^T\right)^{-1} A_0 = QR\left(R^T Q^T Q R\right)^{-1} R^T Q^T$$

$$= QRR^{-1}\left(R^T\right)^{-1} R^T Q^T = QQ^T$$

Therefore,

$$y^{\text{row}} = P_{R(A)} y^p = QQ^T y^p$$

QR$_2$: Also, let $N^T = QR$ where Q is an $n \times (n-r)$. matrix of orthonormal columns and R is a right triangular, $(n-r) \times (n-r)$ full rank matrix.

$$P_{N(A)} = N^T \left(NN^T\right)^{-1} N = QR \left(R^T Q^T QR\right)^{-1} R^T Q^T$$

$$= QQ^T$$

Therefore,

$$y^{null} = P_{N(A)} y^p = QQ^T y^p$$

and

$$y^{\text{row}} = y^p - y^{null} = \left(I - P_{N(A)}\right) y^p$$

QR$_3$: $A_0^T b = y^{\text{row}}$ where $b = \left(A_0 A_0^T\right)^{-1} A_0 y^p$. Therefore, $A_0^T = QR$ implies

$$b = \left(R^T Q^T QR\right)^{-1} R^T Q^T y^p$$

$$= R^{-1} Q^T y^p$$

QR$_4$: $N^T k = y^{null}$ where $k = \left(NN^T\right)^{-1} N^T y^p$. Therefore, $N^T = QR$ implies

$$k = R^{-1} Q^T y^p$$

and $y^{\text{row}} = y^p - y^{null}$.

Pseudoinverse

$\dagger_1$: The (left, right) inverse of A (or A_0) doesn't exist. However, the pseudoinverse, an $n \times m$ matrix, $A^\dagger$ always exists. Pseudo inverse has the following properties: $AA^\dagger A = A$, $A^\dagger AA^\dagger = A^\dagger$, $\left(AA^\dagger\right)^T = AA^\dagger$, and $(A^\dagger A)^T = A^\dagger A$.

$$A^\dagger Ay^p = A^\dagger x = y^{\text{row}}$$

$$AA^\dagger Ay^p = Ay^p = AA^\dagger x = Ay^{\text{row}} = x$$

This implies

$$A^\dagger A = A_0^\dagger A_0 = P_{R(A)}$$

or

$$A_0^\dagger = A_0^T \left(A_0 A_0^T\right)^{-1}$$

Singular value decomposition supplies a systematic method of recovering the pseudoinverse.

$$A = U\Sigma V^T$$

where Σ is an $m \times n$ semi-positive definite, diagonal matrix with diagonal elements the square-root of the eigenvalues of either AA^T or $A^T A$, U is an orthonormal matrix comprised of the eigenvectors of AA^T (by spectral decomposition), and $V^T = \Sigma^\dagger U^T A$ ($\Sigma^\dagger$ is Σ with the reciprocal of the nonzero diagonal elements).

$$A^\dagger = \left(U\Sigma V^T\right)^\dagger = V\Sigma^\dagger U^T$$

Hence,

$$y^{\text{row}} = A^\dagger x = V\Sigma^\dagger U^T x$$

or $\dagger_2$:

$$y^{\text{row}} = A^\dagger A y^p = V\Sigma^\dagger U^T U \Sigma V^T y^P = V \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} V^T y^p$$

$\dagger_3$: Let $N = U\Sigma V^T$, then

$$y^{null} = P_{N(A)} y^p = N^\dagger N y^p$$

$$y^{null} = V \begin{bmatrix} I_{n-r} & 0 \\ 0 & 0 \end{bmatrix} V^T y^p$$

and $y^{\text{row}} = y^p - y^{null}$.

$\dagger_4$: $P_{R(A)} = A^\dagger A = \left(A^\dagger A\right)^T = A^T \left(A^T\right)^\dagger$ and $P_{C(A)} = AA^\dagger = \left(AA^\dagger\right)^T = \left(A^T\right)^\dagger A^T$.[1]

$$A^T b = y^{\text{row}}$$

[1] $\left(A^T\right)^\dagger = \left(A^\dagger\right)^T$. $A^T = V\Sigma^T U^T, A^\dagger = V\Sigma^\dagger U^T, \left(A^T\right)^\dagger = U\left(\Sigma^T\right)^\dagger V^T, \left(A^\dagger\right)^T = U\left(\Sigma^\dagger\right)^T V^T$. Since, Σ is a diagonal matrix, $\left(\Sigma^T\right)^\dagger = \left(\Sigma^\dagger\right)^T$ and the demonstration is complete.

$$b = \left(A^T\right)^\dagger A^T b = \left(A^T\right)^\dagger y^p$$

$$y^{\text{row}} = A^T b = A^T \left(A^T\right)^\dagger A^T b = A^T \left(A^T\right)^\dagger y^p = P_{R(A)} y^p$$

$\dagger_5$: $P_{N(A)} = N^\dagger N = \left(N^\dagger N\right)^T = N^T \left(N^T\right)^\dagger$ and $NN^\dagger = \left(NN^\dagger\right)^T = \left(N^T\right)^\dagger N^T$.

$$N^T k = y^{null}$$

$$k = \left(N^T\right)^\dagger N^T k = \left(N^T\right)^\dagger y^p$$

$$y^{null} = N^T k = N^T \left(N^T\right)^\dagger N^T k = N^T \left(N^T\right)^\dagger y^p = P_{N(A)} y^p$$

and $y^{\text{row}} = y^p - y^{null}$.