1 Propensity score matching and regression

An increasing number of accounting studies employ regression of outcomes on covariates for propensity-score matched samples to identify average treatment effects (average treatment on treated — ATT, average treatment on untreated — $ATUT$, and the unconditional average treatment effect — ATE). Heckman et al's [1998] identification strategy is as follows. Suppose outcome is a function of a subset of covariates, T, while treatment adoption is a function of a perhaps overlapping subset of covariates, Z, where propensity-score is $p \equiv P(Z) = \Pr(D = 1 \mid Z) = P(X) = \Pr(D = 1 \mid X)$ and X is the full set of covariates T and Z. Further, outcome is additively separable into observable, $g_D(T)$ conditional means for potential outcome with treatment D, and unobservable, U_D, components.

$$\begin{aligned} Y_1 &= g_1(T) + U_1 \\ Y_0 &= g_0(T) + U_0 \end{aligned}$$

Then, conditional mean independence of U_0 along with full common support (over $P(Z)$) allows nonparametric identification of $ATT = E[Y_1 - Y_0 \mid D = 1]$.

$$E[U_0 \mid D, P(Z)] = E[U_0 \mid P(Z)]$$

Similarly, conditional mean independence of U_1 along with full common support (over $P(Z)$) allows nonparametric identification of $ATUT = E[Y_1 - Y_0 \mid D = 0]$.

$$E[U_1 \mid D, P(Z)] = E[U_1 \mid P(Z)]$$

And, conditional mean independence of both U_0 and U_1 along with full common support (over $P(Z)$) allows nonparametric identification of $ATE = E[Y_1 - Y_0]$.

The above discussion applies to unconditional average treatment effects and conditional average effects given the propensity score. However, there are other conditional (on the covariates) average treatment effects which the analyst may be interested. A fully saturated design attempts to identify conditional average effects on covariates and propensity scores. This broader spectrum of conditional average treatment effects involves an expanded version of conditional mean independence. For $ATT(T, P(Z))$, the condition is

$$E[Y_0 \mid D, T, P(Z)] = E[Y_0 \mid T, P(Z)]$$

and for $ATUT(T, P(Z))$, the condition is

$$E[Y_1 \mid D, T, P(Z)] = E[Y_1 \mid T, P(Z)]$$

along with full common support (over $(T, P(Z))$). A less ambitious variation identifies average effects conditional on covariates where conditional mean independence for $ATT(T)$, the condition is

$$E[Y_0 \mid D, T] = E[Y_0 \mid T]$$

and for $ATUT\,(T)$, the condition is

$$E\left[Y_1 \mid D, T\right] = E\left[Y_1 \mid T\right].$$

These various designs are discussed below.

Next, we explore the identification strategy for some stylized examples featuring a variety of data generating processes (DGP). Common support for $P\,(Z)$ as well as common support for T is satisfied for all examples but only examples 1 and 5 satisfy common support for $(P\,(Z)\,, T)$ jointly.[1] Observed data are $Y = DY_1 + (1 - D)\,Y_0$ and observable covariates, $T = DT_1 + (1 - D)\,T_0$ (some covariates are unobserved akin to counterfactual outcomes). Covariate selection is a thought experiment (only partially reinforced by the data due to the counterfactual nature of the question) that seeks to determine conditions to which treatment effects are sensitive. A "kitchen sink" approach is unlikely to be effective as it is well known that increasing the explanatory power of the (outcome and/or selection) regressions may increase the selection bias associated with average treatment effects.

For each example we consider three experimental designs. The first two employ linear regression while the third employs nonparametric regression. The examples are not intended to fully explore the merits of nonparametric versus linear regression. A richer set of covariates would aid such a task, however we include one example (example 5) designed to yield some insight into this issue.[2]

Design one is a simple indicator variable regression supposing the DGP can be accommodated via

$$Y = \beta_{0a} + \beta_{1a} D + \beta_{2a} T + \varepsilon_{1a} \tag{1a}$$

Model suggested unconditional average treatment effects are

$$\beta_{1a},$$

the matching strategy determines whether it is ATT, $ATUT$, or ATE.

Next, we consider a design replacing covariates with indicators for propensity score intervals, say intervals above and below $\frac{1}{2}$, envisioning identification of average treatment effects conditional on propensity scores.

$$Y = \beta_{0b} + \beta_{1b} D + \beta_{2b} \Im\,(p) + \beta_{3b} \Im\,(p) \times D + \varepsilon_{1b} \tag{1b}$$

where $\Im\,(p) = 1$ for $p \geq \frac{1}{2}$ and 0 otherwise. Suggested average treatment effects conditional on propensity score are

$$\beta_{1b} \text{ for } p < \frac{1}{2}$$

[1] With full common support for T, propensity-score matching without covariates produces the same (unconditional) average treatment effect (ATE) identification results as regression design 1.

[2] Some examples illustrate limitations of our particular implementation of nonparametric regression. We focus on nonparametric regressions highly concentrated locally at each value of the covariate. Of course, nonparametric regression is highly flexible (though suffers the "curse of dimensionality") so it can emulate linear regression as well as other conditional expectations approaches.

$$\beta_{1b} + \beta_{3b} \text{ for } p \geq \frac{1}{2}.$$

and suggested unconditional average effects are

$$\beta_{1b} + \Pr\left(\Im\left(p\right) = 1 \mid D\right)\beta_{3b}$$

where $D = 1$ refers to ATT, $D = 0$ refers to $ATUT$, and $D = 0, 1$ refers to ATE. Again, the matching strategy determines whether it is ATT, $ATUT$, or ATE.

Design two is a saturated indicator variable regression supposing the DGP can be accommodated via

$$\begin{aligned} Y &= \gamma_{0a} + \gamma_{1a}D + \gamma_{2a}T + \gamma_{3a}T \times D + \gamma_{4a}\Im\left(p\right) \\ &\quad + \gamma_{5a}\Im\left(p\right) \times D + \gamma_{6a}T \times \Im\left(p\right) + \gamma_{7a}T \times \Im\left(p\right) \times D + \varepsilon_{2a} \end{aligned} \tag{2a}$$

Suggested average treatment effects conditional on propensity score below $\frac{1}{2}$ are

$$\gamma_{1a} + \gamma_{3a}E\left[T \mid D, \Im\left(p\right) = 0\right],$$

on propensity score above $\frac{1}{2}$ are

$$\gamma_{1a} + \gamma_{5a} + \left(\gamma_{3a} + \gamma_{7a}\right)E\left[T \mid D, \Im\left(p\right) = 1\right],$$

on covariates are

$$\gamma_{1a} + \gamma_{3a}T + \gamma_{5a}\Pr\left(\Im\left(p\right) = 1 \mid D, T\right) + \gamma_{7a}\Pr\left(\Im\left(p\right) = 1 \mid D, T\right)T,$$

on covariates and propensity score below $\frac{1}{2}$ are

$$\gamma_{1a} + \gamma_{3a}T,$$

and on covariates and propensity score above $\frac{1}{2}$ are

$$\gamma_{1a} + \gamma_{5a} + \left(\gamma_{3a} + \gamma_{7a}\right)T.$$

Suggested unconditional average treatment effects are

$$\gamma_{1a} + \gamma_{3a}E\left[T \mid D\right] + \gamma_{5a}\Pr\left(\Im\left(p\right) = 1 \mid D\right) + \gamma_{7a}E\left[T \times \Im\left(p\right) \mid D\right]$$

or, by iterated expectations over $\Im\left(p\right)$,

$$\begin{aligned} &\Pr\left(\Im\left(p\right) = 0 \mid D\right)\left(\gamma_{1a} + \gamma_{3a}E\left[T \mid D, \Im\left(p\right) = 0\right]\right) \\ &+ \Pr\left(\Im\left(p\right) = 1 \mid D\right)\left(\gamma_{1a} + \gamma_{5a} + \left(\gamma_{3a} + \gamma_{7a}\right)E\left[T \mid D, \Im\left(p\right) = 1\right]\right) \\ = \;& \gamma_{1a} + \gamma_{3a}E\left[T \mid D\right] + \gamma_{5a}\Pr\left(\Im\left(p\right) = 1 \mid D\right) \\ &+ \gamma_{7a}\Pr\left(\Im\left(p\right) = 1 \mid D\right)E\left[T \mid \Im\left(p\right) = 1, D\right] \end{aligned}$$

or, by iterated expectations over T,

$$\sum_{T}\Pr\left(T \mid D\right)\left(\begin{array}{c} \gamma_{1a} + \gamma_{3a}T + \gamma_{5a}\Pr\left(\Im\left(p\right) = 1 \mid D, T\right) \\ + \gamma_{7a}\Pr\left(\Im\left(p\right) = 1 \mid D, T\right)T \end{array}\right)$$

where $D = 1$ for ATT, $D = 0$ for $ATUT$, and $D = 0$ or 1 for ATE.

A reduced design (potentially) accommodates average effects conditional on covariates.

$$Y = \gamma_{0b} + \gamma_{1b} D + \gamma_{2b} T + \gamma_{3b} T \times D + \varepsilon_{2b} \tag{2b}$$

Suggested unconditional average treatment effects are envisioned as

$$\gamma_{1b} + \gamma_{3b} E\left[T \mid D\right],$$

while suggested average treatment effects conditional on the covariates are

$$\gamma_{1b} + \gamma_{3b} T.$$

Design three involves nonparametric regression where the DGP is imagined as two regimes (one with propensity score below $\frac{1}{2}$ and the other with propensity score above $\frac{1}{2}$)

$$Y = \left(1 - \Im\left(p\right)\right) \Im\left(g_D\left(T\right)\right) + \Im\left(p\right) \Im\left(g_D\left(T\right)\right) + \varepsilon_3 \tag{3}$$

where $\Im\left(g_j\left(T\right)\right)$ is an indicator function of D, that is $D = 1$ assigns $g_1\left(T\right)$, $D = 0$ assigns $g_0\left(T\right)$. Suggested average treatment effects conditional on propensity score below $\frac{1}{2}$ are, by iterated expectations,

$$\sum_T \Pr\left(T \mid D, \Im\left(p\right) = 0\right) \left[g_1\left(T\right) - g_0\left(T\right) \mid D, \Im\left(p\right) = 0\right]$$

and propensity score above $\frac{1}{2}$ are

$$\sum_T \Pr\left(T \mid D, \Im\left(p\right) = 1\right) \left[g_1\left(T\right) - g_0\left(T\right) \mid D, \Im\left(p\right) = 1\right],$$

conditional on covariates are

$$\begin{aligned} &\Pr\left(\Im\left(p\right) = 0 \mid D, T\right) \left[g_1\left(T\right) - g_0\left(T\right) \mid D, \Im\left(p\right) = 0\right] \\ &+ \Pr\left(\Im\left(p\right) = 1 \mid D, T\right) \left[g_1\left(T\right) - g_0\left(T\right) \mid D, \Im\left(p\right) = 1\right], \end{aligned}$$

conditional on both propensity score below $\frac{1}{2}$ and covariates are

$$\left[g_1\left(T\right) - g_0\left(T\right) \mid D, \Im\left(p\right) = 0\right],$$

and conditional on both propensity score above $\frac{1}{2}$ and covariates are

$$\left[g_1\left(T\right) - g_0\left(T\right) \mid D, \Im\left(p\right) = 1\right].$$

Hence, suggested unconditional average treatment effects are

$$\begin{aligned} &\Pr\left(\Im\left(p\right) = 0 \mid D\right) \sum_T \Pr\left(T \mid D, \Im\left(p\right) = 0\right) \left[g_1\left(T\right) - g_0\left(T\right) \mid D, \Im\left(p\right) = 0\right] \\ &+ \Pr\left(\Im\left(p\right) = 1 \mid D\right) \sum_T \Pr\left(T \mid D, \Im\left(p\right) = 1\right) \left[g_1\left(T\right) - g_0\left(T\right) \mid D, \Im\left(p\right) = 1\right]. \end{aligned}$$

We model this with little bias so conditional expectations are concentrated locally at each level of the covariate. That is, observed outcome, Y, is averaged at each level of the covariate. Again, the sampling strategy determines whether the average effect is ATT, $ATUT$, or ATE.

ATT matches untreated with the treated subsample, $ATUT$ matches treated with the untreated subsample, and ATE employs randomized matching. In any case, $ATE = \Pr(D=1) \times ATT + \Pr(D=0) \times ATUT$. All examples have the same $(D, P(Z))$ structure except example 5 which is completely symmetric making matching the same for both regimes $D = 0, 1$. For the other examples, the following proportions apply to the ATT subsample

$$8:4:8:4$$

for

$$\begin{array}{rcl} (D=1, P(Z)=0.6) & : & (D=1, P(Z)=0.4): \\ (D=0, P(Z)=0.6) & : & (D=0, P(Z)=0.4) \end{array}$$

and

$$4:8:4:8$$

for

$$\begin{array}{rcl} (D=1, P(Z)=0.6) & : & (D=1, P(Z)=0.4): \\ (D=0, P(Z)=0.6) & : & (D=0, P(Z)=0.4) \end{array}$$

to the $ATUT$ subsample. Focus on the ATT propensity-score matched sample. A large number of random samples preserves the proportions of draws with high and low propensity scores conditional on treatment $D = 1$ at $2:1$, or equivalently for a sample of 12, $8:4$. Then, each high propensity score individual i who adopted no treatment $(D = 0)$, in expectation, is included four times $(E[\#draws\ for\ i \mid D=0, P(Z)=0.6] = np = 8\frac{1}{2} = 4)$ and each low propensity score individual who chose no treatment is included once, in expectation, $(E[\#draws\ for\ i \mid D=0, P(Z)=0.4] = np = 4\frac{1}{4} = 1)$. The ATE sample has equal parts of the above two subsamples since $\Pr(D=1) = \Pr(D=0) = \frac{1}{2}$ for all examples.[3]

NS denotes lack of **observable** common support for the experimental design results; again, for all DGP considered this applies to $(T, P(Z))$ jointly or saturated designs $2a$ and 3. Of course, common support doesn't speak to conditional mean independence which is made even more challenging if covariates as well as outcomes are unobservable. When reporting identification results for the various designs, we indicate by NA when a treatment effect is not defined and also what extrapolation from the design (perhaps, incorrectly) suggests. Iterated expectations employ probabilities drawn from observed support T (not the DGP probabilities associated with T_0 and T_1). ND denotes parameters (particularly, for designs 2a and 3) that are not determined due to collinearity.

[3] Propensity-score matched sample construction is illustrated in example 2.

Example 1 (base case — homogeneous outcome) *Suppose the DGP is*

	D	$P(Z)$	T_1	$g_1(T)$	U_1	Y_1	T_0	$g_0(T)$	U_0	Y_0	Y	T
	1	0.6	4	44	0	44	4	−36	0	−36	44	4
	1	0.6	2	22	0	22	2	−18	0	−18	22	2
	1	0.6	4	44	0	44	4	−36	0	−36	44	4
	1	0.6	2	22	0	22	2	−18	0	−18	22	2
	1	0.4	4	44	0	44	4	−36	0	−36	44	4
	1	0.4	2	22	0	22	2	−18	0	−18	22	2
	0	0.6	4	44	0	44	4	−36	0	−36	−36	4
	0	0.6	2	22	0	22	2	−18	0	−18	−18	2
	0	0.4	4	44	0	44	4	−36	0	−36	−36	4
	0	0.4	2	22	0	22	2	−18	0	−18	−18	2
	0	0.4	4	44	0	44	4	−36	0	−36	−36	4
	0	0.4	2	22	0	22	2	−18	0	−18	−18	2
means	0.5	0.5	3	33	0	33	3	−27	0	−27	3	3

Conditional and unconditional average treatment effects for this DGP are

conditioning	$ATT(\cdot)$	$ATUT(\cdot)$	$ATE(\cdot)$
$\Im(p)=1$	$33-(-27)=60$	$33-(-27)=60$	$33-(-27)=60$
$\Im(p)=0$	$33-(-27)=60$	$33-(-27)=60$	$33-(-27)=60$
$T=2$	$22-(-18)=40$	$22-(-18)=40$	$22-(-18)=40$
$T=4$	$44-(-36)=80$	$44-(-36)=80$	$44-(-36)=80$
$\Im(p)=1, T=2$	$22-(-18)=40$	$22-(-18)=40$	$22-(-18)=40$
$\Im(p)=1, T=4$	$44-(-36)=80$	$44-(-36)=80$	$44-(-36)=80$
$\Im(p)=0, T=2$	$22-(-18)=40$	$22-(-18)=40$	$22-(-18)=40$
$\Im(p)=0, T=4$	$44-(-36)=80$	$44-(-36)=80$	$44-(-36)=80$
none	$33-(-27)=60$	$33-(-27)=60$	$33-(-27)=60$

Design one yields

$$\begin{array}{cc} Y=-30+60D+1T+\varepsilon_{1a} & ATT \text{ p-score matched sample} \\ 60 & \text{suggested } ATT \\ Y=-30+60D+1T+\varepsilon_{1a} & ATUT \text{ p-score matched sample} \\ 60 & \text{suggested } ATUT \\ Y=-30+60D+1T+\varepsilon_{1a} & ATE \text{ p-score matched sample} \\ 60 & \text{suggested } ATE \end{array} \tag{1a}$$

and

$$
\begin{array}{c|l}
Y = -27 + 60D + 0\Im(p) & \\
+0\Im(p) \times D + \varepsilon_{1b} & \textit{ATT p-score matched sample} \\
60 & \textit{suggested ATT } (p = 0.4) \\
60 + 0 = 60 & \textit{suggested ATT } (p = 0.6) \\
60 + 0\left(\frac{2}{3}\right) = 60 & \textit{suggested ATT}
\end{array}
$$

$$
\begin{array}{c|l}
Y = -27 + 60D + 0\Im(p) & \\
+0\Im(p) \times D + \varepsilon_{1b} & \textit{ATUT p-score matched sample} \\
60 & \textit{suggested ATUT } (p = 0.4) \\
60 + 0 = 60 & \textit{suggested ATUT } (p = 0.6) \\
60 + 0\left(\frac{1}{3}\right) = 60 & \textit{suggested ATUT}
\end{array} \tag{1b}
$$

$$
\begin{array}{c|l}
Y = -27 + 60D + 0\Im(p) & \\
+0\Im(p) \times D + \varepsilon_{1b} & \textit{ATE p-score matched sample} \\
60 & \textit{suggested ATE } (p = 0.4) \\
60 + 0 = 60 & \textit{suggested ATE } (p = 0.6) \\
60 + 0\left(\frac{1}{2}\right) = 60 & \textit{suggested ATE}
\end{array}
$$

where p-score refers to propensity-score $P(Z)$*. Design two yields*

$$
\begin{array}{c|l}
Y = 0 + 0D - 9T + 20T \times D & \\
+0\Im(p) + 0\Im(p) \times D & \textit{ATT p-score matched sample} \\
+0T \times \Im(p) & \\
+0T \times \Im(p) \times D + \varepsilon_{2a} & \\
0 + 20(3) = 60 & \textit{suggested ATT } (p = 0.4) \\
0 + 20(3) + 0 + 0(3) = 60 & \textit{suggested ATT } (p = 0.6) \\
0 + 20(2) + 0\left(\frac{2}{3}\right) + 0\left(2 \times \frac{2}{3}\right) = 40 & \textit{suggested ATT } (T = 2) \\
0 + 20(4) + 0\left(\frac{2}{3}\right) + 0\left(4 \times \frac{2}{3}\right) = 80 & \textit{suggested ATT } (T = 4) \\
0 + 20(2) = 40 & \textit{suggested ATT } (p = 0.4, T = 2) \\
0 + 20(2) + 0 + 0(2) = 40 & \textit{suggested ATT } (p = 0.6, T = 2) \\
0 + 20(4) = 80 & \textit{suggested ATT } (p = 0.4, T = 4) \\
0 + 20(4) + 0 + 0(4) = 80 & \textit{suggested ATT } (p = 0.6, T = 4) \\
0 + 20(3) + 0\left(\frac{2}{3}\right) + 0(2) = 60 & \textit{suggested ATT}
\end{array} \tag{2a}
$$

$$
\begin{array}{c}
Y = 0 + 0D - 9T + 20T \times D \\
+0\Im(p) + 0\Im(p) \times D \\
+0T \times \Im(p) \\
+0T \times \Im(p) \times D + \varepsilon_{2a}
\end{array}
\qquad ATUT \text{ p-score matched sample}
$$

$$
\begin{array}{cc}
0 + 20(3) = 60 & \text{suggested } ATUT\,(p = 0.4) \\
0 + 20(3) + 0 + 0(3) = 60 & \text{suggested } ATUT\,(p = 0.6) \\
0 + 20(2) + 0\left(\frac{1}{3}\right) + 0\left(2 \times \frac{1}{3}\right) = 40 & \text{suggested } ATUT\,(T = 2) \\
0 + 20(4) + 0\left(\frac{1}{3}\right) + 0\left(4 \times \frac{1}{3}\right) = 80 & \text{suggested } ATUT\,(T = 4) \\
0 + 20(2) = 40 & \text{suggested } ATUT \left(\begin{array}{c} p = 0.4, \\ T = 2 \end{array} \right) \\
0 + 20(2) + 0 + 0(2) = 40 & \text{suggested } ATUT \left(\begin{array}{c} p = 0.6, \\ T = 2 \end{array} \right) \\
0 + 20(4) = 80 & \text{suggested } ATUT \left(\begin{array}{c} p = 0.4, \\ T = 4 \end{array} \right) \\
0 + 20(4) + 0 + 0(4) = 80 & \text{suggested } ATUT \left(\begin{array}{c} p = 0.6, \\ T = 4 \end{array} \right) \\
0 + 20(3) + 0\left(\frac{1}{3}\right) + 0(1) = 60 & \text{suggested } ATUT
\end{array}
\tag{2a}
$$

$$
\begin{array}{c}
Y = 0 + 0D - 9T + 20T \times D \\
+0\Im(p) + 0\Im(p) \times D \\
+0T \times \Im(p) \\
+0T \times \Im(p) \times D + \varepsilon_{2a}
\end{array}
\qquad ATE \text{ p-score matched sample}
$$

$$
\begin{array}{cc}
0 + 20(3) = 60 & \text{suggested } ATE\,(p = 0.4) \\
0 + 20(3) + 0 + 0(3) = 60 & \text{suggested } ATE\,(p = 0.6) \\
0 + 20(2) + 0\left(\frac{1}{2}\right) + 0\left(2 \times \frac{1}{2}\right) = 40 & \text{suggested } ATE\,(T = 2) \\
0 + 20(4) + 0\left(\frac{1}{2}\right) + 0\left(4 \times \frac{1}{2}\right) = 80 & \text{suggested } ATE\,(T = 4) \\
0 + 20(2) = 40 & \text{suggested } ATE \left(\begin{array}{c} p = 0.4, \\ T = 2 \end{array} \right) \\
0 + 20(2) + 0 + 0(2) = 40 & \text{suggested } ATE \left(\begin{array}{c} p = 0.6, \\ T = 2 \end{array} \right) \\
0 + 20(4) = 80 & \text{suggested } ATE \left(\begin{array}{c} p = 0.4, \\ T = 4 \end{array} \right) \\
0 + 20(4) + 0 + 0(4) = 80 & \text{suggested } ATE \left(\begin{array}{c} p = 0.6, \\ T = 4 \end{array} \right) \\
0 + 20(3) + 0\left(\frac{1}{2}\right) + 0\left(1\frac{1}{2}\right) = 60 & \text{suggested } ATE
\end{array}
\tag{2a}
$$

and

$$
\begin{array}{cc}
Y = 0 + 0D - 9T & \\
+20T \times D + \varepsilon_{2b} & ATT \text{ p-score matched sample} \\
0 + 20\,(2) = 40 & \text{suggested } ATT\,(T = 2) \\
0 + 20\,(4) = 80 & \text{suggested } ATT\,(T = 4) \\
0 + 20\,(3) = 60 & \text{suggested } ATT \\
Y = 0 + 0D - 9T & \\
+20T \times D + \varepsilon_{2b} & ATUT \text{ p-score matched sample} \\
0 + 20\,(2) = 40 & \text{suggested } ATUT\,(T = 2) \\
0 + 20\,(4) = 80 & \text{suggested } ATUT\,(T = 4) \\
0 + 20\,(3) = 60 & \text{suggested } ATUT \\
Y = 0 + 0D - 9T & \\
+20T \times D + \varepsilon_{2b} & ATE \text{ p-score matched sample} \\
0 + 20\,(2) = 40 & \text{suggested } ATE\,(T = 2) \\
0 + 20\,(4) = 80 & \text{suggested } ATE\,(T = 4) \\
0 + 20\,(3) = 60 & \text{suggested } ATE
\end{array}
\tag{2b}
$$

Design 3 yields

$$
\begin{array}{cc}
g_1\,(T = 2, p = 0.6) & \\
-g_0\,(T = 2, p = 0.6) & ATT \text{ p-score matched sample} \\
= 22 - (-18) = 40 & \text{suggested } ATT\,(T = 2, p = 0.6) \\
g_1\,(T = 4, p = 0.6) & \\
-g_0\,(T = 4, p = 0.6) & \\
= 44 - (-36) = 80 & \text{suggested } ATT\,(T = 4, p = 0.6) \\
g_1\,(T = 2, p = 0.4) & \\
-g_0\,(T = 2, p = 0.4) & \\
= 22 - (-18) = 40 & \text{suggested } ATT\,(T = 2, p = 0.4) \\
g_1\,(T = 4, p = 0.4) & \\
-g_0\,(T = 4, p = 0.4) & \\
= 44 - (-36) = 80 & \text{suggested } ATT\,(T = 4, p = 0.4)
\end{array}
\tag{3Tp}
$$

$$
\begin{array}{cc}
\frac{1}{2}ATT\,(T = 2, p = 0.4) & \\
+\frac{1}{2}ATT\,(T = 4, p = 0.4) & ATT \text{ p-score matched sample} \\
= \frac{1}{2}\,(40) & \\
+\frac{1}{2}\,(80) = 60 & \text{suggested } ATT\,(p = 0.4) \\
\frac{1}{2}ATT\,(T = 2, p = 0.6) & \\
+\frac{1}{2}ATT\,(T = 4, p = 0.6) & \\
= \frac{1}{2}\,(40) & \\
+\frac{1}{2}\,(80) = 60 & \text{suggested } ATT\,(p = 0.6)
\end{array}
\tag{3p}
$$

$$
\begin{array}{cc}
\frac{1}{3}ATT\,(T = 2, p = 0.4) & \\
+\frac{2}{3}ATT\,(T = 2, p = 0.6) & ATT \text{ p-score matched sample} \\
= \frac{1}{3}\,(40) + \frac{2}{3}\,(40) = 40 & \text{suggested } ATT\,(T = 2) \\
\frac{1}{3}ATT\,(T = 4, p = 0.4) & \\
+\frac{2}{3}ATT\,(T = 4, p = 0.6) & \\
= \frac{1}{3}\,(80) + \frac{2}{3}\,(80) = 80 & \text{suggested } ATT\,(T = 4)
\end{array}
\tag{3T}
$$

$$\begin{array}{c} \frac{1}{2}ATT\left(T=2\right) \\ +\frac{1}{2}ATT\left(T=4\right) \\ =\frac{1}{2}\left(40\right)+\frac{1}{2}\left(80\right)=60 \end{array} \quad \begin{array}{c} ATT\ p\text{-}score\ matched\ sample \\ suggested\ ATT \end{array} \tag{3}$$

$$\begin{array}{c} g_1\left(T=2,p=0.6\right) \\ -g_0\left(T=2,p=0.6\right) \\ =22-\left(-18\right)=40 \\ g_1\left(T=4,p=0.6\right) \\ -g_0\left(T=4,p=0.6\right) \\ =44-\left(-36\right)=80 \\ g_1\left(T=2,p=0.4\right) \\ -g_0\left(T=2,p=0.4\right) \\ =22-\left(-18\right)=40 \\ g_1\left(T=4,p=0.4\right) \\ -g_0\left(T=4,p=0.4\right) \\ =44-\left(-36\right)=80 \end{array} \quad \begin{array}{c} ATUT\ p\text{-}score\ matched\ sample \\ suggested\ ATUT\left(T=2,p=0.6\right) \\ suggested\ ATUT\left(T=4,p=0.6\right) \\ suggested\ ATUT\left(T=2,p=0.4\right) \\ suggested\ ATUT\left(T=4,p=0.4\right) \end{array} \tag{3Tp}$$

$$\begin{array}{c} \frac{1}{2}ATUT\left(T=2,p=0.4\right) \\ +\frac{1}{2}ATUT\left(T=4,p=0.4\right) \\ =\frac{1}{2}\left(40\right) \\ +\frac{1}{2}\left(80\right)=60 \\ \frac{1}{2}ATUT\left(T=2,p=0.6\right) \\ +\frac{1}{2}ATUT\left(T=4,p=0.6\right) \\ =\frac{1}{2}\left(40\right) \\ +\frac{1}{2}\left(80\right)=60 \end{array} \quad \begin{array}{c} ATUT\ p\text{-}score\ matched\ sample \\ suggested\ ATUT\left(p=0.4\right) \\ suggested\ ATUT\left(p=0.6\right) \end{array} \tag{3p}$$

$$\begin{array}{c} \frac{2}{3}ATUT\left(T=2,p=0.4\right) \\ +\frac{1}{3}ATUT\left(T=2,p=0.6\right) \\ =\frac{2}{3}\left(40\right)+\frac{1}{3}\left(40\right)=40 \\ \frac{2}{3}ATUT\left(T=4,p=0.4\right) \\ +\frac{1}{3}ATUT\left(T=4,p=0.6\right) \\ =\frac{2}{3}\left(80\right)+\frac{1}{3}\left(80\right)=80 \end{array} \quad \begin{array}{c} ATUT\ p\text{-}score\ matched\ sample \\ suggested\ ATUT\left(T=2\right) \\ suggested\ ATUT\left(T=4\right) \end{array} \tag{3T}$$

$$\begin{array}{c} \frac{1}{2}ATUT\left(T=2\right) \\ +\frac{1}{2}ATUT\left(T=4\right) \\ =\frac{1}{2}\left(40\right)+\frac{1}{2}\left(80\right)=60 \end{array} \quad \begin{array}{c} ATUT\ p\text{-}score\ matched\ sample \\ suggested\ ATUT \end{array} \tag{3}$$

and

$$
\begin{array}{ll}
g_1\left(T=2,p=0.6\right) & \\
-g_0\left(T=2,p=0.6\right) & \textit{ATE p-score matched sample} \\
=22-(-18)=40 & \textit{suggested ATE}\left(T=2,p=0.6\right) \\
g_1\left(T=4,p=0.6\right) & \\
-g_0\left(T=4,p=0.6\right) & \\
=44-(-36)=80 & \textit{suggested ATE}\left(T=4,p=0.6\right) \\
g_1\left(T=2,p=0.4\right) & \\
-g_0\left(T=2,p=0.4\right) & \\
=22-(-18)=40 & \textit{suggested ATE}\left(T=2,p=0.4\right) \\
g_1\left(T=4,p=0.4\right) & \\
-g_0\left(T=4,p=0.4\right) & \\
=44-(-36)=80 & \textit{suggested ATE}\left(T=4,p=0.4\right)
\end{array}
\tag{3Tp}
$$

$$
\begin{array}{cc}
\frac{1}{2}ATE\left(T=2,p=0.4\right) & \textit{ATE p-score matched sample} \\
+\frac{1}{2}ATE\left(T=4,p=0.4\right) & \\
=\frac{1}{2}\left(40\right) & \\
+\frac{1}{2}\left(80\right)=60 & \textit{suggested ATE}\left(p=0.4\right) \\
\frac{1}{2}ATE\left(T=2,p=0.6\right) & \\
+\frac{1}{2}ATE\left(T=4,p=0.6\right) & \\
=\frac{1}{2}\left(40\right) & \\
+\frac{1}{2}\left(80\right)=60 & \textit{suggested ATE}\left(p=0.6\right)
\end{array}
\tag{3p}
$$

$$
\begin{array}{cc}
\frac{1}{2}ATE\left(T=2,p=0.4\right) & \textit{ATE p-score matched sample} \\
+\frac{1}{2}ATE\left(T=2,p=0.6\right) & \\
=\frac{1}{2}\left(40\right)+\frac{1}{2}\left(40\right)=40 & \textit{suggested ATE}\left(T=2\right) \\
\frac{1}{2}ATE\left(T=4,p=0.4\right) & \\
+\frac{1}{2}ATE\left(T=4,p=0.6\right) & \\
=\frac{1}{2}\left(80\right)+\frac{1}{2}\left(80\right)=80 & \textit{suggested ATE}\left(T=4\right)
\end{array}
\tag{3T}
$$

$$
\begin{array}{cc}
\frac{1}{2}ATE\left(T=2\right) & \textit{ATE p-score matched sample} \\
+\frac{1}{2}ATE\left(T=4\right) & \\
=\frac{1}{2}\left(40\right)+\frac{1}{2}\left(80\right)=60 & \textit{suggested ATE}
\end{array}
\tag{3}
$$

All designs effectively identify the intended average treatment effects as all variations of conditional mean independence are satisfied along with full common support. This is a prototype setting for linear regression on propensity-score matched samples that accommodates relatively simple designs for identifying all manner of conditional and unconditional average treatment effects. Next, we modify the DGP slightly by adding $T=3$ to the mix but with incomplete support for identifying all conditional average treatment effects.

Example 2 (*DGP* with incomplete covariate support) *Suppose the DGP is*

	D	$P(Z)$	T_1	$g_1(T)$	U_1	Y_1	T_0	$g_0(T)$	U_0	Y_0	Y	T
	1	0.6	4	44	0	44	4	−31	5	−26	44	4
	1	0.6	2	22	0	22	2	−23	−5	−28	22	2
	1	0.6	3	33	0	33	3	−27	−10	−37	33	3
	1	0.6	3	33	0	33	3	−27	−10	−17	33	3
	1	0.4	4	44	0	44	4	−31	−5	−36	44	4
	1	0.4	2	22	0	22	2	−23	5	−18	22	2
	0	0.6	4	44	0	44	3	−27	−10	−37	−37	3
	0	0.6	2	22	0	22	3	−27	−10	−17	−17	3
	0	0.4	3	33	0	33	4	−31	5	−26	−26	4
	0	0.4	2	22	0	22	2	−23	−5	−28	−28	2
	0	0.4	4	44	0	44	4	−31	−5	−36	−36	4
	0	0.4	2	22	0	22	2	−23	5	−18	−18	2
means	0.5	0.5	3	33	0	33	3	−27	0	−27	3	3

Conditional and unconditional average treatment effects for this DGP are

conditioning	$ATT(\cdot)$	$ATUT(\cdot)$	$ATE(\cdot)$
$\Im(p)=1$	$33-(-27)=60$	$33-(-27)=60$	$33-(-27)=60$
$\Im(p)=0$	$33-(-27)=60$	$33-(-27)=60$	$33-(-27)=60$
$T=2$	$22-(-23)=45$	$22-(-23)=45$	$22-(-23)=45$
$T=3$	$33-(-27)=60$	$33-(-27)=60$	$33-(-27)=60$
$T=4$	$44-(-31)=75$	$44-(-31)=75$	$44-(-31)=75$
$\Im(p)=1, T=2$	$22-(-28)=50$	$22-NA=NA$	$22-(-28)=50$
$\Im(p)=1, T=3$	$33-(-27)=60$	$NA-(-27)=NA$	$33-(-27)=60$
$\Im(p)=1, T=4$	$44-(-26)=70$	$44-NA=NA$	$44-(-26)=70$
$\Im(p)=0, T=2$	$22-(-18)=40$	$22-(-23)=45$	$22-\left(-21\frac{1}{3}\right)=43\frac{1}{3}$
$\Im(p)=0, T=3$	$NA-NA=NA$	$33-NA=NA$	$33-NA=NA$
$\Im(p)=0, T=4$	$44-(-36)=80$	$44-(-31)=75$	$44-\left(-32\frac{2}{3}\right)=76\frac{2}{3}$
none	$33-(-27)=60$	$33-(-27)=60$	$33-(-27)=60$

Not all conditional average treatment effects are defined for this DGP. NA refers to conditions in which treatment effects are not defined for the DGP. Unconditional mean effects are averaged over the regions of common support hence excluding any region designated NA. Scrutiny of the sample partially reveals where common support is lacking so even though the DGP is unknown a prudent analyst avoids overstating average treatment effect results. In this case, observable covariate $T=D\times T_1+(1-D)\times T_0$ suggests

$$(T=3, P(Z)=0.4),$$

$$(T=2, P(Z)=0.6)$$

and

$$(T=4, P(Z)=0.6)$$

lack common support but this overstates the case as

$$ATT\,(T=2, P\,(Z)=0.6)$$

and

$$ATT\,(T=4, P\,(Z)=0.6)$$

are defined and also understates the case as

$$ATUT\,(T=3, P\,(Z)=0.6)$$

is not defined. For clarity, we report the propensity-score matched samples for this example.

ATT sample				*ATUT sample*			
D	$P\,(Z)$	Y	T	D	$P\,(Z)$	Y	T
1	0.6	44	4	1	0.6	44	4
1	0.6	22	2	1	0.6	22	2
1	0.6	33	3	1	0.6	33	3
1	0.6	33	3	1	0.6	33	3
1	0.4	44	4	1	0.4	44	4
1	0.4	22	2	1	0.4	22	2
1	0.6	44	4	1	0.4	44	4
1	0.6	22	2	1	0.4	22	2
1	0.6	33	3	1	0.4	44	4
1	0.6	33	3	1	0.4	22	2
1	0.4	44	4	1	0.4	44	4
1	0.4	22	2	1	0.4	22	2
0	0.6	−37	3	0	0.6	−37	3
0	0.6	−17	3	0	0.6	−17	3
0	0.6	−37	3	0	0.4	−26	4
0	0.6	−17	3	0	0.4	−28	2
0	0.6	−37	3	0	0.4	−36	4
0	0.6	−17	3	0	0.4	−18	2
0	0.6	−37	3	0	0.6	−37	3
0	0.6	−17	3	0	0.6	−17	3
0	0.4	−26	4	0	0.4	−26	4
0	0.4	−28	2	0	0.4	−28	2
0	0.4	−36	4	0	0.4	−36	4
0	0.4	−18	2	0	0.4	−18	2

Design one yields

$$\begin{array}{ll} Y = -45 + 60D + 6T + \varepsilon_{1a} & \textit{ATT p-score matched sample} \\ 60 & \textit{suggested ATT} \\ Y = -40 + 60D + 4\tfrac{1}{3}T + \varepsilon_{1a} & \textit{ATUT p-score matched sample} \\ 60 & \textit{suggested ATUT} \\ Y = -42 + 60D + 5T + \varepsilon_{1a} & \textit{ATE p-score matched sample} \\ 60 & \textit{suggested ATE} \end{array} \tag{1a}$$

and

$$
\begin{array}{cc}
Y = -27 + 60D + 0\Im(p) & \textit{ATT p-score matched sample} \\
+0\Im(p) \times D + \varepsilon_{1b} & \\
60 & \textit{suggested ATT}\,(p = 0.4) \\
60 + 0 = 60 & \textit{suggested ATT}\,(p = 0.6) \\
60 + 0\left(\frac{2}{3}\right) = 60 & \textit{suggested ATT}
\end{array}
$$

$$
\begin{array}{cc}
Y = -27 + 60D + 0\Im(p) & \textit{ATUT p-score matched sample} \\
+0\Im(p) \times D + \varepsilon_{1b} & \\
60 & \textit{suggested ATUT}\,(p = 0.4) \\
60 + 0 = 60 & \textit{suggested ATUT}\,(p = 0.6) \\
60 + 0\left(\frac{1}{3}\right) = 60 & \textit{suggested ATUT}
\end{array}
\tag{1b}
$$

$$
\begin{array}{cc}
Y = -27 + 60D + 0\Im(p) & \textit{ATE p-score matched sample} \\
+0\Im(p) \times D + \varepsilon_{1b} & \\
60 & \textit{suggested ATE}\,(p = 0.4) \\
60 + 0 = 60 & \textit{suggested ATE}\,(p = 0.6) \\
60 + 0\left(\frac{1}{2}\right) = 60 & \textit{suggested ATE}
\end{array}
$$

Design two yields

$$
\begin{array}{cc}
Y = -15 + 15D - 4T + 15T \times D & \\
+0\Im(p) + 0\Im(p) \times D & \\
+0T \times \Im(p) & \textit{ATT p-score matched sample} \\
+(ND)\, T \times \Im(p) \times D + \varepsilon_{2a} & \\
15 + 15(3) = 60 & \textit{suggested ATT } (p = 0.4) \\
15 + 15(3) + 0 + (ND)(3) = 60 & \textit{suggested ATT } (p = 0.6) \\
15 + 15(2) + 0\left(\frac{1}{2}\right) & \\
+(ND)\left(2 \times \frac{1}{2}\right) = 45 & \textit{suggested ATT } (T = 2) \\
15 + 15(3) + 0(1) & \\
+(ND)(3 \times 1) = 60 & \textit{suggested ATT } (T = 3) \\
15 + 15(4) + 0\left(\frac{1}{2}\right) & \\
+(ND)\left(4 \times \frac{1}{2}\right) = 75 & \textit{suggested ATT } (T = 4) \\
15 + 15(2) = 45 & \textit{suggested ATT } (p = 0.4, T = 2) \\
15 + 15(2) + 0 & \\
+(ND)(2) = 45\,(NS) & \textit{suggested ATT } (p = 0.6, T = 2) \\
15 + 15(3) = 60\,(NA)\,(NS) & \textit{suggested ATT } (p = 0.4, T = 3) \\
15 + 15(3) + 0 & \\
+(ND)(3) = 60 & \textit{suggested ATT } (p = 0.6, T = 3) \\
15 + 15(4) = 75 & \textit{suggested ATT } (p = 0.4, T = 4) \\
15 + 15(4) + 0 & \\
+(ND)(4) = 75\,(NS) & \textit{suggested ATT } (p = 0.6, T = 4) \\
15 + 15(3) + 0\left(\frac{2}{3}\right) & \\
+(ND)(2) = 60 & \textit{suggested ATT}
\end{array}
\tag{2a}
$$

$$
\begin{array}{c}
Y = -15 + 15D - 4T + 15T \times D \\
+0\Im(p) + 0\Im(p) \times D \\
+0T \times \Im(p) \\
+(ND)\,T \times \Im(p) \times D + \varepsilon_{2a}
\end{array}
\qquad \textit{ATUT p-score matched sample}
$$

$$
15 + 15(3) = 60 \qquad \textit{suggested ATUT } (p = 0.4)
$$

$$
\begin{array}{c}
15 + 15(3) + 0 \\
+(ND)(3) = 60
\end{array}
\qquad \textit{suggested ATUT } (p = 0.6)
$$

$$
\begin{array}{c}
15 + 15(2) + 0(0) \\
+(ND)(2 \times 0) = 45
\end{array}
\qquad \textit{suggested ATUT } (T = 2)
$$

$$
\begin{array}{c}
15 + 15(3) + 0(1) \\
+(ND)(3 \times 1) = 60
\end{array}
\qquad \textit{suggested ATUT } (T = 3)
$$

$$
\begin{array}{c}
15 + 15(4) + 0(0) \\
+(ND)(4 \times 0) = 75
\end{array}
\qquad \textit{suggested ATUT } (T = 4)
$$

$$
15 + 15(2) = 45 \qquad \textit{suggested ATUT} \left(\begin{array}{c} p = 0.4, \\ T = 2 \end{array} \right) \qquad (2a)
$$

$$
\begin{array}{c}
15 + 15(2) + 0 + (ND)(2) \\
= 45\,(NA)\,(NS)
\end{array}
\qquad \textit{suggested ATUT} \left(\begin{array}{c} p = 0.6, \\ T = 2 \end{array} \right)
$$

$$
15 + 15(3) = 60\,(NA)\,(NS) \qquad \textit{suggested ATUT} \left(\begin{array}{c} p = 0.4, \\ T = 3 \end{array} \right)
$$

$$
\begin{array}{c}
15 + 15(3) + 0 + (ND)(3) \\
= 60\,(NA)
\end{array}
\qquad \textit{suggested ATUT} \left(\begin{array}{c} p = 0.6, \\ T = 3 \end{array} \right)
$$

$$
15 + 15(4) = 75 \qquad \textit{suggested ATUT} \left(\begin{array}{c} p = 0.4, \\ T = 4 \end{array} \right)
$$

$$
\begin{array}{c}
15 + 15(4) + 0 + (ND)(4) \\
= 75\,(NA)\,(NS)
\end{array}
\qquad \textit{suggested ATUT} \left(\begin{array}{c} p = 0.6, \\ T = 4 \end{array} \right)
$$

$$
\begin{array}{c}
15 + 15(3) + 0\left(\frac{1}{3}\right) \\
+(ND)(1) = 60
\end{array}
\qquad \textit{suggested ATUT}
$$

$$
\begin{array}{cc}
Y = -15 + 15D - 4T + 15T \times D & \\
+0\Im(p) + 0\Im(p) \times D & \\
+0T \times \Im(p) & ATE \text{ p-score matched sample} \\
+(ND)\, T \times \Im(p) \times D + \varepsilon_{2a} & \\
15 + 15(3) = 60 & \text{suggested } ATE\,(p = 0.4) \\
15 + 15(3) + 0 + (ND)(3) & \\
= 60 & \text{suggested } ATE\,(p = 0.6) \\
15 + 15(2) + 0\left(\frac{1}{4}\right) & \\
+(ND)\left(2 \times \frac{1}{4}\right) = 45 & \text{suggested } ATE\,(T = 2) \\
15 + 15(3) + 0(1) & \\
+(ND)(3 \times 1) = 60 & \text{suggested } ATE\,(T = 3) \\
15 + 15(4) + 0\left(\frac{1}{4}\right) & \\
+(ND)\left(4 \times \frac{1}{4}\right) = 75 & \text{suggested } ATE\,(T = 4) \\
15 + 15(2) = 45 & \text{suggested } ATE \left(\begin{array}{c} p = 0.4, \\ T = 2 \end{array} \right) \\
15 + 15(2) + 0 & \\
+(ND)(2) = 45\,(NS) & \text{suggested } ATE \left(\begin{array}{c} p = 0.6, \\ T = 2 \end{array} \right) \\
15 + 15(3) = 60\,(NA)\,(NS) & \text{suggested } ATE \left(\begin{array}{c} p = 0.4, \\ T = 3 \end{array} \right) \\
15 + 15(3) + 0 & \\
+(ND)(3) = 60 & \text{suggested } ATE \left(\begin{array}{c} p = 0.6, \\ T = 3 \end{array} \right) \\
15 + 15(4) = 75 & \text{suggested } ATE \left(\begin{array}{c} p = 0.4, \\ T = 4 \end{array} \right) \\
15 + 15(4) + 0 & \\
+(ND)(4) = 75\,(NS) & \text{suggested } ATE \left(\begin{array}{c} p = 0.6, \\ T = 4 \end{array} \right) \\
15 + 15(3) + 0\left(\frac{1}{2}\right) & \\
+(ND)\left(1\frac{1}{2}\right) = 60 & \text{suggested } ATE
\end{array}
\tag{2a}
$$

and

$$
\begin{array}{ll}
Y = -15 + 15D - 4T & \\
\quad +15T \times D + \varepsilon_{2b} & \textit{ATT p-score matched sample} \\
\quad 15 + 15\,(2) = 45 & \textit{suggested ATT}\,(T = 2) \\
\quad 15 + 15\,(3) = 60 & \textit{suggested ATT}\,(T = 3) \\
\quad 15 + 15\,(4) = 75 & \textit{suggested ATT}\,(T = 4) \\
\quad 15 + 15\,(3) = 60 & \textit{suggested ATT} \\
Y = -15 + 15D - 4T & \textit{ATUT p-score matched sample} \\
\quad +15T \times D + \varepsilon_{2b} & \\
\quad 15 + 15\,(2) = 45 & \textit{suggested ATUT}\,(T = 2) \\
\quad 15 + 15\,(3) = 60 & \textit{suggested ATUT}\,(T = 3) \\
\quad 15 + 15\,(4) = 75 & \textit{suggested ATUT}\,(T = 4) \\
\quad 15 + 15\,(3) = 60 & \textit{suggested ATUT} \\
Y = -15 + 15D - 4T & \textit{ATE p-score matched sample} \\
\quad +15T \times D + \varepsilon_{2b} & \\
\quad 15 + 15\,(2) = 45 & \textit{suggested ATE}\,(T = 2) \\
\quad 15 + 15\,(3) = 60 & \textit{suggested ATE}\,(T = 3) \\
\quad 15 + 15\,(4) = 75 & \textit{suggested ATE}\,(T = 4) \\
\quad 15 + 15\,(3) = 60 & \textit{suggested ATE}
\end{array}
\tag{2b}
$$

Design 3 yields

$$
\begin{array}{ll}
g_1\,(T = 2, p = 0.6)\,(NS) & \\
-g_0\,(T = 2, p = 0.6)\,(NS) & \textit{ATT p-score matched sample} \\
\quad = 22 - (-23) = 45 & \textit{suggested ATT}\,(T = 2, p = 0.6) \\
\quad g_1\,(T = 3, p = 0.6) & \\
\quad -g_0\,(T = 3, p = 0.6) & \\
\quad = 33 - (-27) = 60 & \textit{suggested ATT}\,(T = 3, p = 0.6) \\
g_1\,(T = 4, p = 0.6)\,(NS) & \\
-g_0\,(T = 4, p = 0.6)\,(NS) & \\
\quad = 44 - (-31) = 75 & \textit{suggested ATT}\,(T = 4, p = 0.6) \\
\quad g_1\,(T = 2, p = 0.4) & \\
\quad -g_0\,(T = 2, p = 0.4) & \\
\quad = 22 - (-23) = 45 & \textit{suggested ATT}\,(T = 2, p = 0.4) \\
g_1\,(T = 3, p = 0.4)\,(NS) & \\
-g_0\,(T = 3, p = 0.4)\,(NS) & \\
\quad = NA - NA = NA & \textit{suggested ATT}\,(T = 3, p = 0.4) \\
\quad g_1\,(T = 4, p = 0.4) & \\
\quad -g_0\,(T = 4, p = 0.4) & \\
\quad = 44 - (-31) = 75 & \textit{suggested ATT}\,(T = 4, p = 0.4)
\end{array}
\tag{3Tp}
$$

$$
\begin{array}{cl}
\frac{1}{2}ATT\left(T=2,p=0.4\right) & \\
+0ATT\left(T=3,p=0.4\right)\left(NS\right) & \textit{ATT p-score matched sample} \\
+\frac{1}{2}ATT\left(T=4,p=0.4\right) & \\
=\frac{1}{2}\left(45\right)+0\left(NA\right) & \\
+\frac{1}{2}\left(75\right)=60 & \textit{suggested ATT}\left(p=0.4\right) \\
\frac{1}{4}ATT\left(T=2,p=0.6\right)\left(NS\right) & \\
+\frac{1}{2}ATT\left(T=3,p=0.6\right) & \\
+\frac{1}{4}ATT\left(T=4,p=0.6\right)\left(NS\right) & \\
=\frac{1}{4}\left(45\right)+\frac{1}{2}\left(60\right) & \\
+\frac{1}{4}\left(75\right)=60 & \textit{suggested ATT}\left(p=0.6\right)
\end{array}
\tag{3p}
$$

$$
\begin{array}{cl}
\frac{1}{2}ATT\left(T=2,p=0.4\right) & \\
+\frac{1}{2}ATT\left(T=2,p=0.6\right)\left(NS\right) & \textit{ATT p-score matched sample} \\
=\frac{1}{2}\left(45\right)+\frac{1}{2}\left(45\right)=45 & \textit{suggested ATT}\left(T=2\right) \\
0ATT\left(T=3,p=0.4\right)\left(NS\right) & \\
+1ATT\left(T=3,p=0.6\right) & \\
=0\left(NA\right)+1\left(60\right)=60 & \textit{suggested ATT}\left(T=3\right) \\
\frac{1}{2}ATT\left(T=4,p=0.4\right) & \\
+\frac{1}{2}ATT\left(T=4,p=0.6\right)\left(NS\right) & \\
=\frac{1}{2}\left(75\right)+\frac{1}{2}\left(75\right)=75 & \textit{suggested ATT}\left(T=4\right)
\end{array}
\tag{3T}
$$

$$
\begin{array}{cl}
\frac{1}{3}ATT\left(T=2\right) & \\
+\frac{1}{3}ATT\left(T=3\right) & \textit{ATT p-score matched sample} \\
+\frac{1}{3}ATT\left(T=4\right) & \\
=\frac{1}{3}\left(45\right)+\frac{1}{3}\left(60\right)+\frac{1}{3}\left(75\right)=60 & \textit{suggested ATT}
\end{array}
\tag{3}
$$

$$
\begin{array}{cl}
g_1\left(T=2,p=0.6\right)\left(NS\right) & \\
-g_0\left(T=2,p=0.6\right)\left(NS\right) & \textit{ATUT p-score matched sample} \\
=NA-NA=NA & \textit{suggested ATUT}\left(T=2,p=0.6\right) \\
g_1\left(T=3,p=0.6\right) & \\
-g_0\left(T=3,p=0.6\right) & \\
=33-\left(-27\right)=60\left(NA\right) & \textit{suggested ATUT}\left(T=3,p=0.6\right) \\
g_1\left(T=4,p=0.6\right)\left(NS\right) & \\
-g_0\left(T=4,p=0.6\right)\left(NS\right) & \\
=NA-NA=NA & \textit{suggested ATUT}\left(T=4,p=0.6\right) \\
g_1\left(T=2,p=0.4\right) & \\
-g_0\left(T=2,p=0.4\right) & \\
=22-\left(-23\right)=45 & \textit{suggested ATUT}\left(T=2,p=0.4\right) \\
g_1\left(T=3,p=0.4\right)\left(NS\right) & \\
-g_0\left(T=3,p=0.4\right)\left(NS\right) & \\
=NA-NA=NA & \textit{suggested ATUT}\left(T=3,p=0.4\right) \\
g_1\left(T=4,p=0.4\right) & \\
-g_0\left(T=4,p=0.4\right) & \\
=44-\left(-31\right)=75 & \textit{suggested ATUT}\left(T=4,p=0.4\right)
\end{array}
\tag{3Tp}
$$

$$
\begin{array}{c}
\frac{1}{2}ATUT\,(T=2,p=0.4) \\
+0ATUT\,(T=3,p=0.4)\,(NS) \\
+\frac{1}{2}ATUT\,(T=4,p=0.4) \\
=\frac{1}{2}\,(45)+0\,(NA) \\
+\frac{1}{2}\,(75)=60 \\
0ATUT\,(T=2,p=0.6)\,(NS) \\
+1ATUT\,(T=3,p=0.6) \\
+0ATUT\,(T=4,p=0.6)\,(NS) \\
=0\,(NA)+1\,(60) \\
+0\,(NA)=60\,(NA)
\end{array}
\begin{array}{c}
ATUT\ p\text{-}score\ matched\ sample \\
\\
\\
suggested\ ATUT\,(p=0.4) \\
\\
\\
\\
\\
suggested\ ATUT\,(p=0.6)
\end{array}
\tag{3p}
$$

$$
\begin{array}{c}
1ATUT\,(T=2,p=0.4) \\
+0ATUT\,(T=2,p=0.6)\,(NS) \\
=1\,(45)+0\,(NA)=45 \\
0ATUT\,(T=3,p=0.4)\,(NS) \\
+1ATUT\,(T=3,p=0.6) \\
=0\,(NA)+1\,(60)=60\,(NA) \\
1ATUT\,(T=4,p=0.4) \\
+0ATUT\,(T=4,p=0.6)\,(NS) \\
=1\,(75)+0\,(NA)=75
\end{array}
\begin{array}{c}
ATUT\ p\text{-}score\ matched\ sample \\
\\
suggested\ ATUT\,(T=2) \\
\\
\\
suggested\ ATUT\,(T=3) \\
\\
\\
suggested\ ATUT\,(T=4)
\end{array}
\tag{3T}
$$

$$
\begin{array}{c}
\frac{1}{3}ATUT\,(T=2) \\
+\frac{1}{3}ATUT\,(T=3) \\
+\frac{1}{3}ATUT\,(T=4) \\
=\frac{1}{3}\,(45)+\frac{1}{3}\,(60)+\frac{1}{3}\,(75)=60
\end{array}
\begin{array}{c}
ATUT\ p\text{-}score\ matched\ sample \\
\\
\\
suggested\ ATUT
\end{array}
\tag{3}
$$

and

$$
\begin{array}{c}
g_1\,(T=2,p=0.6)\,(NS) \\
-g_0\,(T=2,p=0.6)\,(NS) \\
=22-(-23)=45 \\
g_1\,(T=3,p=0.6) \\
-g_0\,(T=3,p=0.6) \\
=33-(-27)=60 \\
g_1\,(T=4,p=0.6)\,(NS) \\
-g_0\,(T=4,p=0.6)\,(NS) \\
=44-(-31)=75 \\
g_1\,(T=2,p=0.4) \\
-g_0\,(T=2,p=0.4) \\
=22-(-23)=45 \\
g_1\,(T=3,p=0.4)\,(NS) \\
-g_0\,(T=3,p=0.4)\,(NS) \\
=NA-NA=NA \\
g_1\,(T=4,p=0.4) \\
-g_0\,(T=4,p=0.4) \\
=44-(-31)=75
\end{array}
\begin{array}{c}
ATE\ p\text{-}score\ matched\ sample \\
\\
suggested\ ATE\,(T=2,p=0.6) \\
\\
\\
suggested\ ATE\,(T=3,p=0.6) \\
\\
\\
suggested\ ATE\,(T=4,p=0.6) \\
\\
\\
suggested\ ATE\,(T=2,p=0.4) \\
\\
\\
suggested\ ATE\,(T=3,p=0.4) \\
\\
\\
suggested\ ATE\,(T=4,p=0.4)
\end{array}
\tag{3Tp}
$$

$$
\begin{array}{cl}
\frac{1}{2}ATE\,(T=2,p=0.4) & \\
+0ATE\,(T=3,p=0.4)\,(NS) & ATE\ p\text{-}score\ matched\ sample \\
+\frac{1}{2}ATE\,(T=4,p=0.4) & \\
=\frac{1}{2}\,(45)+0\,(NA) & \\
+\frac{1}{2}\,(75)=60 & suggested\ ATE\,(p=0.4) \\
\frac{1}{6}ATE\,(T=2,p=0.6)\,(NS) & \\
+\frac{2}{3}ATE\,(T=3,p=0.6) & \\
+\frac{1}{6}ATE\,(T=4,p=0.6)\,(NS) & \\
=\frac{1}{6}\,(45)+\frac{2}{3}\,(60) & \\
+\frac{1}{6}\,(75)=60 & suggested\ ATE\,(p=0.6)
\end{array}
\tag{3p}
$$

$$
\begin{array}{cl}
\frac{3}{4}ATE\,(T=2,p=0.4) & \\
+\frac{1}{4}ATE\,(T=2,p=0.6)\,(NS) & ATE\ p\text{-}score\ matched\ sample \\
=\frac{3}{4}\,(45)+\frac{1}{4}\,(45)=45 & suggested\ ATE\,(T=2) \\
0ATE\,(T=3,p=0.4)\,(NS) & \\
+1ATE\,(T=3,p=0.6) & \\
=0\,(NA)+1\,(60)=60 & suggested\ ATE\,(T=3) \\
\frac{3}{4}ATE\,(T=4,p=0.4) & \\
+\frac{1}{4}ATE\,(T=4,p=0.6)\,(NS) & \\
=\frac{3}{4}\,(75)+\frac{1}{4}\,(75)=75 & suggested\ ATE\,(T=4)
\end{array}
\tag{3T}
$$

$$
\begin{array}{cl}
\frac{1}{3}ATE\,(T=2) & \\
+\frac{1}{3}ATE\,(T=3) & ATE\ p\text{-}score\ matched\ sample \\
+\frac{1}{3}ATE\,(T=4) & \\
=\frac{1}{3}\,(45)+\frac{1}{3}\,(60)+\frac{1}{3}\,(75)=60 & suggested\ ATE
\end{array}
\tag{3}
$$

Since conditional mean independence is satisfied only for the propensity score, unconditional average treatment effects and average effects conditional on propensity score are identified. However, full support (both for the DGP and observable T) for (T,p) conditions is lacking and conditional mean independence is not satisfied for all regions of common (T,p) support. Dropping observations for which observable (T,p) common support is lacking produces the same identification results for saturated designs 2a and 3.

Example 3 (heterogeneous *DGP* with incomplete support) *Suppose the DGP is*

	D	$P(Z)$	T_1	$g_1(T)$	U_1	Y_1	T_0	$g_0(T)$	U_0	Y_0	Y	T
	1	0.6	4	19	25	44	4	-11	-25	-36	44	4
	1	0.6	2	17	-15	2	2	-13	15	2	2	2
	1	0.6	5	20	15	35	5	-10	-15	-25	35	5
	1	0.6	1	16	-5	11	1	-14	5	-9	11	1
	1	0.4	3	18	-5	13	3	-12	5	-7	13	3
	1	0.4	3	18	-5	13	3	-12	5	-7	13	3
	0	0.6	3	18	5	23	3	-12	-5	-17	-17	3
	0	0.6	3	18	5	23	3	-12	-5	-17	-17	3
	0	0.4	1	16	15	31	1	-14	-15	-29	-29	1
	0	0.4	5	20	-25	-5	5	-10	25	15	15	5
	0	0.4	2	17	-15	2	2	-13	15	2	2	2
	0	0.4	4	19	5	24	4	-11	-5	-16	-16	4
means	0.5	0.5	3	18	0	18	3	-12	0	-12	$4\frac{2}{3}$	3

Even though $T_0 = T_1 = T$ there is no common support for (T, p) conditions. Hence, any inferences drawn for (T, p) conditions involve extrapolation beyond the data. Also, experimental designs based on observable common support exclude saturated designs 2*a and* 3*. Conditional and unconditional average treatment effects for this DGP are*

conditioning	$ATT(\cdot)$	$ATUT(\cdot)$	$ATE(\cdot)$
$\Im(p) = 1$	$23 - (-17) = 40$	$23 - (-17) = 40$	$23 - (-17) = 40$
$\Im(p) = 0$	$13 - (-7) = 20$	$13 - (-7) = 20$	$13 - (-7) = 20$
$T = 1$	$11 - (-9) = 20$	$31 - (-29) = 60$	$21 - (-19) = 40$
$T = 2$	$2 - 2 = 0$	$2 - 2 = 0$	$2 - 2 = 0$
$T = 3$	$13 - (-7) = 20$	$23 - (-17) = 40$	$18 - (-12) = 30$
$T = 4$	$44 - (-36) = 80$	$24 - (-16) = 40$	$34 - (-26) = 60$
$T = 5$	$35 - (-25) = 60$	$-5 - 15 = -20$	$15 - (-5) = 20$
$\Im(p) = 1, T = 1$	$11 - (-9) = 20$	$NA - NA = NA$	$11 - (-9) = 20$
$\Im(p) = 1, T = 2$	$2 - 2 = 0$	$NA - NA = NA$	$2 - 2 = 0$
$\Im(p) = 1, T = 3$	$NA - NA = NA$	$23 - (-17) = 40$	$23 - (-17) = 40$
$\Im(p) = 1, T = 4$	$44 - (-36) = 80$	$NA - NA = NA$	$44 - (-36) = 80$
$\Im(p) = 1, T = 5$	$35 - (-25) = 60$	$NA - NA = NA$	$35 - (-25) = 60$
$\Im(p) = 0, T = 1$	$NA - NA = NA$	$31 - (-29) = 60$	$31 - (-29) = 60$
$\Im(p) = 0, T = 2$	$NA - NA = NA$	$2 - 2 = 0$	$2 - 2 = 0$
$\Im(p) = 0, T = 3$	$13 - (-7) = 20$	$NA - NA = NA$	$13 - (-7) = 20$
$\Im(p) = 0, T = 4$	$NA - NA = NA$	$24 - (-16) = 40$	$24 - (-16) = 40$
$\Im(p) = 0, T = 5$	$NA - NA = NA$	$-5 - 15 = -20$	$-5 - 15 = -20$
none	$19\frac{2}{3} - \left(-13\frac{2}{3}\right) = 33\frac{1}{3}$	$16\frac{1}{3} - \left(-10\frac{1}{3}\right) = 26\frac{2}{3}$	$18 - (-12) = 30$

Again, not all conditional average treatment effects are defined for this DGP.

Design one yields

$$\begin{array}{cc}
Y = -38\frac{1}{3} + 33\frac{1}{3}D + 8\frac{1}{3}T + \varepsilon_{1a} & \textit{ATT p-score matched sample} \\
33\frac{1}{3} & \textit{suggested ATT} \\
Y = -33\frac{1}{3} + 26\frac{2}{3}D + 7\frac{2}{3}T + \varepsilon_{1a} & \textit{ATUT p-score matched sample} \\
26\frac{2}{3} & \textit{suggested ATUT} \\
Y = -36 + 30D + 8T + \varepsilon_{1a} & \textit{ATE p-score matched sample} \\
30 & \textit{suggested ATE}
\end{array} \tag{1a}$$

and

$$\begin{array}{cc}
Y = -7 + 20D - 10\Im(p) & \textit{ATT p-score matched sample} \\
+20\Im(p) \times D + \varepsilon_{1b} & \\
20 & \textit{suggested ATT } (p = 0.4) \\
20 + 20 = 40 & \textit{suggested ATT } (p = 0.6) \\
20 + 20\left(\frac{2}{3}\right) = 33\frac{1}{3} & \textit{suggested ATT} \\
Y = -7 + 20D - 10\Im(p) & \textit{ATUT p-score matched sample} \\
+20\Im(p) \times D + \varepsilon_{1b} & \\
20 & \textit{suggested ATUT } (p = 0.4) \\
20 + 20 = 40 & \textit{suggested ATUT } (p = 0.6) \\
20 + 20\left(\frac{1}{3}\right) = 26\frac{2}{3} & \textit{suggested ATUT} \\
Y = -7 + 20D - 10\Im(p) & \textit{ATE p-score matched sample} \\
+20\Im(p) \times D + \varepsilon_{1b} & \\
20 & \textit{suggested ATE } (p = 0.4) \\
20 + 20 = 40 & \textit{suggested ATE } (p = 0.6) \\
20 + 20\left(\frac{1}{2}\right) = 30 & \textit{suggested ATE}
\end{array} \tag{1b}$$

Design two yields

$$
\begin{array}{c|l}
Y = -28 + 14D + 7T + 2T \times D & \\
-10\Im(p) + 20\Im(p) \times D & \textit{ATT p-score matched sample} \\
+ (ND)\, T \times \Im(p) & \\
+ (ND)\, T \times \Im(p) \times D + \varepsilon_{2a} & \\
14 + 2 \times 3 = 20 & \textit{suggested ATT}\ (p = 0.4) \\
14 + 2(3) + 20 & \\
+ (ND)(3) = 40 & \textit{suggested ATT}\ (p = 0.6) \\
14 + 2(1) + 20(1) & \\
+ (ND)(1 \times 1) = 36 & \textit{suggested ATT}\ (T = 1) \\
14 + 2(2) + 20(1) & \\
+ (ND)(2 \times 1) = 38 & \textit{suggested ATT}\ (T = 2) \\
14 + 2(3) + 20(0) & \\
+ (ND)(3 \times 0) = 20 & \textit{suggested ATT}\ (T = 3) \\
14 + 2(4) + 20(1) & \\
+ (ND)(4 \times 1) = 42 & \textit{suggested ATT}\ (T = 4) \\
14 + 2(5) + 20(1) & \\
+ (ND)(5 \times 1) = 44 & \textit{suggested ATT}\ (T = 5) \\
14 + 2(1) = 16\,(NA)\,(NS) & \textit{suggested ATT}\ (p = 0.4, T = 1) \\
14 + 2(1) + 20 & \\
+ (ND)(1) = 36\,(NS) & \textit{suggested ATT}\ (p = 0.6, T = 1) \\
14 + 2(2) = 18\,(NA)\,(NS) & \textit{suggested ATT}\ (p = 0.4, T = 2) \\
14 + 2(2) + 20 & \\
+ (ND)(2) = 38\,(NS) & \textit{suggested ATT}\ (p = 0.6, T = 2) \\
14 + 2(3) = 20\,(NS) & \textit{suggested ATT}\ (p = 0.4, T = 3) \\
14 + 2(3) + 20 + (ND)(3) & \\
= 40\,(NA)\,(NS) & \textit{suggested ATT}\ (p = 0.6, T = 3) \\
14 + 2(4) = 22\,(NA)\,(NS) & \textit{suggested ATT}\ (p = 0.4, T = 4) \\
14 + 2(4) + 20 & \\
+ (ND)(4) = 42\,(NS) & \textit{suggested ATT}\ (p = 0.6, T = 4) \\
14 + 2(5) = 24\,(NA)\,(NS) & \textit{suggested ATT}\ (p = 0.4, T = 5) \\
14 + 2(5) + 20 & \\
+ (ND)(5) = 44\,(NS) & \textit{suggested ATT}\ (p = 0.6, T = 5) \\
14 + 2 \times 3 + 20\left(\frac{2}{3}\right) & \\
+ (ND)(2) = 33\frac{1}{3} & \textit{suggested ATT}
\end{array}
\tag{2a}
$$

$$
\begin{array}{c}
Y = -28 + 14D + 7T + 2T \times D \\
-10\Im(p) + 20\Im(p) \times D \\
+ (ND)\, T \times \Im(p) \\
+ (ND)\, T \times \Im(p) \times D + \varepsilon_{2a} \\
14 + 2 \times 3 = 20 \\
14 + 2(3) + 20 + (ND)(3) = 40 \\
\\
14 + 2(1) + 20(0) \\
+ (ND)(1 \times 0) = 16 \\
\\
14 + 2(2) + 20(0) \\
+ (ND)(2 \times 0) = 18 \\
\\
14 + 2(3) + 20(1) \\
+ (ND)(3 \times 1) = 40 \\
\\
14 + 2(4) + 20(0) \\
+ (ND)(4 \times 0) = 22 \\
\\
14 + 2(5) + 20(0) \\
+ (ND)(5 \times 0) = 24 \\
\\
14 + 2(1) = 16(NS) \\
14 + 2(1) + 20 + (ND)(1) \\
= 36(NA)(NS) \\
14 + 2(2) = 18(NS) \\
14 + 2(2) + 20 + (ND)(2) \\
= 38(NA)(NS) \\
14 + 2(3) = 20(NA)(NS) \\
14 + 2(3) + 20 + (ND)(3) \\
= 40(NS) \\
14 + 2(4) = 22(NS) \\
14 + 2(4) + 20 + (ND)(4) \\
= 42(NA)(NS) \\
14 + 2(5) = 24(NS) \\
14 + 2(5) + 20 + (ND)(5) \\
= 44(NA)(NS) \\
\\
14 + 2 \times 3 + 20\left(\frac{1}{3}\right) \\
+ (ND)(1) = 26\frac{2}{3}
\end{array}
$$

$$
\begin{array}{c}
ATUT \text{ p-score matched sample} \\
\\
\text{suggested } ATUT\,(p = 0.4) \\
\text{suggested } ATUT\,(p = 0.6) \\
\\
\text{suggested } ATUT\,(T = 1) \\
\\
\text{suggested } ATUT\,(T = 2) \\
\\
\text{suggested } ATUT\,(T = 3) \\
\\
\text{suggested } ATUT\,(T = 4) \\
\\
\text{suggested } ATUT\,(T = 5) \\
\\
\text{suggested } ATUT\,(p = 0.4, T = 1) \\
\text{suggested } ATUT\,(p = 0.6, T = 1) \\
\text{suggested } ATUT\,(p = 0.4, T = 2) \\
\text{suggested } ATUT\,(p = 0.6, T = 2) \\
\text{suggested } ATUT\,(p = 0.4, T = 3) \\
\text{suggested } ATUT\,(p = 0.6, T = 3) \\
\text{suggested } ATUT\,(p = 0.4, T = 4) \\
\text{suggested } ATUT\,(p = 0.6, T = 4) \\
\text{suggested } ATUT\,(p = 0.4, T = 5) \\
\text{suggested } ATUT\,(p = 0.6, T = 5) \\
\\
\text{suggested } ATUT
\end{array}
\tag{2a}
$$

$$
\begin{array}{c}
Y = -28 + 14D + 7T + 2T \times D \\
-10\Im(p) + 20\Im(p) \times D \\
+ (ND)\, T \times \Im(p) \\
+ (ND)\, T \times \Im(p) \times D + \varepsilon_{2a}
\end{array}
\qquad ATE \text{ p-score matched sample}
$$

$$
14 + 2 \times 3 = 20 \qquad \text{suggested } ATE\,(p = 0.4)
$$

$$
\begin{array}{c}
14 + 2\,(3) + 20 \\
+ (ND)\,(3) = 40
\end{array}
\qquad \text{suggested } ATE\,(p = 0.6)
$$

$$
\begin{array}{c}
14 + 2\,(1) + 20\left(\frac{1}{2}\right) \\
+ (ND)\left(1 \times \frac{1}{2}\right) = 26
\end{array}
\qquad \text{suggested } ATE\,(T = 1)
$$

$$
\begin{array}{c}
14 + 2\,(2) + 20\left(\frac{1}{2}\right) \\
+ (ND)\left(2 \times \frac{1}{2}\right) = 28
\end{array}
\qquad \text{suggested } ATE\,(T = 2)
$$

$$
\begin{array}{c}
14 + 2\,(3) + 20\left(\frac{1}{2}\right) \\
+ (ND)\left(3 \times \frac{1}{2}\right) = 30
\end{array}
\qquad \text{suggested } ATE\,(T = 3)
$$

$$
\begin{array}{c}
14 + 2\,(4) + 20\left(\frac{1}{2}\right) \\
+ (ND)\left(4 \times \frac{1}{2}\right) = 32
\end{array}
\qquad \text{suggested } ATE\,(T = 4)
$$

$$
\begin{array}{c}
14 + 2\,(5) + 20\left(\frac{1}{2}\right) \\
+ (ND)\left(5 \times \frac{1}{2}\right) = 34
\end{array}
\qquad \text{suggested } ATE\,(T = 5)
\tag{2a}
$$

$$
\begin{array}{ll}
14 + 2\,(1) = 16\,(NS) & \text{suggested } ATE\,(p = 0.4, T = 1) \\
\begin{array}{c} 14 + 2\,(1) + 20 \\ + (ND)\,(1) = 36\,(NS) \end{array} & \text{suggested } ATE\,(p = 0.6, T = 1) \\
14 + 2\,(2) = 18\,(NS) & \text{suggested } ATE\,(p = 0.4, T = 2) \\
\begin{array}{c} 14 + 2\,(2) + 20 \\ + (ND)\,(2) = 38\,(NS) \end{array} & \text{suggested } ATE\,(p = 0.6, T = 2) \\
14 + 2\,(3) = 20\,(NS) & \text{suggested } ATE\,(p = 0.4, T = 3) \\
\begin{array}{c} 14 + 2\,(3) + 20 \\ + (ND)\,(3) = 40\,(NS) \end{array} & \text{suggested } ATE\,(p = 0.6, T = 3) \\
14 + 2\,(4) = 22\,(NS) & \text{suggested } ATE\,(p = 0.4, T = 4) \\
\begin{array}{c} 14 + 2\,(4) + 20 \\ + (ND)\,(4) = 42\,(NS) \end{array} & \text{suggested } ATE\,(p = 0.6, T = 4) \\
14 + 2\,(5) = 24\,(NS) & \text{suggested } ATE\,(p = 0.4, T = 5) \\
\begin{array}{c} 14 + 2\,(5) + 20 \\ + (ND)\,(5) = 44\,(NS) \end{array} & \text{suggested } ATE\,(p = 0.6, T = 5)
\end{array}
$$

$$
\begin{array}{c}
14 + 2 \times 3 + 20\left(\frac{1}{2}\right) \\
+ (ND)\left(1\frac{1}{2}\right) = 30
\end{array}
\qquad \text{suggested } ATE
$$

and

$$
\begin{aligned}
Y &= -34\tfrac{2}{3} + 27\tfrac{1}{3}D + 7T \\
&\quad +2T \times D + \varepsilon_{2b} && \textit{ATT p-score matched sample} \\
27\tfrac{1}{3} + 2\,(1) &= 29\tfrac{1}{3} && \textit{suggested ATT}\,(T=1) \\
27\tfrac{1}{3} + 2\,(2) &= 31\tfrac{1}{3} && \textit{suggested ATT}\,(T=2) \\
27\tfrac{1}{3} + 2\,(3) &= 33\tfrac{1}{3} && \textit{suggested ATT}\,(T=3) \\
27\tfrac{1}{3} + 2\,(4) &= 35\tfrac{1}{3} && \textit{suggested ATT}\,(T=4) \\
27\tfrac{1}{3} + 2\,(5) &= 37\tfrac{1}{3} && \textit{suggested ATT}\,(T=5) \\
27\tfrac{1}{3} + 2\,(3) &= 33\tfrac{1}{3} && \textit{suggested ATT} \\
Y &= -31\tfrac{1}{3} + 20\tfrac{2}{3}D + 7T \\
&\quad +2T \times D + \varepsilon_{2b} && \textit{ATUT p-score matched sample} \\
20\tfrac{2}{3} + 2\,(1) &= 22\tfrac{2}{3} && \textit{suggested ATUT}\,(T=1) \\
20\tfrac{2}{3} + 2\,(2) &= 24\tfrac{2}{3} && \textit{suggested ATUT}\,(T=2) \\
20\tfrac{2}{3} + 2\,(3) &= 26\tfrac{2}{3} && \textit{suggested ATUT}\,(T=3) \\
20\tfrac{2}{3} + 2\,(4) &= 28\tfrac{2}{3} && \textit{suggested ATUT}\,(T=4) \\
20\tfrac{2}{3} + 2\,(2) &= 30\tfrac{2}{3} && \textit{suggested ATUT}\,(T=5) \\
20\tfrac{2}{3} + 2\,(3) &= 26\tfrac{2}{3} && \textit{suggested ATUT} \\
Y &= -33 + 24D + 7T \\
&\quad +2T \times D + \varepsilon_{2b} && \textit{ATE p-score matched sample} \\
24 + 2\,(1) &= 26 && \textit{suggested ATE}\,(T=1) \\
24 + 2\,(2) &= 28 && \textit{suggested ATE}\,(T=2) \\
24 + 2\,(3) &= 30 && \textit{suggested ATE}\,(T=3) \\
24 + 2\,(4) &= 32 && \textit{suggested ATE}\,(T=4) \\
24 + 2\,(2) &= 34 && \textit{suggested ATE}\,(T=5) \\
24 + 2\,(3) &= 30 && \textit{suggested ATE}
\end{aligned}
\tag{2b}
$$

Design 3 yields

$$
\begin{array}{cl}
g_1\left(T=1,p=0.6\right)\left(NS\right) & \\
-g_0\left(T=1,p=0.6\right)\left(NS\right) & \textit{ATT p-score matched sample} \\
=11-(-29)=40 & \textit{suggested ATT}\left(T=1,p=0.6\right) \\
g_1\left(T=2,p=0.6\right)\left(NS\right) & \\
-g_0\left(T=2,p=0.6\right)\left(NS\right) & \\
=2-(2)=0 & \textit{suggested ATT}\left(T=2,p=0.6\right) \\
g_1\left(T=3,p=0.6\right)\left(NS\right) & \\
-g_0\left(T=3,p=0.6\right)\left(NS\right) & \\
=13-(-17)=30\left(NA\right) & \textit{suggested ATT}\left(T=3,p=0.6\right) \\
g_1\left(T=4,p=0.6\right)\left(NS\right) & \\
-g_0\left(T=4,p=0.6\right)\left(NS\right) & \\
=44-(-16)=60 & \textit{suggested ATT}\left(T=4,p=0.6\right) \\
g_1\left(T=5,p=0.6\right)\left(NS\right) & \\
-g_0\left(T=5,p=0.6\right)\left(NS\right) & \\
=35-(15)=20 & \textit{suggested ATT}\left(T=5,p=0.6\right)
\end{array}
\tag{3Tp}
$$

$$
\begin{array}{cl}
g_1\left(T=1,p=0.4\right)\left(NS\right) & \\
-g_0\left(T=1,p=0.4\right)\left(NS\right) & \\
=11-(-29)=40\left(NA\right) & \textit{suggested ATT}\left(T=1,p=0.4\right) \\
g_1\left(T=2,p=0.4\right)\left(NS\right) & \\
-g_0\left(T=2,p=0.4\right)\left(NS\right) & \\
=2-(2)=0\left(NA\right) & \textit{suggested ATT}\left(T=2,p=0.4\right) \\
g_1\left(T=3,p=0.4\right)\left(NS\right) & \\
-g_0\left(T=3,p=0.4\right)\left(NS\right) & \\
=13-(-17)=30 & \textit{suggested ATT}\left(T=3,p=0.4\right) \\
g_1\left(T=4,p=0.4\right)\left(NS\right) & \\
-g_0\left(T=4,p=0.4\right)\left(NS\right) & \\
=44-(-16)=60\left(NA\right) & \textit{suggested ATT}\left(T=4,p=0.4\right) \\
g_1\left(T=5,p=0.4\right)\left(NS\right) & \\
-g_0\left(T=5,p=0.4\right)\left(NS\right) & \\
=35-(15)=20\left(NA\right) & \textit{suggested ATT}\left(T=5,p=0.4\right)
\end{array}
\tag{3Tp}
$$

$$
\begin{array}{rl}
0ATT\,(T=1,p=0.4)\,(NS) & \\
+0ATT\,(T=2,p=0.4)\,(NS) & \\
+1ATT\,(T=3,p=0.4)\,(NS) & \textit{ATT p-score matched sample} \\
+0ATT\,(T=4,p=0.4)\,(NS) & \\
+0ATT\,(T=5,p=0.4)\,(NS) & \\
=0\,(40)+0\,(0) & \\
+1\,(30)+0\,(60) & \textit{suggested ATT}\,(p=0.4) \\
+0\,(20)=30 & \\
\frac{1}{4}ATT\,(T=1,p=0.6)\,(NS) & \\
+\frac{1}{4}ATT\,(T=2,p=0.6)\,(NS) & \\
+0ATT\,(T=3,p=0.6)\,(NS) & \\
+\frac{1}{4}ATT\,(T=4,p=0.6)\,(NS) & \\
+\frac{1}{4}ATT\,(T=5,p=0.6)\,(NS) & \\
=\frac{1}{4}\,(40)+\frac{1}{4}\,(0) & \\
+0\,(30)+\frac{1}{4}\,(60) & \textit{suggested ATT}\,(p=0.6) \\
+\frac{1}{4}\,(20)=30 &
\end{array}
\tag{3p}
$$

$$
\begin{array}{rl}
0ATT\,(T=1,p=0.4)\,(NS) & \textit{ATT p-score matched sample} \\
+1ATT\,(T=1,p=0.6)\,(NS) & \\
=0\,(40)+1\,(40)=40 & \textit{suggested ATT}\,(T=1) \\
0ATT\,(T=2,p=0.4)\,(NS) & \\
+1ATT\,(T=2,p=0.6)\,(NS) & \\
=0\,(0)+1\,(0)=0 & \textit{suggested ATT}\,(T=2) \\
1ATT\,(T=3,p=0.4)\,(NS) & \\
+0ATT\,(T=3,p=0.6)\,(NS) & \\
=1\,(30)+0\,(30)=30 & \textit{suggested ATT}\,(T=3) \\
0ATT\,(T=4,p=0.4)\,(NS) & \\
+1ATT\,(T=4,p=0.6)\,(NS) & \\
=0\,(60)+1\,(60)=60 & \textit{suggested ATT}\,(T=4) \\
0ATT\,(T=5,p=0.4)\,(NS) & \\
+1ATT\,(T=5,p=0.6)\,(NS) & \\
=0\,(20)+1\,(20)=20 & \textit{suggested ATT}\,(T=5)
\end{array}
\tag{3T}
$$

$$
\begin{array}{cl}
\frac{1}{6}ATT\,(T=1) & \\
+\frac{1}{6}ATT\,(T=2) & \\
+\frac{1}{3}ATT\,(T=3) & \textit{ATT p-score matched sample} \\
+\frac{1}{6}ATT\,(T=4) & \\
+\frac{1}{6}ATT\,(T=5) & \\
=\frac{1}{6}\,(40)+\frac{1}{6}\,(0) & \\
+\frac{1}{3}\,(30)+\frac{1}{6}\,(60) & \textit{suggested ATT} \\
+\frac{1}{6}\,(20)=30 &
\end{array}
\tag{3}
$$

$$
\begin{array}{cl}
g_1\,(T=1,p=0.4)\,(NS) & \\
-g_0\,(T=1,p=0.4)\,(NS) & \\
=11-(-29)=40 & \textit{suggested ATUT}\,(T=1,p=0.4) \\
g_1\,(T=2,p=0.4)\,(NS) & \\
-g_0\,(T=2,p=0.4)\,(NS) & \\
=2-(2)=0 & \textit{suggested ATUT}\,(T=2,p=0.4) \\
g_1\,(T=3,p=0.4)\,(NS) & \\
-g_0\,(T=3,p=0.4)\,(NS) & \\
=13-(-17)=30\,(NA) & \textit{suggested ATUT}\,(T=3,p=0.4) \\
g_1\,(T=4,p=0.4)\,(NS) & \\
-g_0\,(T=4,p=0.4)\,(NS) & \\
=44-(-16)=60 & \textit{suggested ATUT}\,(T=4,p=0.4) \\
g_1\,(T=5,p=0.4)\,(NS) & \\
-g_0\,(T=5,p=0.4)\,(NS) & \\
=35-(15)=20 & \textit{suggested ATUT}\,(T=5,p=0.4)
\end{array}
\tag{3Tp}
$$

$$
\begin{array}{cl}
g_1\,(T=1,p=0.6)\,(NS) & \\
-g_0\,(T=1,p=0.6)\,(NS) & \\
=11-(-29)=40\,(NA) & \textit{suggested ATUT}\,(T=1,p=0.6) \\
g_1\,(T=2,p=0.6)\,(NS) & \\
-g_0\,(T=2,p=0.6)\,(NS) & \\
=2-(2)=0\,(NA) & \textit{suggested ATUT}\,(T=2,p=0.6) \\
g_1\,(T=3,p=0.6)\,(NS) & \\
-g_0\,(T=3,p=0.6)\,(NS) & \\
=13-(-17)=30 & \textit{suggested ATUT}\,(T=3,p=0.6) \\
g_1\,(T=4,p=0.6)\,(NS) & \\
-g_0\,(T=4,p=0.6)\,(NS) & \\
=44-(-16)=60\,(NA) & \textit{suggested ATUT}\,(T=4,p=0.6) \\
g_1\,(T=5,p=0.6)\,(NS) & \\
-g_0\,(T=5,p=0.6)\,(NS) & \\
=35-(15)=20\,(NA) & \textit{suggested ATUT}\,(T=5,p=0.6)
\end{array}
\tag{3Tp}
$$

$$
\begin{array}{c}
\frac{1}{4}ATUT\left(T=1,p=0.4\right)\left(NS\right) \\
+\frac{1}{4}ATUT\left(T=2,p=0.4\right)\left(NS\right) \\
+0ATUT\left(T=3,p=0.4\right)\left(NS\right) \\
+\frac{1}{4}ATUT\left(T=4,p=0.4\right)\left(NS\right) \\
+\frac{1}{4}ATUT\left(T=5,p=0.4\right)\left(NS\right) \\
=\frac{1}{4}\left(40\right)+\frac{1}{4}\left(0\right) \\
+0\left(30\right)+\frac{1}{4}\left(60\right) \\
+\frac{1}{4}\left(20\right)=30
\end{array}
\begin{array}{l}
ATUT\ p\text{-}score\ matched\ sample \\
\\
\\
\\
\\
\\
suggested\ ATUT\left(p=0.4\right)
\end{array}
\tag{3p}
$$

$$
\begin{array}{c}
0ATUT\left(T=1,p=0.6\right)\left(NS\right) \\
+0ATUT\left(T=2,p=0.6\right)\left(NS\right) \\
+1ATUT\left(T=3,p=0.6\right)\left(NS\right) \\
+0ATUT\left(T=4,p=0.6\right)\left(NS\right) \\
+0ATUT\left(T=5,p=0.6\right)\left(NS\right) \\
=0\left(40\right)+0\left(0\right) \\
+1\left(30\right)+0\left(60\right) \\
+0\left(20\right)=30
\end{array}
\begin{array}{l}
\\
\\
\\
\\
\\
\\
suggested\ ATUT\left(p=0.6\right)
\end{array}
$$

$$
\begin{array}{c}
1ATUT\left(T=1,p=0.4\right)\left(NS\right) \\
+0ATUT\left(T=1,p=0.6\right)\left(NS\right) \\
=1\left(40\right)+0\left(40\right)=40 \\
1ATUT\left(T=2,p=0.4\right)\left(NS\right) \\
+0ATUT\left(T=2,p=0.6\right)\left(NS\right) \\
=1\left(0\right)+0\left(0\right)=0 \\
0ATUT\left(T=3,p=0.4\right)\left(NS\right) \\
+1ATUT\left(T=3,p=0.6\right)\left(NS\right) \\
=0\left(30\right)+1\left(30\right)=30 \\
1ATUT\left(T=4,p=0.4\right)\left(NS\right) \\
+0ATUT\left(T=4,p=0.6\right)\left(NS\right) \\
=1\left(60\right)+0\left(60\right)=60 \\
1ATUT\left(T=5,p=0.4\right)\left(NS\right) \\
+0ATUT\left(T=5,p=0.6\right)\left(NS\right) \\
=1\left(20\right)+0\left(20\right)=20
\end{array}
\begin{array}{l}
ATUT\ p\text{-}score\ matched\ sample \\
suggested\ ATUT\left(T=1\right) \\
\\
suggested\ ATUT\left(T=2\right) \\
\\
suggested\ ATUT\left(T=3\right) \\
\\
suggested\ ATUT\left(T=4\right) \\
\\
suggested\ ATUT\left(T=5\right)
\end{array}
\tag{3T}
$$

$$
\begin{array}{c}
\frac{1}{6}ATUT\,(T=1) \\
+\frac{1}{6}ATUT\,(T=2) \\
+\frac{1}{3}ATUT\,(T=3) \\
+\frac{1}{6}ATUT\,(T=4) \\
+\frac{1}{6}ATUT\,(T=5) \\
=\frac{1}{6}\,(40)+\frac{1}{6}\,(0) \\
+\frac{1}{3}\,(30)+\frac{1}{6}\,(60) \\
+\frac{1}{6}\,(20)=30
\end{array}
\qquad
\begin{array}{l}
ATUT \text{ p-score matched sample} \\
\\
\\
\\
\\
\\
\text{suggested } ATUT
\end{array}
\tag{3}
$$

$$
\begin{array}{c}
g_1\,(T=1,p=0.6)\,(NS) \\
-g_0\,(T=1,p=0.6)\,(NS) \\
=11-(-29)=40 \\
g_1\,(T=2,p=0.6)\,(NS) \\
-g_0\,(T=2,p=0.6)\,(NS) \\
=2-(2)=0 \\
g_1\,(T=3,p=0.6)\,(NS) \\
-g_0\,(T=3,p=0.6)\,(NS) \\
=13-(-17)=30 \\
g_1\,(T=4,p=0.6)\,(NS) \\
-g_0\,(T=4,p=0.6)\,(NS) \\
=44-(-16)=60 \\
g_1\,(T=5,p=0.6)\,(NS) \\
-g_0\,(T=5,p=0.6)\,(NS) \\
=35-(15)=20
\end{array}
\qquad
\begin{array}{l}
ATE \text{ p-score matched sample} \\
\text{suggested } ATE\,(T=1,p=0.6) \\
\\
\text{suggested } ATE\,(T=2,p=0.6) \\
\\
\text{suggested } ATE\,(T=3,p=0.6) \\
\\
\text{suggested } ATE\,(T=4,p=0.6) \\
\\
\text{suggested } ATE\,(T=5,p=0.6)
\end{array}
\tag{3Tp}
$$

$$
\begin{array}{c}
g_1\,(T=1,p=0.6)\,(NS) \\
-g_0\,(T=1,p=0.6)\,(NS) \\
=11-(-29)=40 \\
g_1\,(T=2,p=0.4)\,(NS) \\
-g_0\,(T=2,p=0.4)\,(NS) \\
=2-(2)=0 \\
g_1\,(T=3,p=0.4)\,(NS) \\
-g_0\,(T=3,p=0.4)\,(NS) \\
=13-(-17)=30 \\
g_1\,(T=4,p=0.4)\,(NS) \\
-g_0\,(T=4,p=0.4)\,(NS) \\
=44-(-16)=60 \\
g_1\,(T=5,p=0.6)\,(NS) \\
-g_0\,(T=5,p=0.6)\,(NS) \\
=35-(15)=20
\end{array}
\qquad
\begin{array}{l}
\\
\text{suggested } ATE\,(T=1,p=0.4) \\
\\
\text{suggested } ATE\,(T=2,p=0.4) \\
\\
\text{suggested } ATE\,(T=3,p=0.4) \\
\\
\text{suggested } ATE\,(T=4,p=0.4) \\
\\
\text{suggested } ATE\,(T=5,p=0.4)
\end{array}
\tag{3Tp}
$$

$$
\begin{aligned}
&\tfrac{1}{6} ATE\,(T=1, p=0.4)\,(NS) \\
+&\tfrac{1}{6} ATE\,(T=2, p=0.4)\,(NS) \\
+&\tfrac{1}{3} ATE\,(T=3, p=0.4)\,(NS) \quad && ATE \text{ p-score matched sample} \\
+&\tfrac{1}{6} ATE\,(T=4, p=0.4)\,(NS) \\
+&\tfrac{1}{6} ATE\,(T=5, p=0.4)\,(NS) \\
&= \tfrac{1}{6}(40) + \tfrac{1}{6}(0) \\
&+\tfrac{1}{3}(30) + \tfrac{1}{6}(60) && \text{suggested } ATE\,(p=0.4) \\
&+\tfrac{1}{6}(20) = 30 \\
&\tfrac{1}{6} ATE\,(T=1, p=0.6)\,(NS) \\
+&\tfrac{1}{6} ATE\,(T=2, p=0.6)\,(NS) \\
+&\tfrac{1}{3} ATE\,(T=3, p=0.6)\,(NS) \\
+&\tfrac{1}{6} ATE\,(T=4, p=0.6)\,(NS) \\
+&\tfrac{1}{6} ATE\,(T=5, p=0.6)\,(NS) \\
&= \tfrac{1}{6}(40) + \tfrac{1}{6}(0) \\
&+\tfrac{1}{3}(30) + \tfrac{1}{6}(60) && \text{suggested } ATE\,(p=0.6) \\
&+\tfrac{1}{6}(20) = 30
\end{aligned}
\tag{3p}
$$

$$
\begin{array}{ll}
\frac{1}{2}ATE\left(T=1,p=0.4\right)\left(NS\right) & \\
+\frac{1}{2}ATE\left(T=1,p=0.6\right)\left(NS\right) & ATE\ p\text{-}score\ matched\ sample \\
=\frac{1}{2}\left(40\right)+\frac{1}{2}\left(40\right)=40 & suggested\ ATE\left(T=1\right) \\
\frac{1}{2}ATE\left(T=2,p=0.4\right)\left(NS\right) & \\
+\frac{1}{2}ATE\left(T=2,p=0.6\right)\left(NS\right) & \\
=\frac{1}{2}\left(0\right)+\frac{1}{2}\left(0\right)=0 & suggested\ ATE\left(T=2\right) \\
\frac{1}{2}ATE\left(T=3,p=0.4\right)\left(NS\right) & \\
+\frac{1}{2}ATE\left(T=3,p=0.6\right)\left(NS\right) & \\
=\frac{1}{2}\left(30\right)+\frac{1}{2}\left(30\right)=30 & suggested\ ATE\left(T=3\right) \\
\frac{1}{2}ATE\left(T=4,p=0.4\right)\left(NS\right) & \\
+\frac{1}{2}ATE\left(T=4,p=0.6\right)\left(NS\right) & \\
=\frac{1}{2}\left(60\right)+\frac{1}{2}\left(60\right)=60 & suggested\ ATE\left(T=4\right) \\
\frac{1}{2}ATE\left(T=5,p=0.4\right)\left(NS\right) & \\
+\frac{1}{2}ATE\left(T=5,p=0.6\right)\left(NS\right) & \\
=\frac{1}{2}\left(20\right)+\frac{1}{2}\left(20\right)=20 & suggested\ ATE\left(T=5\right)
\end{array}
\tag{3T}
$$

$$
\begin{array}{ll}
\frac{1}{6}ATE\left(T=1\right) & \\
+\frac{1}{6}ATE\left(T=2\right) & \\
+\frac{1}{3}ATE\left(T=3\right) & ATE\ p\text{-}score\ matched\ sample \\
+\frac{1}{6}ATE\left(T=4\right) & \\
+\frac{1}{6}ATE\left(T=5\right) & \\
=\frac{1}{6}\left(40\right)+\frac{1}{6}\left(0\right) & \\
+\frac{1}{3}\left(30\right)+\frac{1}{6}\left(60\right) & suggested\ ATE \\
+\frac{1}{6}\left(20\right)=30 &
\end{array}
\tag{3}
$$

As mean independence conditional on propensity score is satisfied by the DGP, any design conditioning on propensity score identifies average treatment effects. This conclusion includes nonparametric regression (results are not reported). However, nonparametric saturated-design 3 fails as it chases the nonlinearity in covariate T and mean independence conditional on T is not satisfied by this DGP. Remarkably, the linear saturated-design 2a does not suffer the same malady. We speculate it is because mean independence of outcome conditional on covariate means is satisfied by this DGP even though conditional mean independence fails at all levels of T. This conjecture is the subject of the next example. As discussed earlier, dropping observations for which observable common sup-

port for (T, p) *is lacking results in no data for saturated designs* 2*a and* 3*.*

Example 4 (heterogeneous *DGP* with unbalanced covariate means) *Suppose the DGP is*

	D	$P(Z)$	T_1	$g_1(T)$	U_1	Y_1	T_0	$g_0(T)$	U_0	Y_0	Y	T
	1	0.6	4	19	25	44	4	−19	−15	−34	44	4
	1	0.6	2	17	−15	2	2	−17	5	−12	2	2
	1	0.6	5	20	15	35	5	−20	−25	−45	35	5
	1	0.6	3	18	−5	13	1	−16	15	−1	13	3
	1	0.4	3	18	−5	13	3	−18	5	−13	13	3
	1	0.4	1	16	−5	11	3	−18	5	−13	11	1
	0	0.6	3	18	5	23	5	−20	−5	−25	−25	5
	0	0.6	1	16	5	21	1	−16	−5	−21	−21	1
	0	0.4	3	18	15	33	3	−18	15	−3	−3	3
	0	0.4	5	20	−25	−5	3	−18	−5	−23	−23	3
	0	0.4	2	17	−15	2	2	−17	25	8	8	2
	0	0.4	4	19	5	24	4	−19	−15	−34	−34	4
means	0.5	0.5	3	18	0	18	3	−18	0	−18	$1\frac{2}{3}$	3

Only $(T = 3, p = 0.4)$ *and* $(T = 5, p = 0.6)$ *have observable common support among* (T, p) *conditions and conditional mean independence is not satisfied for these* (T, p) *conditions. Conditional and unconditional average treatment effects for*

this DGP are

conditioning	$ATT(\cdot)$	$ATUT(\cdot)$	$ATE(\cdot)$
$\Im(p)=1$	$23\frac{1}{2}-(-23)=46\frac{1}{2}$	$22-(-23)=45$	$23-(-23)=46$
$\Im(p)=0$	$12-(-13)=25$	$13\frac{1}{2}-(-13)=26\frac{1}{2}$	$13-(-13)=26$
$T=1$	$11-(-1)=12$	$21-(-21)=42$	$16-(-11)=27$
$T=2$	$2-(-12)=14$	$2-8=-6$	$2-(-2)=4$
$T=3$	$13-(-13)=26$	$28-(-13)=41$	$20\frac{1}{2}-(-13)=33\frac{1}{2}$
$T=4$	$44-(-34)=78$	$24-(-34)=58$	$34-(-34)=68$
$T=5$	$35-(-45)=80$	$-5-(-25)=20$	$15-(-35)=50$
$\Im(p)=1, T=1$	$NA-(-1)=NA$	$21-(-21)=42$	$21-(-11)=32$
$\Im(p)=1, T=2$	$2-(-12)=14$	$NA-NA=NA$	$2-(-12)=14$
$\Im(p)=1, T=3$	$13-NA=NA$	$23-NA=NA$	$18-NA=NA$
$\Im(p)=1, T=4$	$44-(-34)=78$	$NA-NA=NA$	$44-(-34)=78$
$\Im(p)=1, T=5$	$35-(-45)=80$	$NA-(-25)=NA$	$35-(-35)=70$
$\Im(p)=0, T=1$	$11-NA=NA$	$NA-NA=NA$	$11-NA=NA$
$\Im(p)=0, T=2$	$NA-NA=NA$	$2-8=-6$	$2-8=-6$
$\Im(p)=0, T=3$	$13-(-13)=26$	$33-(-13)=46$	$23-(-13)=36$
$\Im(p)=0, T=4$	$NA-NA=NA$	$24-(-34)=58$	$24-(-34)=58$
$\Im(p)=0, T=5$	$NA-NA=NA$	$-5-NA=NA$	$-5-NA=NA$
none	$19\frac{2}{3}-\left(-19\frac{2}{3}\right)=39\frac{1}{3}$	$16\frac{1}{3}-\left(-16\frac{1}{3}\right)=32\frac{2}{3}$	$18-(-18)=36$

Again, not all conditional average treatment effects are defined for this DGP. Further, mean independence of Y_1 conditional on propensity score is not satisfied for this DGP, hence unconditional and conditional on $P(Z)$ $ATUT$ and ATE are likely unidentified. Design one yields

$$\begin{array}{cc}
Y=-25\frac{5}{9}+39\frac{1}{3}D+1\frac{26}{27}T+\varepsilon_{1a} & ATT \text{ p-score matched sample} \\
39\frac{1}{3} & \text{suggested } ATT \\
Y=-17\frac{31}{39}+32\frac{16}{39}D+\frac{19}{39}T+\varepsilon_{1a} & ATUT \text{ p-score matched sample} \\
32\frac{16}{39} & \text{suggested } ATUT \\
Y=-22\frac{1}{3}+36\frac{1}{9}D+1\frac{4}{9}T+\varepsilon_{1a} & ATE \text{ p-score matched sample} \\
36\frac{1}{9} & \text{suggested } ATE
\end{array} \tag{1a}$$

and

$$
\begin{array}{cc}
Y = -13 + 25D - 10\Im(p) & \\
+21\frac{1}{2}\Im(p) \times D + \varepsilon_{1b} & \textit{ATT p-score matched sample} \\
25 & \textit{suggested ATT } (p = 0.4) \\
25 + 21\frac{1}{2} = 46\frac{1}{2} & \textit{suggested ATT } (p = 0.6) \\
25 + 21\frac{1}{2}\left(\frac{2}{3}\right) = 39\frac{1}{3} & \textit{suggested ATT} \\
Y = -13 + 25D - 10\Im(p) & \\
+21\frac{1}{2}\Im(p) \times D + \varepsilon_{1b} & \textit{ATUT p-score matched sample} \\
25 & \textit{suggested ATUT } (p = 0.4) \\
25 + 21\frac{1}{2} = 46\frac{1}{2} & \textit{suggested ATUT } (p = 0.6) \\
25 + 21\frac{1}{2}\left(\frac{1}{3}\right) = 32\frac{1}{6} & \textit{suggested ATUT} \\
Y = -13 + 25D - 10\Im(p) & \\
+21\frac{1}{2}\Im(p) \times D + \varepsilon_{1b} & \textit{ATE p-score matched sample} \\
25 & \textit{suggested ATE } (p = 0.4) \\
25 + 21\frac{1}{2} = 46\frac{1}{2} & \textit{suggested ATE } (p = 0.6) \\
25 + 21\frac{1}{2}\left(\frac{1}{2}\right) = 35\frac{3}{4} & \textit{suggested ATE}
\end{array}
\tag{1b}
$$

Design two yields

$$
\begin{array}{cl}
Y = 50 - 40D - 21T & \\
+22T \times D - 70\Im(p) & \\
+38\Im(p) \times D & \textit{ATT p-score matched sample} \\
+20T \times \Im(p) & \\
-8T \times \Im(p) \times D + \varepsilon_{2a} & \\
-40 + 22(2) = 4 & \textit{suggested ATT } (p = 0.4) \\
-40 + 22\left(3\tfrac{1}{2}\right) + 38 & \\
-8\left(3\tfrac{1}{2}\right) = 47 & \textit{suggested ATT } (p = 0.6) \\
-40 + 22(1) + 38(0) & \\
-8(1 \times 0) = -18 & \textit{suggested ATT } (T = 1) \\
-40 + 22(2) + 38(1) & \\
-8(2 \times 1) = 26 & \textit{suggested ATT } (T = 2) \\
-40 + 22(3) + 38\left(\tfrac{1}{2}\right) & \\
-8\left(3 \times \tfrac{1}{2}\right) = 33 & \textit{suggested ATT } (T = 3) \\
-40 + 22(4) + 38(1) & \\
-8(4 \times 1) = 54 & \textit{suggested ATT } (T = 4) \\
-40 + 22(5) + 38(1) & \\
-8(5 \times 1) = 68 & \textit{suggested ATT } (T = 5)
\end{array}
\tag{2a}
$$

$$
\begin{array}{cc}
-40+22\,(1)=-18\,(NA)\,(NS) & \textit{suggested } ATT\,(p=0.4,T=1) \\
-40+22\,(1)+38 & \\
-8\,(1)=12\,(NA)\,(NS) & \textit{suggested } ATT\,(p=0.6,T=1) \\
-40+22\,(2)=4\,(NA)\,(NS) & \textit{suggested } ATT\,(p=0.4,T=2) \\
-40+22\,(2)+38 & \\
-8\,(2)=26\,(NS) & \textit{suggested } ATT\,(p=0.6,T=2) \\
-40+22\,(3)=26 & \textit{suggested } ATT\,(p=0.4,T=3) \\
-40+22\,(3)+38 & \\
-8\,(3)=40\,(NA)\,(NS) & \textit{suggested } ATT\,(p=0.6,T=3) \\
-40+22\,(4)=48\,(NA)\,(NS) & \textit{suggested } ATT\,(p=0.4,T=4) \\
-40+22\,(4)+38 & \\
-8\,(4)=54\,(NS) & \textit{suggested } ATT\,(p=0.6,T=4) \\
-40+22\,(5)=70\,(NA)\,(NS) & \textit{suggested } ATT\,(p=0.4,T=5) \\
-40+22\,(5)+38 & \\
-8\,(5)=68 & \textit{suggested } ATT\,(p=0.6,T=5) \\
-40+22\,(3)+38\left(\frac{2}{3}\right) & \\
-8\left(2\frac{1}{3}\right)=32\frac{2}{3} & \textit{suggested } ATT
\end{array}
\tag{2a}
$$

$$
\begin{aligned}
Y = 50 - 40D - 21T & \\
+22T \times D - 70\Im(p) & \\
+38\Im(p) \times D & \quad ATUT \text{ } p\text{-score matched sample} \\
+20T \times \Im(p) & \\
-8T \times \Im(p) \times D + \varepsilon_{2a} & \\
-40 + 22(3) = 26 & \quad \textit{suggested } ATUT\,(p = 0.4) \\
-40 + 22(3) + 38 - 8(3) = 40 & \quad \textit{suggested } ATUT\,(p = 0.6) \\
-40 + 22(1) + 38(1) & \\
-8(1 \times 1) = 12 & \quad \textit{suggested } ATUT\,(T = 1) \\
-40 + 22(2) + 38(0) & \\
-8(2 \times 0) = 4 & \quad \textit{suggested } ATUT\,(T = 2) \\
-40 + 22(3) + 38(0) & \\
-8(3 \times 0) = 26 & \quad \textit{suggested } ATUT\,(T = 3) \\
-40 + 22(4) + 38(0) & \\
-8(4 \times 0) = 48 & \quad \textit{suggested } ATUT\,(T = 4) \\
-40 + 22(5) + 38(1) & \\
-8(5 \times 1) = 68 & \quad \textit{suggested } ATUT\,(T = 5)
\end{aligned}
\tag{2a}
$$

$$
\begin{array}{cl}
-40 + 22\,(1) = -18\,(NA)\,(NS) & \textit{suggested } ATUT\,(p = 0.4, T = 1) \\
\begin{array}{c} -40 + 22\,(1) + 38 - 8\,(1) \\ = 12\,(NS) \end{array} & \textit{suggested } ATUT\,(p = 0.6, T = 1) \\
-40 + 22\,(2) = 4\,(NS) & \textit{suggested } ATUT\,(p = 0.4, T = 2) \\
\begin{array}{c} -40 + 22\,(2) + 38 - 8\,(2) \\ = 26\,(NA)\,(NS) \end{array} & \textit{suggested } ATUT\,(p = 0.6, T = 2) \\
-40 + 22\,(3) = 26 & \textit{suggested } ATUT\,(p = 0.4, T = 3) \\
\begin{array}{c} -40 + 22\,(3) + 38 - 8\,(3) \\ = 40\,(NA)\,(NS) \end{array} & \textit{suggested } ATUT\,(p = 0.6, T = 3) \\
-40 + 22\,(4) = 48\,(NS) & \textit{suggested } ATUT\,(p = 0.4, T = 4) \\
\begin{array}{c} -40 + 22\,(4) + 38 - 8\,(4) \\ = 54\,(NA)\,(NS) \end{array} & \textit{suggested } ATUT\,(p = 0.6, T = 4) \\
-40 + 22\,(5) = 70\,(NA)\,(NS) & \textit{suggested } ATUT\,(p = 0.4, T = 5) \\
\begin{array}{c} -40 + 22\,(5) + 38 - 8\,(5) \\ = 68\,(NA) \end{array} & \textit{suggested } ATUT\,(p = 0.6, T = 5) \\
-40 + 22\,(3) + 38\left(\frac{1}{3}\right) & \\
-8\,(1) = 30\frac{2}{3} & \textit{suggested } ATUT
\end{array}
\tag{2a}
$$

$$
\begin{array}{cl}
Y = 50 - 40D - 21T & \\
+22T \times D - 70\Im(p) & \\
+38\Im(p) \times D & \textit{ATE p-score matched sample} \\
+20T \times \Im(p) & \\
-8T \times \Im(p) \times D + \varepsilon_{2a} & \\
-40 + 22\left(2\frac{2}{3}\right) = 18\frac{2}{3} & \textit{suggested ATE}\,(p = 0.4) \\
-40 + 22\left(3\frac{1}{3}\right) + 38 & \\
-8\left(3\frac{1}{3}\right) = 44\frac{2}{3} & \textit{suggested ATE}\,(p = 0.6) \\
-40 + 22\,(1) + 38\left(\frac{1}{2}\right) & \\
-8\left(1 \times \frac{1}{2}\right) = -3 & \textit{suggested ATE}\,(T = 1) \\
-40 + 22\,(2) + 38\left(\frac{1}{2}\right) & \\
-8\left(2 \times \frac{1}{2}\right) = 15 & \textit{suggested ATE}\,(T = 2) \\
-40 + 22\,(3) + 38\left(\frac{1}{4}\right) & \\
-8\left(3 \times \frac{1}{4}\right) = 29\frac{1}{2} & \textit{suggested ATE}\,(T = 3) \\
-40 + 22\,(4) + 38\left(\frac{1}{2}\right) & \\
-8\left(4 \times \frac{1}{2}\right) = 51 & \textit{suggested ATE}\,(T = 4) \\
-40 + 22\,(5) + 38\,(1) & \\
-8\,(5 \times 1) = 68 & \textit{suggested ATE}\,(T = 5)
\end{array}
\tag{2a}
$$

$$
\begin{array}{cc}
-40+22\,(1)=-18\,(NA)\,(NS) & \textit{suggested } ATE\,(p=0.4,T=1) \\
\begin{array}{c}-40+22\,(1)+38-8\,(1)\\=12\,(NS)\end{array} & \textit{suggested } ATE\,(p=0.6,T=1) \\
-40+22\,(2)=4\,(NS) & \textit{suggested } ATE\,(p=0.4,T=2) \\
\begin{array}{c}-40+22\,(2)+38-8\,(2)\\=26\,(NS)\end{array} & \textit{suggested } ATE\,(p=0.6,T=2) \\
-40+22\,(3)=26 & \textit{suggested } ATE\,(p=0.4,T=3) \\
\begin{array}{c}-40+22\,(3)+38-8\,(3)\\=40\,(NA)\,(NS)\end{array} & \textit{suggested } ATE\,(p=0.6,T=3) \\
-40+22\,(4)=48\,(NS) & \textit{suggested } ATE\,(p=0.4,T=4) \\
\begin{array}{c}-40+22\,(4)+38-8\,(4)\\=54\,(NS)\end{array} & \textit{suggested } ATE\,(p=0.6,T=4) \\
-40+22\,(5)=70\,(NA)\,(NS) & \textit{suggested } ATE\,(p=0.4,T=5) \\
-40+22\,(5)+38-8\,(5)=68 & \textit{suggested } ATE\,(p=0.6,T=5) \\
\begin{array}{c}-40+22\,(3)+38\left(\frac{1}{2}\right)\\-8\left(1\frac{2}{3}\right)=31\frac{2}{3}\end{array} & \textit{suggested } ATE
\end{array}
\tag{2a}
$$

and

$$
\begin{array}{rl}
Y = -13\frac{7}{51} + 5\frac{41}{51}D - 2\frac{3}{17}T & \\
\quad +11\frac{3}{17}T \times D + \varepsilon_{2b} & \textit{ATT p-score matched sample} \\
5\frac{41}{51} + 11\frac{3}{17}(1) = 16\frac{50}{51} & \textit{suggested ATT } (T=1) \\
5\frac{41}{51} + 11\frac{3}{17}(2) = 28\frac{8}{51} & \textit{suggested ATT } (T=2) \\
5\frac{41}{51} + 11\frac{3}{17}(3) = 39\frac{1}{3} & \textit{suggested ATT } (T=3) \\
5\frac{41}{51} + 11\frac{3}{17}(4) = 50\frac{26}{51} & \textit{suggested ATT } (T=4) \\
5\frac{41}{51} + 11\frac{3}{17}(5) = 61\frac{35}{51} & \textit{suggested ATT } (T=5) \\
5\frac{41}{51} + 11\frac{3}{17}(3) = 39\frac{1}{3} & \textit{suggested ATT} \\
Y = -1\frac{1}{3} + 1\frac{29}{57}D - 5T & \\
\quad +11\frac{5}{19}T \times D + \varepsilon_{2b} & \textit{ATUT p-score matched sample} \\
1\frac{29}{57} + 11\frac{5}{19}(1) = 12\frac{44}{57} & \textit{suggested ATUT } (T=1) \\
1\frac{29}{57} + 11\frac{5}{19}(2) = 24\frac{2}{57} & \textit{suggested ATUT } (T=2) \\
1\frac{29}{57} + 11\frac{5}{19}(3) = 35\frac{17}{57} & \textit{suggested ATUT } (T=3) \\
1\frac{29}{57} + 11\frac{5}{19}(4) = 46\frac{32}{57} & \textit{suggested ATUT } (T=4) \\
1\frac{29}{57} + 11\frac{5}{19}(5) = 57\frac{47}{57} & \textit{suggested ATUT } (T=5) \\
1\frac{29}{57} + 11\frac{5}{19}(3) = 35\frac{17}{57} & \textit{suggested ATUT} \\
Y = -8\frac{1}{3} + 5D - 3\frac{2}{9}T & \\
\quad +10\frac{8}{9}T \times D + \varepsilon_{2b} & \textit{ATE p-score matched sample} \\
5 + 10\frac{8}{9}(1) = 15\frac{8}{9} & \textit{suggested ATE } (T=1) \\
5 + 10\frac{8}{9}(2) = 26\frac{7}{9} & \textit{suggested ATE } (T=2) \\
5 + 10\frac{8}{9}(3) = 37\frac{2}{3} & \textit{suggested ATE } (T=3) \\
5 + 10\frac{8}{9}(4) = 48\frac{5}{9} & \textit{suggested ATE } (T=4) \\
5 + 10\frac{8}{9}(2) = 59\frac{4}{9} & \textit{suggested ATE } (T=5) \\
5 + 10\frac{8}{9}(3) = 37\frac{2}{3} & \textit{suggested ATE}
\end{array}
\tag{2b}
$$

Design 3 yields

$$
\begin{array}{ll}
g_1\left(T=1,p=0.6\right)\left(NS\right) & \\
-g_0\left(T=1,p=0.6\right)\left(NS\right) & \textit{ATT p-score matched sample} \\
=11-\left(-21\right)=32\left(NA\right) & \textit{suggested ATT}\left(T=1,p=0.6\right) \\
g_1\left(T=2,p=0.6\right)\left(NS\right) & \\
-g_0\left(T=2,p=0.6\right)\left(NS\right) & \\
=2-8=-6 & \textit{suggested ATT}\left(T=2,p=0.6\right) \\
g_1\left(T=3,p=0.6\right)\left(NS\right) & \\
-g_0\left(T=3,p=0.6\right)\left(NS\right) & \\
=13-\left(-13\right)=26\left(NA\right) & \textit{suggested ATT}\left(T=3,p=0.6\right) \\
g_1\left(T=4,p=0.6\right)\left(NS\right) & \\
-g_0\left(T=4,p=0.6\right)\left(NS\right) & \\
=44-\left(-34\right)=78 & \textit{suggested ATT}\left(T=4,p=0.6\right) \\
g_1\left(T=5,p=0.6\right) & \\
-g_0\left(T=5,p=0.6\right) & \\
=35-\left(-25\right)=60 & \textit{suggested ATT}\left(T=5,p=0.6\right)
\end{array}
\tag{3Tp}
$$

$$
\begin{aligned}
& g_1\left(T=1, p=0.4\right)(NS) \\
& -g_0\left(T=1, p=0.4\right)(NS) \\
& =11-(-21)=32\,(NA) \qquad \textit{suggested } ATT\left(T=1, p=0.4\right) \\
& g_1\left(T=2, p=0.4\right)(NS) \\
& -g_0\left(T=2, p=0.4\right)(NS) \\
& =2-8=-6\,(NA) \qquad \textit{suggested } ATT\left(T=2, p=0.4\right) \\
& g_1\left(T=3, p=0.4\right) \\
& -g_0\left(T=3, p=0.4\right) \\
& =13-(-13)=26 \qquad \textit{suggested } ATT\left(T=3, p=0.4\right) \\
& g_1\left(T=4, p=0.4\right)(NS) \\
& -g_0\left(T=4, p=0.4\right)(NS) \\
& =44-(-34)=78\,(NA) \qquad \textit{suggested } ATT\left(T=4, p=0.4\right) \\
& g_1\left(T=5, p=0.4\right)(NS) \\
& -g_0\left(T=5, p=0.4\right)(NS) \\
& =35-(-25)=60\,(NA) \qquad \textit{suggested } ATT\left(T=5, p=0.4\right)
\end{aligned}
\tag{3Tp}
$$

$$
\begin{aligned}
&\tfrac{1}{2}ATT\,(T=1,p=0.4)\,(NS)\\
&+0ATT\,(T=2,p=0.4)\,(NS)\\
&\quad+\tfrac{1}{2}ATT\,(T=3,p=0.4) && \textit{ATT p-score matched sample}\\
&+0ATT\,(T=4,p=0.4)\,(NS)\\
&+0ATT\,(T=5,p=0.4)\,(NS)\\
&\qquad=\tfrac{1}{2}\,(32)+0\,(-6)\\
&\qquad\quad+\tfrac{1}{2}\,(26)+0\,(78) && \textit{suggested ATT}\,(p=0.4)\\
&\qquad\quad+0\,(60)=29\\
&0ATT\,(T=1,p=0.6)\,(NS)\\
&+\tfrac{1}{4}ATT\,(T=2,p=0.6)\,(NS)\\
&+\tfrac{1}{4}ATT\,(T=3,p=0.6)\,(NS)\\
&+\tfrac{1}{4}ATT\,(T=4,p=0.6)\,(NS)\\
&\quad+\tfrac{1}{4}ATT\,(T=5,p=0.6)\\
&\qquad=0\,(32)+\tfrac{1}{4}\,(-6)\\
&\qquad\quad+\tfrac{1}{4}\,(26)+\tfrac{1}{4}\,(78) && \textit{suggested ATT}\,(p=0.6)\\
&\qquad\quad+\tfrac{1}{4}\,(60)=30\tfrac{1}{2}
\end{aligned}
\tag{3p}
$$

$$
\begin{array}{rl}
1ATT\left(T=1,p=0.4\right)\left(NS\right) & \\
+0ATT\left(T=1,p=0.6\right)\left(NS\right) & \textit{ATT p-score matched sample} \\
=1\left(32\right)+0\left(32\right)=32 & \textit{suggested ATT}\left(T=1\right) \\
0ATT\left(T=2,p=0.4\right)\left(NS\right) & \\
+1ATT\left(T=2,p=0.6\right) & \\
=0\left(-6\right)+1\left(-6\right)=-6 & \textit{suggested ATT}\left(T=2\right) \\
\frac{1}{2}ATT\left(T=3,p=0.4\right) & \\
+\frac{1}{2}ATT\left(T=3,p=0.6\right)\left(NS\right) & \\
=\frac{1}{2}\left(26\right)+\frac{1}{2}\left(26\right)=26 & \textit{suggested ATT}\left(T=3\right) \\
0ATT\left(T=4,p=0.4\right)\left(NS\right) & \\
+1ATT\left(T=4,p=0.6\right)\left(NS\right) & \\
=0\left(78\right)+1\left(78\right)=78 & \textit{suggested ATT}\left(T=4\right) \\
0ATT\left(T=5,p=0.4\right)\left(NS\right) & \\
+1ATT\left(T=5,p=0.6\right) & \\
=0\left(60\right)+1\left(60\right)=60 & \textit{suggested ATT}\left(T=5\right)
\end{array}
\tag{3T}
$$

$$
\begin{array}{rl}
\frac{1}{6}ATT\left(T=1\right) & \\
+\frac{1}{6}ATT\left(T=2\right) & \\
+\frac{1}{3}ATT\left(T=3\right) & \textit{ATT p-score matched sample} \\
+\frac{1}{6}ATT\left(T=4\right) & \\
+\frac{1}{6}ATT\left(T=5\right) & \\
=\frac{1}{6}\left(32\right)+\frac{1}{6}\left(-6\right) & \\
+\frac{1}{3}\left(26\right)+\frac{1}{6}\left(78\right) & \textit{suggested ATT} \\
+\frac{1}{6}\left(60\right)=36 &
\end{array}
\tag{3}
$$

$$
\begin{array}{ll}
g_1\left(T=1,p=0.4\right)\left(NS\right) & \\
-g_0\left(T=1,p=0.4\right)\left(NS\right) & \\
=11-\left(-21\right)=32\left(NA\right) & \textit{suggested } ATUT\left(T=1,p=0.4\right) \\
g_1\left(T=2,p=0.4\right)\left(NS\right) & \\
-g_0\left(T=2,p=0.4\right)\left(NS\right) & \\
=2-8=-6 & \textit{suggested } ATUT\left(T=2,p=0.4\right) \\
g_1\left(T=3,p=0.4\right) & \\
-g_0\left(T=3,p=0.4\right) & \\
=13-\left(-13\right)=26 & \textit{suggested } ATUT\left(T=3,p=0.4\right) \\
g_1\left(T=4,p=0.4\right)\left(NS\right) & \\
-g_0\left(T=4,p=0.4\right)\left(NS\right) & \\
=44-\left(-34\right)=78 & \textit{suggested } ATUT\left(T=4,p=0.4\right) \\
g_1\left(T=5,p=0.4\right)\left(NS\right) & \\
-g_0\left(T=5,p=0.4\right)\left(NS\right) & \\
=35-\left(-25\right)=60\left(NA\right) & \textit{suggested } ATUT\left(T=5,p=0.4\right)
\end{array}
\tag{3Tp}
$$

$$
\begin{array}{ll}
g_1\left(T=1,p=0.6\right)\left(NS\right) & \\
-g_0\left(T=1,p=0.6\right)\left(NS\right) & \\
=11-\left(-21\right)=32 & \textit{suggested } ATUT\left(T=1,p=0.6\right) \\
g_1\left(T=2,p=0.6\right)\left(NS\right) & \\
-g_0\left(T=2,p=0.6\right)\left(NS\right) & \\
=2-8=-6\left(NA\right) & \textit{suggested } ATUT\left(T=2,p=0.6\right) \\
g_1\left(T=3,p=0.6\right)\left(NS\right) & \\
-g_0\left(T=3,p=0.6\right)\left(NS\right) & \\
=13-\left(-13\right)=26\left(NA\right) & \textit{suggested } ATUT\left(T=3,p=0.6\right) \\
g_1\left(T=4,p=0.6\right)\left(NS\right) & \\
-g_0\left(T=4,p=0.6\right)\left(NS\right) & \\
=44-\left(-34\right)=78\left(NA\right) & \textit{suggested } ATUT\left(T=4,p=0.6\right) \\
g_1\left(T=5,p=0.6\right) & \\
-g_0\left(T=5,p=0.6\right) & \\
=35-\left(-25\right)=60\left(NA\right) & \textit{suggested } ATUT\left(T=5,p=0.6\right)
\end{array}
\tag{3Tp}
$$

$$
\begin{aligned}
&0ATUT\,(T=1,p=0.4)\,(NS)\\
&+\tfrac{1}{4}ATUT\,(T=2,p=0.4)\,(NS)\\
&+\tfrac{1}{2}ATUT\,(T=3,p=0.4) && \textit{ATUT p-score matched sample}\\
&+\tfrac{1}{4}ATUT\,(T=4,p=0.4)\,(NS)\\
&+0ATUT\,(T=5,p=0.4)\,(NS)\\
&=0\,(32)+\tfrac{1}{4}\,(-6)\\
&+\tfrac{1}{2}\,(26)+\tfrac{1}{4}\,(78) && \textit{suggested ATUT}\,(p=0.4)\\
&+0\,(60)=31
\end{aligned}
\tag{3p}
$$

$$
\begin{aligned}
&\tfrac{1}{2}ATUT\,(T=1,p=0.6)\,(NS)\\
&+0ATUT\,(T=2,p=0.6)\,(NS)\\
&+0ATUT\,(T=3,p=0.6)\,(NS)\\
&+0ATUT\,(T=4,p=0.6)\,(NS)\\
&+\tfrac{1}{2}ATUT\,(T=5,p=0.6)\\
&=1\,(32)+0\,(-6)\\
&+0\,(26)+0\,(78) && \textit{suggested ATUT}\,(p=0.6)\\
&+\tfrac{1}{2}\,(60)=46
\end{aligned}
$$

$$
\begin{array}{rl}
0ATUT\,(T=1,p=0.4)\,(NS) & \\
+1ATUT\,(T=1,p=0.6)\,(NS) & \textit{ATUT p-score matched sample} \\
=0\,(32)+1\,(32)=32 & \textit{suggested ATUT}\,(T=1) \\
1ATUT\,(T=2,p=0.4)\,(NS) & \\
+0ATUT\,(T=2,p=0.6)\,(NS) & \\
=1\,(-6)+0\,(-6)=-6 & \textit{suggested ATUT}\,(T=2) \\
1ATUT\,(T=3,p=0.4) & \\
+0ATUT\,(T=3,p=0.6)\,(NS) & \\
=1\,(26)+0\,(26)=26 & \textit{suggested ATUT}\,(T=3) \\
1ATUT\,(T=4,p=0.4)\,(NS) & \\
+0ATUT\,(T=4,p=0.6)\,(NS) & \\
=1\,(78)+0\,(78)=78 & \textit{suggested ATUT}\,(T=4) \\
0ATUT\,(T=5,p=0.4)\,(NS) & \\
+1ATUT\,(T=5,p=0.6) & \\
=0\,(60)+1\,(60)=60 & \textit{suggested ATUT}\,(T=5)
\end{array}
\tag{3T}
$$

$$
\begin{array}{rl}
\frac{1}{6}ATUT\,(T=1) & \\
+\frac{1}{6}ATUT\,(T=2) & \\
+\frac{1}{3}ATUT\,(T=3) & \textit{ATUT p-score matched sample} \\
+\frac{1}{6}ATUT\,(T=4) & \\
+\frac{1}{6}ATUT\,(T=5) & \\
=\frac{1}{6}\,(32)+\frac{1}{6}\,(-6) & \\
+\frac{1}{3}\,(26)+\frac{1}{6}\,(78) & \textit{suggested ATUT} \\
+\frac{1}{6}\,(60)=36 &
\end{array}
\tag{3}
$$

$$
\begin{array}{ll}
g_1\left(T=1,p=0.6\right)\left(NS\right) & \\
-g_0\left(T=1,p=0.6\right)\left(NS\right) & \textit{ATE p-score matched sample} \\
=11-\left(-21\right)=32 & \textit{suggested } ATE\left(T=1,p=0.6\right) \\
g_1\left(T=2,p=0.6\right)\left(NS\right) & \\
-g_0\left(T=2,p=0.6\right)\left(NS\right) & \\
=2-8=-6 & \textit{suggested } ATE\left(T=2,p=0.6\right) \\
g_1\left(T=3,p=0.6\right)\left(NS\right) & \\
-g_0\left(T=3,p=0.6\right)\left(NS\right) & \\
=13-\left(-13\right)=26\left(NA\right) & \textit{suggested } ATE\left(T=3,p=0.6\right) \\
g_1\left(T=4,p=0.6\right)\left(NS\right) & \\
-g_0\left(T=4,p=0.6\right)\left(NS\right) & \\
=44-\left(-34\right)=78 & \textit{suggested } ATE\left(T=4,p=0.6\right) \\
g_1\left(T=5,p=0.6\right) & \\
-g_0\left(T=5,p=0.6\right) & \\
=35-\left(-25\right)=60 & \textit{suggested } ATE\left(T=5,p=0.6\right)
\end{array}
\tag{3Tp}
$$

$$
\begin{array}{ll}
g_1\left(T=1,p=0.4\right)\left(NS\right) & \\
-g_0\left(T=1,p=0.4\right)\left(NS\right) & \\
=11-\left(-21\right)=32\left(NA\right) & \textit{suggested } ATE\left(T=1,p=0.4\right) \\
g_1\left(T=2,p=0.4\right)\left(NS\right) & \\
-g_0\left(T=2,p=0.4\right)\left(NS\right) & \\
=2-8=-6 & \textit{suggested } ATE\left(T=2,p=0.4\right) \\
g_1\left(T=3,p=0.4\right) & \\
-g_0\left(T=3,p=0.4\right) & \\
=13-\left(-13\right)=26 & \textit{suggested } ATE\left(T=3,p=0.4\right) \\
g_1\left(T=4,p=0.4\right)\left(NS\right) & \\
-g_0\left(T=4,p=0.4\right)\left(NS\right) & \\
=44-\left(-34\right)=78 & \textit{suggested } ATE\left(T=4,p=0.4\right) \\
g_1\left(T=5,p=0.4\right)\left(NS\right) & \\
-g_0\left(T=5,p=0.4\right)\left(NS\right) & \\
=35-\left(-25\right)=60\left(NA\right) & \textit{suggested } ATE\left(T=5,p=0.4\right)
\end{array}
\tag{3Tp}
$$

$$
\begin{aligned}
&\tfrac{1}{6}ATE\,(T=1,p=0.4)\,(NS)\\
+&\tfrac{1}{6}ATE\,(T=2,p=0.4)\,(NS)\\
&+\tfrac{1}{2}ATE\,(T=3,p=0.4) && ATE\ p\text{-}score\ matched\ sample\\
+&\tfrac{1}{6}ATE\,(T=4,p=0.4)\,(NS)\\
+&0ATE\,(T=5,p=0.4)\,(NS)\\
&=\tfrac{1}{6}\,(32)+\tfrac{1}{6}\,(-6)\\
&+\tfrac{1}{2}\,(26)+\tfrac{1}{6}\,(78) && suggested\ ATE\,(p=0.4)\\
&+0\,(60)=30\tfrac{1}{3}\\
&\tfrac{1}{6}ATE\,(T=1,p=0.6)\,(NS) && (3\mathrm{p})\\
+&\tfrac{1}{6}ATE\,(T=2,p=0.6)\,(NS)\\
+&\tfrac{1}{6}ATE\,(T=3,p=0.6)\,(NS)\\
+&\tfrac{1}{6}ATE\,(T=4,p=0.6)\,(NS)\\
&+\tfrac{1}{3}ATE\,(T=5,p=0.6)\\
&=\tfrac{1}{6}\,(32)+\tfrac{1}{6}\,(-6)\\
&+\tfrac{1}{6}\,(26)+\tfrac{1}{6}\,(78) && suggested\ ATE\,(p=0.6)\\
&+\tfrac{1}{3}\,(60)=41\tfrac{2}{3}
\end{aligned}
$$

$$
\begin{array}{rl}
\frac{1}{2}ATE\left(T=1,p=0.4\right)\left(NS\right) & \\
+\frac{1}{2}ATE\left(T=1,p=0.6\right)\left(NS\right) & \textit{ATE p-score matched sample} \\
=\frac{1}{2}\left(32\right)+\frac{1}{2}\left(32\right)=32 & \textit{suggested ATE}\left(T=1\right) \\
\frac{1}{2}ATE\left(T=2,p=0.4\right)\left(NS\right) & \\
+\frac{1}{2}ATE\left(T=2,p=0.6\right)\left(NS\right) & \\
=\frac{1}{2}\left(-6\right)+\frac{1}{2}\left(-6\right)=-6 & \textit{suggested ATE}\left(T=2\right) \\
\frac{3}{4}ATE\left(T=3,p=0.4\right) & \\
+\frac{1}{4}ATE\left(T=3,p=0.6\right)\left(NS\right) & \\
=\frac{3}{4}\left(26\right)+\frac{1}{4}\left(26\right)=26 & \textit{suggested ATE}\left(T=3\right) \\
\frac{1}{2}ATE\left(T=4,p=0.4\right)\left(NS\right) & \\
+\frac{1}{2}ATE\left(T=4,p=0.6\right)\left(NS\right) & \\
=\frac{1}{2}\left(78\right)+\frac{1}{2}\left(78\right)=78 & \textit{suggested ATE}\left(T=4\right) \\
0ATE\left(T=5,p=0.4\right)\left(NS\right) & \\
+1ATE\left(T=5,p=0.6\right) & \\
=0\left(60\right)+1\left(60\right)=60 & \textit{suggested ATE}\left(T=5\right)
\end{array}
\tag{3T}
$$

$$
\begin{array}{rl}
\frac{1}{6}ATE\left(T=1\right) & \\
+\frac{1}{6}ATE\left(T=2\right) & \\
+\frac{1}{3}ATE\left(T=3\right) & \textit{ATE p-score matched sample} \\
+\frac{1}{6}ATE\left(T=4\right) & \\
+\frac{1}{6}ATE\left(T=5\right) & \\
=\frac{1}{6}\left(32\right)+\frac{1}{6}\left(-6\right) & \\
+\frac{1}{3}\left(26\right)+\frac{1}{6}\left(78\right) & \textit{suggested ATE} \\
+\frac{1}{6}\left(60\right)=36 &
\end{array}
\tag{3}
$$

Average treatment effects are not identified except for ATT. Lack of common support is subtle and delicate in this case. Dropping observations for which observable common support for (T,p) is lacking results in a drastically smaller sample (only 5 of 12 potential observations are included). Both designs 2a and 3 produce the same identification results for ATT, ATUT and ATE propensity-

score matched samples.

$$\begin{aligned} Y &= 5-25D-6T+17T\times D+(ND)\,\Im(p) \\ &\quad +(ND)\,\Im(p)\times D+(ND)\,T\times\Im(p) \\ &\quad +(ND)\,T\times\Im(p)\times D+\varepsilon_{2a} \qquad (1) \\ ATT\,(T=3) &= ATUT\,(T=3)=ATE\,(T=3) \qquad (2a) \\ &= -25+17\,(3)=26 \\ ATT\,(T=5) &= ATUT\,(T=5)=ATE\,(T=5) \\ &= -25+17\,(5)=60 \end{aligned}$$

and

$$\begin{aligned} ATT\,(T=3) &= ATUT\,(T=3)=ATE\,(T=3) \\ &= 13-(-13)=26 \qquad (3) \\ ATT\,(T=5) &= ATUT\,(T=5)=ATE\,(T=5) \\ &= 35-(-25)=60 \end{aligned}$$

As conditional mean independence is not satisfied by this limited DGP, average treatment effects are not identified with the seemingly fortuitous exception of $ATT\,(T=3)$.

Example 5 (nonlinear conditional expectations) *Suppose the DGP is*

	D	$P(Z)$	T_1	$g_1(T)$	U_1	Y_1	T_0	$g_0(T)$	U_0	Y_0	Y	T
	1	0.6	2	$2\frac{1}{3}$	$-\frac{1}{3}$	2	2	$-\frac{1}{3}$	$\frac{1}{3}$	0	2	2
	1	0.6	3	$1\frac{1}{3}$	$\frac{2}{3}$	2	3	$-1\frac{1}{3}$	$-\frac{2}{3}$	-2	2	3
	1	0.6	4	$6\frac{1}{3}$	$-\frac{1}{3}$	6	4	$3\frac{2}{3}$	$\frac{1}{3}$	4	6	4
	1	0.4	2	$2\frac{1}{3}$	$-\frac{1}{3}$	2	2	$-\frac{1}{3}$	$\frac{1}{3}$	0	2	2
	1	0.4	3	$1\frac{1}{3}$	$\frac{2}{3}$	2	3	$-1\frac{1}{3}$	$-\frac{2}{3}$	-2	2	3
	1	0.4	4	$6\frac{1}{3}$	$-\frac{1}{3}$	6	4	$3\frac{2}{3}$	$\frac{1}{3}$	4	6	4
	0	0.6	2	$2\frac{1}{3}$	$-\frac{1}{3}$	2	2	$-\frac{1}{3}$	$\frac{1}{3}$	0	0	2
	0	0.6	3	$1\frac{1}{3}$	$\frac{2}{3}$	2	3	$-1\frac{1}{3}$	$-\frac{2}{3}$	-2	-2	3
	0	0.6	4	$6\frac{1}{3}$	$-\frac{1}{3}$	6	4	$3\frac{2}{3}$	$\frac{1}{3}$	4	4	4
	0	0.4	2	$2\frac{1}{3}$	$-\frac{1}{3}$	2	2	$-\frac{1}{3}$	$\frac{1}{3}$	0	0	2
	0	0.4	3	$1\frac{1}{3}$	$\frac{2}{3}$	2	3	$-1\frac{1}{3}$	$-\frac{2}{3}$	-2	-2	3
	0	0.4	4	$6\frac{1}{3}$	$-\frac{1}{3}$	6	4	$3\frac{2}{3}$	$\frac{1}{3}$	4	4	4
means	0.5	0.5	3	$3\frac{1}{3}$	0	$3\frac{1}{3}$	3	$\frac{2}{3}$	0	$\frac{2}{3}$	2	3

where unlike the foregoing examples propensity-score, $P(Z)$*, is balanced across treatment regimes* $D=0,1$*. Conditional and unconditional average treatment*

effects for this DGP are

conditioning	$ATT(\cdot)$	$ATUT(\cdot)$	$ATE(\cdot)$
$\Im(p)=1$	$3\frac{1}{3}-\frac{2}{3}=2\frac{2}{3}$	$3\frac{1}{3}-\frac{2}{3}=2\frac{2}{3}$	$3\frac{1}{3}-\frac{2}{3}=2\frac{2}{3}$
$\Im(p)=0$	$3\frac{1}{3}-\frac{2}{3}=2\frac{2}{3}$	$3\frac{1}{3}-\frac{2}{3}=2\frac{2}{3}$	$3\frac{1}{3}-\frac{2}{3}=2\frac{2}{3}$
$T=2$	$2-0=2$	$2-0=2$	$2-0=2$
$T=3$	$2-(-2)=4$	$2-(-2)=4$	$2-(-2)=4$
$T=4$	$6-4=2$	$6-4=2$	$6-4=2$
$\Im(p)=1, T=2$	$2-0=2$	$2-0=2$	$2-0=2$
$\Im(p)=1, T=3$	$2-(-2)=4$	$2-(-2)=4$	$2-(-2)=4$
$\Im(p)=1, T=4$	$6-4=2$	$6-4=2$	$6-4=2$
$\Im(p)=0, T=2$	$2-0=2$	$2-0=2$	$2-0=2$
$\Im(p)=0, T=3$	$2-(-2)=4$	$2-(-2)=4$	$2-(-2)=4$
$\Im(p)=0, T=4$	$6-4=2$	$6-4=2$	$6-4=2$
none	$3\frac{1}{3}-\frac{2}{3}=2\frac{2}{3}$	$3\frac{1}{3}-\frac{2}{3}=2\frac{2}{3}$	$3\frac{1}{3}-\frac{2}{3}=2\frac{2}{3}$

Design one yields

$$\begin{array}{cc} Y=-5\frac{1}{3}+2\frac{2}{3}D+2T+\varepsilon_{1a} & ATT \text{ p-score matched sample} \\ 2\frac{2}{3} & \text{suggested } ATT \\ Y=-5\frac{1}{3}+2\frac{2}{3}D+2T+\varepsilon_{1a} & ATUT \text{ p-score matched sample} \\ 2\frac{2}{3} & \text{suggested } ATUT \\ Y=-5\frac{1}{3}+2\frac{2}{3}D+2T+\varepsilon_{1a} & ATE \text{ p-score matched sample} \\ 2\frac{2}{3} & \text{suggested } ATE \end{array} \tag{1a}$$

and

$$
\begin{array}{cc}
Y = \frac{2}{3} + 2\frac{2}{3}D + 0\Im(p) & \\
+0\Im(p) \times D + \varepsilon_{1b} & \textit{ATT p-score matched sample} \\
2\frac{2}{3} & \textit{suggested ATT } (p = 0.4) \\
2\frac{2}{3} + 0 = 2\frac{2}{3} & \textit{suggested ATT } (p = 0.6) \\
2\frac{2}{3} + 0\left(\frac{1}{2}\right) = 2\frac{2}{3} & \textit{suggested ATT} \\
Y = \frac{2}{3} + 2\frac{2}{3}D + 0\Im(p) & \\
+0\Im(p) \times D + \varepsilon_{1b} & \textit{ATUT p-score matched sample} \\
2\frac{2}{3} & \textit{suggested ATUT } (p = 0.4) \\
2\frac{2}{3} + 0 = 2\frac{2}{3} & \textit{suggested ATUT } (p = 0.6) \\
2\frac{2}{3} + 0\left(\frac{1}{2}\right) = 2\frac{2}{3} & \textit{suggested ATUT} \\
Y = \frac{2}{3} + 2\frac{2}{3}D + 0\Im(p) & \\
+0\Im(p) \times D + \varepsilon_{1b} & \textit{ATE p-score matched sample} \\
2\frac{2}{3} & \textit{suggested ATE } (p = 0.4) \\
2\frac{2}{3} + 0 = 2\frac{2}{3} & \textit{suggested ATE } (p = 0.6) \\
2\frac{2}{3} + 0\left(\frac{1}{2}\right) = 2\frac{2}{3} & \textit{suggested ATE}
\end{array}
\tag{1b}
$$

Design two yields

$$
\begin{array}{cc}
Y = -5\frac{1}{3} + 2\frac{2}{3}D + 2T + 0T \times D & \\
+0\Im(p) + 0\Im(p) \times D & \\
+0T \times \Im(p) & \textit{ATT p-score matched sample} \\
+0T \times \Im(p) \times D + \varepsilon_{2a} & \\
2\frac{2}{3} + 0(3) = 2\frac{2}{3} & \textit{suggested ATT } (p = 0.4) \\
2\frac{2}{3} + 0(3) + 0 + 0(3) = 2\frac{2}{3} & \textit{suggested ATT } (p = 0.6) \\
2\frac{2}{3} + 0(2) + 0\left(\frac{1}{2}\right) + 0\left(2 \times \frac{1}{2}\right) = 2\frac{2}{3} & \textit{suggested ATT } (T = 2) \\
2\frac{2}{3} + 0(3) + 0\left(\frac{1}{2}\right) + 0\left(3 \times \frac{1}{2}\right) = 2\frac{2}{3} & \textit{suggested ATT } (T = 3) \\
2\frac{2}{3} + 0(4) + 0\left(\frac{1}{2}\right) + 0\left(4 \times \frac{1}{2}\right) = 2\frac{2}{3} & \textit{suggested ATT } (T = 4) \\
2\frac{2}{3} + 0(2) = 2\frac{2}{3} & \textit{suggested ATT } (p = 0.4, T = 2) \\
2\frac{2}{3} + 0(2) + 0 + 0(2) = 2\frac{2}{3} & \textit{suggested ATT } (p = 0.6, T = 2) \\
2\frac{2}{3} + 0(3) = 2\frac{2}{3} & \textit{suggested ATT } (p = 0.4, T = 3) \\
2\frac{2}{3} + 0(3) + 0 + 0(3) = 2\frac{2}{3} & \textit{suggested ATT } (p = 0.6, T = 3) \\
2\frac{2}{3} + 0(4) = 2\frac{2}{3} & \textit{suggested ATT } (p = 0.4, T = 4) \\
2\frac{2}{3} + 0(4) + 0 + 0(4) = 2\frac{2}{3} & \textit{suggested ATT } (p = 0.6, T = 4) \\
2\frac{2}{3} + 0(3) + 0\left(\frac{1}{2}\right) + 0\left(1\frac{1}{2}\right) = 2\frac{2}{3} & \textit{suggested ATT}
\end{array}
\tag{2a}
$$

$$
\begin{array}{cc}
Y = -5\frac{1}{3} + 2\frac{2}{3}D + 2T + 0T \times D & \\
+0\Im\left(p\right) + 0\Im\left(p\right) \times D & \\
+0T \times \Im\left(p\right) & \textit{ATUT p-score matched sample} \\
+0T \times \Im\left(p\right) \times D + \varepsilon_{2a} & \\
2\frac{2}{3} + 0\left(3\right) = 2\frac{2}{3} & \textit{suggested ATUT}\left(p = 0.4\right) \\
2\frac{2}{3} + 0\left(3\right) + 0 + 0\left(3\right) = 2\frac{2}{3} & \textit{suggested ATUT}\left(p = 0.6\right) \\
2\frac{2}{3} + 0\left(2\right) + 0\left(\frac{1}{2}\right) + 0\left(2 \times \frac{1}{2}\right) = 2\frac{2}{3} & \textit{suggested ATUT}\left(T = 2\right) \\
2\frac{2}{3} + 0\left(3\right) + 0\left(\frac{1}{2}\right) + 0\left(3 \times \frac{1}{2}\right) = 2\frac{2}{3} & \textit{suggested ATUT}\left(T = 3\right) \\
2\frac{2}{3} + 0\left(4\right) + 0\left(\frac{1}{2}\right) + 0\left(4 \times \frac{1}{2}\right) = 2\frac{2}{3} & \textit{suggested ATUT}\left(T = 4\right) \\
2\frac{2}{3} + 0\left(2\right) = 2\frac{2}{3} & \textit{suggested ATUT}\left(\begin{array}{c} p = 0.4, \\ T = 2 \end{array}\right) \\
2\frac{2}{3} + 0\left(2\right) + 0 + 0\left(2\right) = 2\frac{2}{3} & \textit{suggested ATUT}\left(\begin{array}{c} p = 0.6, \\ T = 2 \end{array}\right) \\
2\frac{2}{3} + 0\left(3\right) = 2\frac{2}{3} & \textit{suggested ATUT}\left(\begin{array}{c} p = 0.4, \\ T = 3 \end{array}\right) \\
2\frac{2}{3} + 0\left(3\right) + 0 + 0\left(3\right) = 2\frac{2}{3} & \textit{suggested ATUT}\left(\begin{array}{c} p = 0.6, \\ T = 3 \end{array}\right) \\
2\frac{2}{3} + 0\left(4\right) = 2\frac{2}{3} & \textit{suggested ATUT}\left(\begin{array}{c} p = 0.4, \\ T = 4 \end{array}\right) \\
2\frac{2}{3} + 0\left(4\right) + 0 + 0\left(4\right) = 2\frac{2}{3} & \textit{suggested ATUT}\left(\begin{array}{c} p = 0.6, \\ T = 4 \end{array}\right) \\
2\frac{2}{3} + 0\left(3\right) + 0\left(\frac{1}{2}\right) + 0\left(1\frac{1}{2}\right) = 2\frac{2}{3} & \textit{suggested ATUT}
\end{array}
\tag{2a}
$$

$$
\begin{aligned}
Y = -5\tfrac{1}{3} + 2\tfrac{2}{3}D + 2T + 0T \times D & \\
+0\Im(p) + 0\Im(p) \times D & \\
+0T \times \Im(p) & \quad ATE \text{ p-score matched sample} \\
+0T \times \Im(p) \times D + \varepsilon_{2a} & \\
2\tfrac{2}{3} + 0(3) = 2\tfrac{2}{3} & \quad \text{suggested } ATE\,(p = 0.4) \\
2\tfrac{2}{3} + 0(3) + 0 + 0(3) = 2\tfrac{2}{3} & \quad \text{suggested } ATE\,(p = 0.6) \\
2\tfrac{2}{3} + 0(2) + 0\left(\tfrac{1}{2}\right) + 0\left(2 \times \tfrac{1}{2}\right) = 2\tfrac{2}{3} & \quad \text{suggested } ATE\,(T = 2) \\
2\tfrac{2}{3} + 0(3) + 0\left(\tfrac{1}{2}\right) + 0\left(3 \times \tfrac{1}{2}\right) = 2\tfrac{2}{3} & \quad \text{suggested } ATE\,(T = 3) \\
2\tfrac{2}{3} + 0(4) + 0\left(\tfrac{1}{2}\right) + 0\left(4 \times \tfrac{1}{2}\right) = 2\tfrac{2}{3} & \quad \text{suggested } ATE\,(T = 4) \\
2\tfrac{2}{3} + 0(2) = 2\tfrac{2}{3} & \quad \text{suggested } ATE \begin{pmatrix} p = 0.4, \\ T = 2 \end{pmatrix} \qquad \text{(2a)} \\
2\tfrac{2}{3} + 0(2) + 0 + 0(2) = 2\tfrac{2}{3} & \quad \text{suggested } ATE \begin{pmatrix} p = 0.6, \\ T = 2 \end{pmatrix} \\
2\tfrac{2}{3} + 0(3) = 2\tfrac{2}{3} & \quad \text{suggested } ATE \begin{pmatrix} p = 0.4, \\ T = 3 \end{pmatrix} \\
2\tfrac{2}{3} + 0(3) + 0 + 0(3) = 2\tfrac{2}{3} & \quad \text{suggested } ATE \begin{pmatrix} p = 0.6, \\ T = 3 \end{pmatrix} \\
2\tfrac{2}{3} + 0(4) = 2\tfrac{2}{3} & \quad \text{suggested } ATE \begin{pmatrix} p = 0.4, \\ T = 4 \end{pmatrix} \\
2\tfrac{2}{3} + 0(4) + 0 + 0(4) = 2\tfrac{2}{3} & \quad \text{suggested } ATE \begin{pmatrix} p = 0.6, \\ T = 4 \end{pmatrix} \\
2\tfrac{2}{3} + 0(3) + 0\left(\tfrac{1}{2}\right) + 0\left(1\tfrac{1}{2}\right) = 2\tfrac{2}{3} & \quad \text{suggested } ATE
\end{aligned}
$$

and

$$
\begin{aligned}
&Y = -5\tfrac{1}{3} + 2\tfrac{2}{3}D + 0T && \\
&\quad +0T \times D + \varepsilon_{2b} && ATT \text{ p-score matched sample} \\
&\quad 2\tfrac{2}{3} + 0\,(2) = 2\tfrac{2}{3} && \text{suggested } ATT\,(T=2) \\
&\quad 2\tfrac{2}{3} + 0\,(3) = 2\tfrac{2}{3} && \text{suggested } ATT\,(T=3) \\
&\quad 2\tfrac{2}{3} + 0\,(4) = 2\tfrac{2}{3} && \text{suggested } ATT\,(T=4) \\
&\quad 2\tfrac{2}{3} + 0\,(3) = 2\tfrac{2}{3} && \text{suggested } ATT \\
&Y = -5\tfrac{1}{3} + 2\tfrac{2}{3}D + 0T && \\
&\quad +0T \times D + \varepsilon_{2b} && ATUT \text{ p-score matched sample} \\
&\quad 2\tfrac{2}{3} + 0\,(2) = 2\tfrac{2}{3} && \text{suggested } ATUT\,(T=2) \\
&\quad 2\tfrac{2}{3} + 0\,(3) = 2\tfrac{2}{3} && \text{suggested } ATUT\,(T=3) \\
&\quad 2\tfrac{2}{3} + 0\,(4) = 2\tfrac{2}{3} && \text{suggested } ATUT\,(T=4) \\
&\quad 2\tfrac{2}{3} + 0\,(3) = 2\tfrac{2}{3} && \text{suggested } ATUT \\
&Y = -5\tfrac{1}{3} + 2\tfrac{2}{3}D + 0T && \\
&\quad +0T \times D + \varepsilon_{2b} && ATE \text{ p-score matched sample} \\
&\quad 2\tfrac{2}{3} + 0\,(2) = 2\tfrac{2}{3} && \text{suggested } ATE\,(T=2) \\
&\quad 2\tfrac{2}{3} + 0\,(3) = 2\tfrac{2}{3} && \text{suggested } ATE\,(T=3) \\
&\quad 2\tfrac{2}{3} + 0\,(4) = 2\tfrac{2}{3} && \text{suggested } ATE\,(T=4) \\
&\quad 2\tfrac{2}{3} + 0\,(3) = 2\tfrac{2}{3} && \text{suggested } ATE
\end{aligned}
\tag{2b}
$$

Design 3 yields

$$
\begin{array}{ll}
g_1\left(T=2,p=0.6\right) & \\
-g_0\left(T=2,p=0.6\right) & \textit{ATT p-score matched sample} \\
=2-0=2 & \textit{suggested } ATT\left(T=2,p=0.6\right) \\
g_1\left(T=3,p=0.6\right) & \\
-g_0\left(T=3,p=0.6\right) & \\
=2-(-2)=4 & \textit{suggested } ATT\left(T=3,p=0.6\right) \\
g_1\left(T=4,p=0.6\right) & \\
-g_0\left(T=4,p=0.6\right) & \\
=4-2=2 & \textit{suggested } ATT\left(T=4,p=0.6\right) \\
g_1\left(T=2,p=0.4\right) & \\
-g_0\left(T=2,p=0.4\right) & \\
=2-0=2 & \textit{suggested } ATT\left(T=2,p=0.4\right) \\
g_1\left(T=3,p=0.4\right) & \\
-g_0\left(T=3,p=0.4\right) & \\
=2-(-2)=4 & \textit{suggested } ATT\left(T=3,p=0.4\right) \\
g_1\left(T=4,p=0.4\right) & \\
-g_0\left(T=4,p=0.4\right) & \\
=4-2=2 & \textit{suggested } ATT\left(T=4,p=0.4\right)
\end{array}
\tag{3Tp}
$$

$$
\begin{array}{ll}
\frac{1}{3}ATT\left(T=2,p=0.4\right) & \\
+\frac{1}{3}ATT\left(T=3,p=0.4\right) & \textit{ATT p-score matched sample} \\
+\frac{1}{3}ATT\left(T=4,p=0.4\right) & \\
=\frac{1}{3}\left(2\right)+\frac{1}{3}\left(4\right) & \\
+\frac{1}{3}\left(2\right)=2\frac{2}{3} & \textit{suggested } ATT\left(p=0.4\right) \\
\frac{1}{3}ATT\left(T=2,p=0.6\right) & \\
+\frac{1}{3}ATT\left(T=3,p=0.6\right) & \\
+\frac{1}{3}ATT\left(T=4,p=0.6\right) & \\
=\frac{1}{3}\left(2\right)+\frac{1}{3}\left(4\right) & \\
+\frac{1}{3}\left(2\right)=2\frac{2}{3} & \textit{suggested } ATT\left(p=0.6\right)
\end{array}
\tag{3p}
$$

$$
\begin{array}{ll}
\frac{1}{2}ATT\left(T=2,p=0.4\right) & \\
+\frac{1}{2}ATT\left(T=2,p=0.6\right) & \textit{ATT p-score matched sample} \\
=\frac{1}{2}\left(2\right)+\frac{1}{2}\left(2\right)=2 & \textit{suggested ATT}\left(T=2\right) \\
\frac{1}{2}ATT\left(T=3,p=0.4\right) & \\
+\frac{1}{2}ATT\left(T=3,p=0.6\right) & \\
=\frac{1}{2}\left(4\right)+\frac{1}{2}\left(4\right)=4 & \textit{suggested ATT}\left(T=3\right) \\
\frac{1}{2}ATT\left(T=4,p=0.4\right) & \\
+\frac{1}{2}ATT\left(T=4,p=0.6\right) & \\
=\frac{1}{2}\left(2\right)+\frac{1}{2}\left(2\right)=2 & \textit{suggested ATT}\left(T=4\right)
\end{array}
\tag{3T}
$$

$$
\begin{array}{ll}
\frac{1}{3}ATT\left(T=2\right) & \\
+\frac{1}{3}ATT\left(T=3\right) & \textit{ATT p-score matched sample} \\
+\frac{1}{3}ATT\left(T=4\right) & \\
=\frac{1}{3}\left(2\right)+\frac{1}{3}\left(4\right)+\frac{1}{3}\left(2\right)=2\frac{2}{3} & \textit{suggested ATT}
\end{array}
\tag{3}
$$

$$
\begin{array}{ll}
g_1\left(T=2,p=0.6\right) & \textit{ATUT p-score matched sample} \\
-g_0\left(T=2,p=0.6\right) & \\
=2-0=2 & \textit{suggested ATUT}\left(T=2,p=0.6\right) \\
g_1\left(T=3,p=0.6\right) & \\
-g_0\left(T=3,p=0.6\right) & \\
=2-\left(-2\right)=4 & \textit{suggested ATUT}\left(T=3,p=0.6\right) \\
g_1\left(T=4,p=0.6\right) & \\
-g_0\left(T=4,p=0.6\right) & \\
=4-2=2 & \textit{suggested ATUT}\left(T=4,p=0.6\right) \\
g_1\left(T=2,p=0.4\right) & \\
-g_0\left(T=2,p=0.4\right) & \\
=2-0=2 & \textit{suggested ATUT}\left(T=2,p=0.4\right) \\
g_1\left(T=3,p=0.4\right) & \\
-g_0\left(T=3,p=0.4\right) & \\
=2-\left(-2\right)=4 & \textit{suggested ATUT}\left(T=3,p=0.4\right) \\
g_1\left(T=4,p=0.4\right) & \\
-g_0\left(T=4,p=0.4\right) & \\
=4-2=2 & \textit{suggested ATUT}\left(T=4,p=0.4\right)
\end{array}
\tag{3Tp}
$$

$$
\begin{array}{ll}
\frac{1}{3}ATUT\,(T=2,p=0.4) & \\
+\frac{1}{3}ATUT\,(T=3,p=0.4) & \textit{ATUT p-score matched sample} \\
+\frac{1}{3}ATUT\,(T=4,p=0.4) & \\
\quad=\frac{1}{3}\,(2)+\frac{1}{3}\,(4) & \\
\quad+\frac{1}{3}\,(2)=2\frac{2}{3} & \textit{suggested ATUT}\,(p=0.4) \\
\frac{1}{3}ATUT\,(T=2,p=0.6) & \\
+\frac{1}{3}ATUT\,(T=3,p=0.6) & \\
+\frac{1}{3}ATUT\,(T=4,p=0.6) & \\
\quad=\frac{1}{3}\,(2)+\frac{1}{3}\,(4) & \\
\quad+\frac{1}{3}\,(2)=2\frac{2}{3} & \textit{suggested ATUT}\,(p=0.6)
\end{array}
\tag{3p}
$$

$$
\begin{array}{ll}
\frac{1}{2}ATUT\,(T=2,p=0.4) & \\
+\frac{1}{2}ATUT\,(T=2,p=0.6) & \textit{ATUT p-score matched sample} \\
\quad=\frac{1}{2}\,(2)+\frac{1}{2}\,(2)=2 & \textit{suggested ATUT}\,(T=2) \\
\frac{1}{2}ATUT\,(T=3,p=0.4) & \\
+\frac{1}{2}ATUT\,(T=3,p=0.6) & \\
\quad=\frac{1}{2}\,(4)+\frac{1}{2}\,(4)=4 & \textit{suggested ATUT}\,(T=3) \\
\frac{1}{2}ATUT\,(T=4,p=0.4) & \\
+\frac{1}{2}ATUT\,(T=4,p=0.6) & \\
\quad=\frac{1}{2}\,(2)+\frac{1}{2}\,(2)=2 & \textit{suggested ATUT}\,(T=4)
\end{array}
\tag{3T}
$$

$$
\begin{array}{ll}
\frac{1}{3}ATUT\,(T=2) & \\
+\frac{1}{3}ATUT\,(T=3) & \textit{ATUT p-score matched sample} \\
+\frac{1}{3}ATUT\,(T=4) & \\
=\frac{1}{3}\,(2)+\frac{1}{3}\,(4)+\frac{1}{3}\,(2)=2\frac{2}{3} & \textit{suggested ATUT}
\end{array}
\tag{3}
$$

and

$$
\begin{array}{ll}
g_1\left(T=2,p=0.6\right) & \\
-g_0\left(T=2,p=0.6\right) & \textit{ATE p-score matched sample} \\
=2-0=2 & \textit{suggested } ATE\left(T=2,p=0.6\right) \\
g_1\left(T=3,p=0.6\right) & \\
-g_0\left(T=3,p=0.6\right) & \\
=2-(-2)=4 & \textit{suggested } ATE\left(T=3,p=0.6\right) \\
g_1\left(T=4,p=0.6\right) & \\
-g_0\left(T=4,p=0.6\right) & \\
=4-2=2 & \textit{suggested } ATE\left(T=4,p=0.6\right) \\
g_1\left(T=2,p=0.4\right) & \\
-g_0\left(T=2,p=0.4\right) & \\
=2-0=2 & \textit{suggested } ATE\left(T=2,p=0.4\right) \\
g_1\left(T=3,p=0.4\right) & \\
-g_0\left(T=3,p=0.4\right) & \\
=2-(-2)=4 & \textit{suggested } ATE\left(T=3,p=0.4\right) \\
g_1\left(T=4,p=0.4\right) & \\
-g_0\left(T=4,p=0.4\right) & \\
=4-2=2 & \textit{suggested } ATE\left(T=4,p=0.4\right)
\end{array}
\tag{3Tp}
$$

$$
\begin{array}{ll}
\frac{1}{3}ATE\left(T=2,p=0.4\right) & \\
+\frac{1}{3}ATE\left(T=3,p=0.4\right) & \textit{ATE p-score matched sample} \\
+\frac{1}{3}ATE\left(T=4,p=0.4\right) & \\
=\frac{1}{3}\left(2\right)+\frac{1}{3}\left(4\right) & \\
+\frac{1}{3}\left(2\right)=2\frac{2}{3} & \textit{suggested } ATE\left(p=0.4\right) \\
\frac{1}{3}ATE\left(T=2,p=0.6\right) & \\
+\frac{1}{3}ATE\left(T=3,p=0.6\right) & \\
+\frac{1}{3}ATE\left(T=4,p=0.6\right) & \\
=\frac{1}{3}\left(2\right)+\frac{1}{3}\left(4\right) & \\
+\frac{1}{3}\left(2\right)=2\frac{2}{3} & \textit{suggested } ATE\left(p=0.6\right)
\end{array}
\tag{3p}
$$

$$
\begin{array}{ll}
\frac{1}{2}ATE\left(T=2,p=0.4\right) & \\
+\frac{1}{2}ATE\left(T=2,p=0.6\right) & \textit{ATE p-score matched sample} \\
=\frac{1}{2}\left(2\right)+\frac{1}{2}\left(2\right)=2 & \textit{suggested ATE}\left(T=2\right) \\
\frac{1}{2}ATE\left(T=3,p=0.4\right) & \\
+\frac{1}{2}ATE\left(T=3,p=0.6\right) & \\
=\frac{1}{2}\left(4\right)+\frac{1}{2}\left(4\right)=4 & \textit{suggested ATE}\left(T=3\right) \\
\frac{1}{2}ATE\left(T=4,p=0.4\right) & \\
+\frac{1}{2}ATE\left(T=4,p=0.6\right) & \\
=\frac{1}{2}\left(2\right)+\frac{1}{2}\left(2\right)=2 & \textit{suggested ATE}\left(T=4\right)
\end{array}
\tag{3T}
$$

$$
\begin{array}{ll}
\frac{1}{3}ATE\left(T=2\right) & \\
+\frac{1}{3}ATE\left(T=3\right) & \textit{ATE p-score matched sample} \\
+\frac{1}{3}ATE\left(T=4\right) & \\
=\frac{1}{3}\left(2\right)+\frac{1}{3}\left(4\right)+\frac{1}{3}\left(2\right)=2\frac{2}{3} & \textit{suggested ATE}
\end{array}
\tag{3}
$$

As potential outcomes and treatment effects are nonlinear in covariate T, only the nonparametric design identifies average treatment effects conditional on T. Unconditional average treatment effects are identified for this DGP by all designs considered but it is prudent to bear in mind the implications of unbalanced covariates and covariate means in the foregoing examples.

The examples suggest that common support for the covariates across the $D = 0$ and $D = 1$ subpopulations is quite subtle in regression of outcomes on covariates for propensity-score matched samples. Covariate balance seems to be a critical component of common support. Identification failure in example 4 is not a result of heterogenous outcome but rather can be readily replicated with homogeneous outcome where covariate means are unbalanced. Covariate imbalance manifests itself in violation of potential outcome conditional mean independence. Conditional average treatment effect identification rests on mean independence given the aforementioned conditions. Finally, nonlinearity in conditional expectations points to nonparametric identification rather than reliance on linear regression.

1.1 Double robustness and states of knowledge

Composite methods like regression on propensity-score matched samples are said to be "doubly robust" (*DR*, for instance, Bang and Robins [2005]). The idea is that if either the outcome model or propensity-score model are well specified then average treatment effects are robust to mis-specification of the other. Further, their analysis and simulations suggest that bias (and imprecision) is much smaller for *DR* strategies than regression or matching alone when models

are modestly mis-specified. However, the forgoing examples don't address or illustrate these advantages. Next, we explore two more examples: one in which propensity-score is properly specified but a linear outcome model is employed in the face of inherently nonlinear outcome in T and a second in which the DGP follows a linear outcome model but propensity-score is mis-specified.

Proper specification means logically consistent assessment based on the analyst's state of knowledge. In other words, we're not referring to some unknowable "objective truth." Relying on a false pretext of some unknowable objective truth stands in the way of meshing thought and data experiments inherent to causal effects analysis. On the other hand, logical consistency is no small matter but rather the bedrock of science — the unrelenting search for richer and deeper understanding. Probability assignment associated with counterfactual potential outcomes is a clear-cut case in point. We utilize our background knowledge to assign the probability distributions. While this can be daunting, it is manageable. Hence, when we speak of mis-specification we mean our analysis involves some logical inconsistency with our state of knowledge. In the case of DR strategies, this inconsistency/mis-specification may arise through the outcome model or the selection model as illustrated below.

Example 6 (DR with mis-specified outcome model) *Suppose the DGP is*

	D	$P(Z)$	T_1	Y_1	T_0	Y_0	Y	T
	1	0.6	3	0	3	0	0	3
	1	0.6	4	6	4	-6	6	4
	1	0.6	3	0	3	0	0	3
	1	0.6	4	6	4	-6	6	4
	1	0.6	3	0	3	0	0	3
	1	0.6	4	6	4	-6	6	4
	1	0.4	2	2	2	-2	2	2
	1	0.4	3	0	3	0	0	3
	0	0.6	3	0	3	0	0	3
	0	0.6	4	6	4	-6	-6	4
	0	0.4	2	2	2	-2	-2	2
	0	0.4	3	0	3	0	0	3
means	$\frac{2}{3}$	$\frac{8}{15}$	$3\frac{1}{6}$	$2\frac{1}{3}$	$3\frac{1}{6}$	$-2\frac{1}{3}$	1	$3\frac{1}{6}$

Conditional and unconditional average treatment effects for this DGP are

conditioning	$ATT(\cdot)$	$ATUT(\cdot)$	$ATE(\cdot)$
$\Im(p)=1$	$3-(-3)=6$	$3-(-3)=6$	$3-(-3)=6$
$\Im(p)=0$	$1-(-1)=2$	$1-(-1)=2$	$1-(-1)=2$
$T=2$	$2-(-2)=4$	$2-(-2)=4$	$2-(-2)=4$
$T=3$	$0-0=0$	$0-0=0$	$0-0=0$
$T=4$	$6-(-6)=12$	$6-(-6)=12$	$6-(-6)=12$
$\Im(p)=1, T=2$	$NA-NA=NA$	$NA-NA=NA$	$NA-NA=NA$
$\Im(p)=1, T=3$	$0-0=0$	$0-0=0$	$0-0=0$
$\Im(p)=1, T=4$	$6-(-6)=12$	$6-(-6)=12$	$6-(-6)=12$
$\Im(p)=0, T=2$	$2-0=2$	$2-0=2$	$2-0=2$
$\Im(p)=0, T=3$	$0-0=0$	$0-0=0$	$0-0=0$
$\Im(p)=0, T=4$	$NA-NA=NA$	$NA-NA=NA$	$NA-NA=NA$
none	$2\frac{1}{2}-\left(-2\frac{1}{2}\right)=5$	$2-(-2)=4$	$2\frac{1}{3}-\left(-2\frac{1}{3}\right)=4\frac{2}{3}$

As in example 5, potential outcomes for this DGP are clearly nonlinear in T, hence a linear outcome model is mis-specified but propensity-score is properly specified. Outcome mis-specification can arise from omitted or extraneous covariates in T or in functional form for $E[Y \mid T]$. ATT propensity-score matched samples have the following proportions

$$6:2:6:2$$

for

$$\begin{array}{rcl}(D=1,\Im(p)=1) & : & (D=1,\Im(p)=0): \\ (D=0,\Im(p)=1) & : & (D=0,\Im(p)=0)\end{array}$$

$ATUT$ propensity-score matched samples have proportions

$$6:6:6:6$$

for

$$\begin{array}{rcl}(D=1,\Im(p)=1) & : & (D=1,\Im(p)=0): \\ (D=0,\Im(p)=1) & : & (D=0,\Im(p)=0)\end{array}$$

and ATE propensity-score matched samples employment of ATT sample: $ATUT$

sample is $2:1$. *Design one yields*

$$\begin{array}{cc}
Y=-2\frac{1}{2}+5D+0T+\varepsilon_{1a} & \textit{ATT p-score matched sample} \\
5 & \textit{suggested ATT} \\
Y=-2+4D+0T+\varepsilon_{1a} & \textit{ATUT p-score matched sample} \\
4 & \textit{suggested ATUT} \\
Y=-2\frac{1}{3}+4\frac{2}{3}D+0T+\varepsilon_{1a} & \textit{ATE p-score matched sample} \\
4\frac{2}{3} & \textit{suggested ATE} \\
Y=-5\frac{9}{11}+4\frac{2}{11}D+1\frac{3}{11}T+\varepsilon_{1a} & \textit{unmatched sample} \\
4\frac{2}{11} & \textit{suggested ATE}
\end{array} \tag{1a}$$

and

$$\begin{array}{cc}
Y=-1+2D-2\Im(p) & \\
+4\Im(p)\times D+\varepsilon_{1b} & \textit{ATT p-score matched sample} \\
2 & \textit{suggested ATT}\,(p=0.4) \\
2+4=6 & \textit{suggested ATT}\,(p=0.6) \\
2+4\left(\frac{3}{4}\right)=5 & \textit{suggested ATT} \\
Y=-1+2D-2\Im(p) & \\
+4\Im(p)\times D+\varepsilon_{1b} & \textit{ATUT p-score matched sample} \\
2 & \textit{suggested ATUT}\,(p=0.4) \\
2+4=6 & \textit{suggested ATUT}\,(p=0.6) \\
2+4\left(\frac{1}{2}\right)=4 & \textit{suggested ATUT} \\
Y=-1+2D-2\Im(p) & \\
+4\Im(p)\times D+\varepsilon_{1b} & \textit{ATE p-score matched sample} \\
2 & \textit{suggested ATE}\,(p=0.4) \\
2+4=6 & \textit{suggested ATE}\,(p=0.6) \\
2+4\left(\frac{2}{3}\right)=4\frac{2}{3} & \textit{suggested ATE} \\
Y=-1+2D-2\Im(p) & \\
+4\Im(p)\times D+\varepsilon_{1b} & \textit{unmatched sample} \\
2 & \textit{suggested ATE}\,(p=0.4) \\
2+4=6 & \textit{suggested ATE}\,(p=0.6) \\
2+4\left(\frac{2}{3}\right)=4\frac{2}{3} & \textit{suggested ATE}
\end{array} \tag{1b}$$

Design one illustrates the advantage of DR strategies. Unmatched design 1a fails to identify ATE as potential outcomes are nonlinear in T. However, design 1a based on propensity-score matched samples identify average treatment effects as propensity-score is well specified. Design two yields

$$
\begin{array}{cc}
Y = -6 + 12D + 2T - 4T \times D & \\
+24\Im(p) - 48\Im(p) \times D & \textit{ATT p-score matched sample} \\
-8T \times \Im(p) & \\
+16T \times \Im(p) \times D + \varepsilon_{2a} & \\
12 - 4\left(2\tfrac{1}{2}\right) = 2 & \textit{suggested ATT } (p = 0.4) \\
12 - 4\left(3\tfrac{1}{2}\right) - 48 + 16\left(3\tfrac{1}{2}\right) = 6 & \textit{suggested ATT } (p = 0.6) \\
12 - 4(2) - 48(0) + 16(2 \times 0) = 4 & \textit{suggested ATT } (T = 2) \\
12 - 4(3) - 48\left(\tfrac{3}{4}\right) & \\
+16\left(3 \times \tfrac{3}{4}\right) = 0 & \textit{suggested ATT } (T = 3) \\
12 - 4(4) - 48(1) + 16(4 \times 1) = 12 & \textit{suggested ATT } (T = 4) \\
12 - 4(2) = 4 & \textit{suggested ATT } (p = 0.4, T = 2) \\
12 - 4(2) - 48 & \\
+16(2) = -12\,(NA) & \textit{suggested ATT } (p = 0.6, T = 2) \\
12 - 4(3) = 0 & \textit{suggested ATT } (p = 0.4, T = 3) \\
12 - 4(3) - 48 + 16(3) = 0 & \textit{suggested ATT } (p = 0.6, T = 3) \\
12 - 4(4) = -4\,(NA) & \textit{suggested ATT } (p = 0.4, T = 4) \\
12 - 4(4) - 48 + 16(4) = 12 & \textit{suggested ATT } (p = 0.6, T = 4) \\
12 - 4\left(3\tfrac{1}{4}\right) - 48\left(\tfrac{3}{4}\right) & \\
+16\left(2\tfrac{5}{8}\right) = 5 & \textit{suggested ATT}
\end{array}
\tag{2a}
$$

$$
\begin{array}{cc}
Y = -6 + 12D + 2T - 4T \times D & \\
+24\Im(p) - 48\Im(p) \times D & \textit{ATUT p-score matched sample} \\
-8T \times \Im(p) & \\
+16T \times \Im(p) \times D + \varepsilon_{2a} & \\
12 - 4\left(2\tfrac{1}{2}\right) = 2 & \textit{suggested ATUT}\,(p = 0.4) \\
12 - 4\left(3\tfrac{1}{2}\right) - 48 + 16\left(3\tfrac{1}{2}\right) = 6 & \textit{suggested ATUT}\,(p = 0.6) \\
12 - 4(2) - 48(0) + 16(2 \times 0) = 4 & \textit{suggested ATUT}\,(T = 2) \\
12 - 4(3) - 48\left(\tfrac{1}{2}\right) + 16\left(3 \times \tfrac{1}{2}\right) = 0 & \textit{suggested ATUT}\,(T = 3) \\
12 - 4(4) - 48(1) + 16(4 \times 1) = 10 & \textit{suggested ATUT}\,(T = 4) \\
12 - 4(2) = 4 & \textit{suggested ATUT}\begin{pmatrix} p = 0.4, \\ T = 2 \end{pmatrix} \\
12 - 4(2) - 48 + 16(2) = -12\,(NA) & \textit{suggested ATUT}\begin{pmatrix} p = 0.6, \\ T = 2 \end{pmatrix} \\
12 - 4(3) = 0 & \textit{suggested ATUT}\begin{pmatrix} p = 0.4, \\ T = 3 \end{pmatrix} \\
12 - 4(3) - 48 + 16(3) = 0 & \textit{suggested ATUT}\begin{pmatrix} p = 0.6, \\ T = 3 \end{pmatrix} \\
12 - 4(4) = -4\,(NA) & \textit{suggested ATUT}\begin{pmatrix} p = 0.4, \\ T = 4 \end{pmatrix} \\
12 - 4(4) - 48 + 16(4) = 12 & \textit{suggested ATUT}\begin{pmatrix} p = 0.6, \\ T = 4 \end{pmatrix} \\
12 - 4(3) - 48\left(\tfrac{1}{2}\right) + 16\left(1\tfrac{3}{4}\right) = 4 & \textit{suggested ATUT}
\end{array}
\tag{2a}
$$

$$
\begin{array}{cc}
Y = -6 + 12D + 2T - 4T \times D & \\
+24\Im(p) - 48\Im(p) \times D & \\
-8T \times \Im(p) & ATE \text{ p-score matched sample} \\
+16T \times \Im(p) \times D + \varepsilon_{2a} & \\
12 - 4\left(2\tfrac{1}{2}\right) = 2 & \text{suggested } ATE\,(p = 0.4) \\
12 - 4\left(3\tfrac{1}{2}\right) - 48 & \\
+16\left(3\tfrac{1}{2}\right) = 6 & \text{suggested } ATE\,(p = 0.6) \\
12 - 4(2) - 48(0) & \\
+16(2 \times 0) = 4 & \text{suggested } ATE\,(T = 2) \\
12 - 4(3) - 48\left(\tfrac{2}{3}\right) & \\
+16\left(3 \times \tfrac{2}{3}\right) = 0 & \text{suggested } ATE\,(T = 3) \\
12 - 4(4) - 48(1) & \\
+16(4 \times 1) = 12 & \text{suggested } ATE\,(T = 4) \\
12 - 4(2) = 4 & \text{suggested } ATE \left(\begin{array}{c} p = 0.4, \\ T = 2 \end{array} \right) \\
12 - 4(2) - 48 & \\
+16(2) = -12\,(NA) & \text{suggested } ATE \left(\begin{array}{c} p = 0.6, \\ T = 2 \end{array} \right) \\
12 - 4(3) = 0 & \text{suggested } ATE \left(\begin{array}{c} p = 0.4, \\ T = 3 \end{array} \right) \\
12 - 4(3) - 48 + 16(3) = 0 & \text{suggested } ATE \left(\begin{array}{c} p = 0.6, \\ T = 3 \end{array} \right) \\
12 - 4(4) = -4\,(NA) & \text{suggested } ATE \left(\begin{array}{c} p = 0.4, \\ T = 4 \end{array} \right) \\
12 - 4(4) - 48 + 16(4) = 12 & \text{suggested } ATE \left(\begin{array}{c} p = 0.6, \\ T = 4 \end{array} \right) \\
12 - 4\left(3\tfrac{1}{6}\right) - 48\left(\tfrac{2}{3}\right) & \\
+16\left(2\tfrac{1}{3}\right) = 4\tfrac{2}{3} & \text{suggested } ATE
\end{array}
\tag{2a}
$$

$$
\begin{array}{cc}
Y = 4 + 2D - 2T + 0T \times D & \\
+0\Im(p) - 24\Im(p) \times D & \textit{unmatched sample} \\
+8T \times \Im(p) \times D + \varepsilon_{2a} & \\
2 + 0\left(2\tfrac{1}{2}\right) = 2 & \textit{suggested } ATE\,(p = 0.4) \\
2 + 0\left(3\tfrac{1}{2}\right) - 24 + 8\left(3\tfrac{1}{2}\right) = 6 & \textit{suggested } ATE\,(p = 0.6) \\
2 + 0\,(2) - 24\,(0) + 8\,(2 \times 0) = 2 & \textit{suggested } ATE\,(T = 2) \\
2 + 0\,(3) - 24\left(\tfrac{2}{3}\right) & \\
+8\left(3 \times \tfrac{2}{3}\right) = 2 & \textit{suggested } ATE\,(T = 3) \\
2 + 0\,(4) - 24\,(1) + 8\,(4 \times 1) = 10 & \textit{suggested } ATE\,(T = 4) \\
2 + 0\,(2) = 2 & \textit{suggested } ATE \begin{pmatrix} p = 0.4, \\ T = 2 \end{pmatrix} \\
2 + 0\,(2) - 24 & \\
+8\,(2) = -6\,(NA) & \textit{suggested } ATE \begin{pmatrix} p = 0.6, \\ T = 2 \end{pmatrix} \\
2 + 0\,(3) = 2 & \textit{suggested } ATE \begin{pmatrix} p = 0.4, \\ T = 3 \end{pmatrix} \\
2 + 0\,(3) - 24 + 8\,(3) = 2 & \textit{suggested } ATE \begin{pmatrix} p = 0.6, \\ T = 3 \end{pmatrix} \\
2 + 0\,(4) = 2\,(NA) & \textit{suggested } ATE \begin{pmatrix} p = 0.4, \\ T = 4 \end{pmatrix} \\
2 + 0\,(4) - 24 + 8\,(4) = 10 & \textit{suggested } ATE \begin{pmatrix} p = 0.6, \\ T = 4 \end{pmatrix} \\
2 + 0\left(3\tfrac{1}{6}\right) - 24\left(\tfrac{2}{3}\right) & \\
+8\left(2\tfrac{1}{3}\right) = 4\tfrac{2}{3} & \textit{suggested } ATE
\end{array}
\tag{2a}
$$

and

$$
\begin{array}{cc}
Y = 7\frac{5}{7} - 15\frac{3}{7}D - 3\frac{1}{7}T & \\
+6\frac{2}{7}T \times D + \varepsilon_{2b} & \textit{ATT p-score matched sample} \\
-15\frac{3}{7} + 6\frac{2}{7}(2) = -2\frac{6}{7} & \textit{suggested ATT } (T=2) \\
-15\frac{3}{7} + 6\frac{2}{7}(3) = 3\frac{3}{7} & \textit{suggested ATT } (T=3) \\
-15\frac{3}{7} + 6\frac{2}{7}(4) = 9\frac{5}{7} & \textit{suggested ATT } (T=4) \\
-15\frac{3}{7} + 6\frac{2}{7}\left(3\frac{1}{4}\right) = 5 & \textit{suggested ATT} \\
Y = 4 - 8D - 2T & \\
+4T \times D + \varepsilon_{2b} & \textit{ATUT p-score matched sample} \\
-8 + 4(2) = 0 & \textit{suggested ATUT } (T=2) \\
-8 + 4(3) = 4 & \textit{suggested ATUT } (T=3) \\
-8 + 4(4) = 8 & \textit{suggested ATUT } (T=4) \\
-8 + 4(3) = 4 & \textit{suggested ATUT} \\
Y = 6\frac{4}{17} - 12\frac{8}{17}D - 2\frac{12}{17}T & \\
+5\frac{8}{17}T \times D + \varepsilon_{2b} & \textit{ATE p-score matched sample} \\
-12\frac{8}{17} + 5\frac{8}{17}(2) = -1\frac{11}{17} & \textit{suggested ATE } (T=2) \\
-12\frac{8}{17} + 5\frac{8}{17}(3) = 3\frac{13}{17} & \textit{suggested ATE } (T=3) \\
-12\frac{8}{17} + 5\frac{8}{17}(4) = 9\frac{3}{17} & \textit{suggested ATE } (T=4) \\
-12\frac{8}{17} + 5\frac{8}{17}\left(3\frac{1}{6}\right) = 4\frac{2}{3} & \textit{suggested ATE} \\
Y = 4 - 11\frac{5}{7}D - 2T & \\
+5\frac{1}{7}T \times D + \varepsilon_{2b} & \textit{unmatched sample} \\
-11\frac{5}{7} + 5\frac{1}{7}(2) = -1\frac{3}{7} & \textit{suggested ATE } (T=2) \\
-11\frac{5}{7} + 5\frac{1}{7}(3) = 3\frac{5}{7} & \textit{suggested ATE } (T=3) \\
-11\frac{5}{7} + 5\frac{1}{7}(4) = 8\frac{6}{7} & \textit{suggested ATE } (T=4) \\
-11\frac{5}{7} + 5\frac{1}{7}\left(3\frac{1}{6}\right) = 4\frac{5}{7} & \textit{suggested ATE}
\end{array}
\tag{2b}
$$

Design 3 yields

$$
\begin{array}{ll}
g_1\left(T=2,p=0.6\right) & \\
-g_0\left(T=2,p=0.6\right) & \textit{ATT p-score matched sample} \\
=2-(-2)=4 & \textit{suggested } ATT\left(T=2,p=0.6\right) \\
g_1\left(T=3,p=0.6\right) & \\
-g_0\left(T=3,p=0.6\right) & \\
=0-0=0 & \textit{suggested } ATT\left(T=3,p=0.6\right) \\
g_1\left(T=4,p=0.6\right) & \\
-g_0\left(T=4,p=0.6\right) & \\
=6-(-6)=12 & \textit{suggested } ATT\left(T=4,p=0.6\right) \\
g_1\left(T=2,p=0.4\right) & \\
-g_0\left(T=2,p=0.4\right) & \\
=2-(-2)=4 & \textit{suggested } ATT\left(T=2,p=0.4\right) \\
g_1\left(T=3,p=0.4\right) & \\
-g_0\left(T=3,p=0.4\right) & \\
=0-0=0 & \textit{suggested } ATT\left(T=3,p=0.4\right) \\
g_1\left(T=4,p=0.4\right) & \\
-g_0\left(T=4,p=0.4\right) & \\
=6-(-6)=12 & \textit{suggested } ATT\left(T=4,p=0.4\right)
\end{array}
\tag{3Tp}
$$

$$
\begin{array}{ll}
\frac{1}{2}ATT\left(T=2,p=0.4\right) & \\
+\frac{1}{2}ATT\left(T=3,p=0.4\right) & \textit{ATT p-score matched sample} \\
+0ATT\left(T=4,p=0.4\right) & \\
=\frac{1}{2}\left(4\right)+\frac{1}{2}\left(0\right) & \\
+0\left(12\right)=2 & \textit{suggested } ATT\left(p=0.4\right) \\
0ATT\left(T=2,p=0.6\right) & \\
+\frac{1}{2}ATT\left(T=3,p=0.6\right) & \\
+\frac{1}{2}ATT\left(T=4,p=0.6\right) & \\
=0\left(4\right)+\frac{1}{2}\left(0\right) & \\
+\frac{1}{2}\left(12\right)=6 & \textit{suggested } ATT\left(p=0.6\right)
\end{array}
\tag{3p}
$$

$$
\begin{array}{ll}
1ATT\,(T=2,p=0.4) & \\
+0ATT\,(T=2,p=0.6) & \textit{ATT p-score matched sample} \\
=1\,(4)+0\,(NA)=4 & \textit{suggested ATT}\,(T=2) \\
\frac{1}{4}ATT\,(T=3,p=0.4) & \\
+\frac{3}{4}ATT\,(T=3,p=0.6) & \\
=\frac{1}{4}\,(0)+\frac{3}{4}\,(0)=0 & \textit{suggested ATT}\,(T=3) \\
0ATT\,(T=4,p=0.4) & \\
+1ATT\,(T=4,p=0.6) & \\
=0\,(NA)+1\,(12)=12 & \textit{suggested ATT}\,(T=4)
\end{array}
\tag{3T}
$$

$$
\begin{array}{ll}
\frac{1}{8}ATT\,(T=2) & \\
+\frac{1}{2}ATT\,(T=3) & \textit{ATT p-score matched sample} \\
+\frac{3}{8}ATT\,(T=4) & \\
=\frac{1}{8}\,(4)+\frac{1}{2}\,(0)+\frac{3}{8}\,(12)=5 & \textit{suggested ATT}
\end{array}
\tag{3}
$$

$$
\begin{array}{ll}
g_1\,(T=2,p=0.6) & \textit{ATUT p-score matched sample} \\
-g_0\,(T=2,p=0.6) & \\
=2-(-2)=4 & \textit{suggested ATUT}\,(T=2,p=0.6) \\
g_1\,(T=3,p=0.6) & \\
-g_0\,(T=3,p=0.6) & \\
=0-0=0 & \textit{suggested ATUT}\,(T=3,p=0.6) \\
g_1\,(T=4,p=0.6) & \\
-g_0\,(T=4,p=0.6) & \\
=6-(-6)=12 & \textit{suggested ATUT}\,(T=4,p=0.6) \\
g_1\,(T=2,p=0.4) & \\
-g_0\,(T=2,p=0.4) & \\
=2-(-2)=4 & \textit{suggested ATUT}\,(T=2,p=0.4) \\
g_1\,(T=3,p=0.4) & \\
-g_0\,(T=3,p=0.4) & \\
=0-0=0 & \textit{suggested ATUT}\,(T=3,p=0.4) \\
g_1\,(T=4,p=0.4) & \\
-g_0\,(T=4,p=0.4) & \\
=6-(-6)=12 & \textit{suggested ATUT}\,(T=4,p=0.4)
\end{array}
\tag{3Tp}
$$

$$
\begin{array}{ll}
\frac{1}{2}ATUT\,(T=2,p=0.4) & \\
+\frac{1}{2}ATUT\,(T=3,p=0.4) & \textit{ATUT p-score matched sample} \\
+0ATUT\,(T=4,p=0.4) & \\
=\frac{1}{2}\,(4)+\frac{1}{2}\,(0) & \\
+0\,(12)=2 & \textit{suggested ATUT}\,(p=0.4) \\
0ATUT\,(T=2,p=0.6) & \\
+\frac{1}{2}ATUT\,(T=3,p=0.6) & \\
+\frac{1}{2}ATUT\,(T=4,p=0.6) & \\
=0\,(4)+\frac{1}{2}\,(0) & \\
+\frac{1}{2}\,(12)=6 & \textit{suggested ATUT}\,(p=0.6)
\end{array}
\tag{3p}
$$

$$
\begin{array}{ll}
1ATUT\,(T=2,p=0.4) & \\
+0ATUT\,(T=2,p=0.6) & \textit{ATUT p-score matched sample} \\
=1\,(4)+0\,(NA)=2 & \textit{suggested ATUT}\,(T=2) \\
\frac{1}{2}ATUT\,(T=3,p=0.4) & \\
+\frac{1}{2}ATUT\,(T=3,p=0.6) & \\
=\frac{1}{2}\,(0)+\frac{1}{2}\,(0)=0 & \textit{suggested ATUT}\,(T=3) \\
0ATUT\,(T=4,p=0.4) & \\
+1ATUT\,(T=4,p=0.6) & \\
=0\,(NA)+1\,(12)=12 & \textit{suggested ATUT}\,(T=4)
\end{array}
\tag{3T}
$$

$$
\begin{array}{ll}
\frac{1}{4}ATUT\,(T=2) & \\
+\frac{1}{2}ATUT\,(T=3) & \textit{ATUT p-score matched sample} \\
+\frac{1}{4}ATUT\,(T=4) & \\
=\frac{1}{4}\,(4)+\frac{1}{2}\,(0)+\frac{1}{4}\,(12)=4 & \textit{suggested ATUT}
\end{array}
\tag{3}
$$

$$
\begin{array}{ll}
g_1\left(T=2,p=0.6\right) & \\
-g_0\left(T=2,p=0.6\right) & \textit{ATE p-score matched sample} \\
=2-(-2)=4 & \textit{suggested } ATE\left(T=2,p=0.6\right) \\
g_1\left(T=3,p=0.6\right) & \\
-g_0\left(T=3,p=0.6\right) & \\
=0-0=0 & \textit{suggested } ATE\left(T=3,p=0.6\right) \\
g_1\left(T=4,p=0.6\right) & \\
-g_0\left(T=4,p=0.6\right) & \\
=6-(-6)=12 & \textit{suggested } ATE\left(T=4,p=0.6\right) \\
g_1\left(T=2,p=0.4\right) & \\
-g_0\left(T=2,p=0.4\right) & \\
=2-(-2)=4 & \textit{suggested } ATE\left(T=2,p=0.4\right) \\
g_1\left(T=3,p=0.4\right) & \\
-g_0\left(T=3,p=0.4\right) & \\
=0-0=0 & \textit{suggested } ATE\left(T=3,p=0.4\right) \\
g_1\left(T=4,p=0.4\right) & \\
-g_0\left(T=4,p=0.4\right) & \\
=6-(-6)=12 & \textit{suggested } ATE\left(T=4,p=0.4\right)
\end{array}
\tag{3Tp}
$$

$$
\begin{array}{ll}
\frac{1}{2}ATE\left(T=2,p=0.4\right) & \\
+\frac{1}{2}ATE\left(T=3,p=0.4\right) & \textit{ATE p-score matched sample} \\
+0ATE\left(T=4,p=0.4\right) & \\
=\frac{1}{2}\left(4\right)+\frac{1}{2}\left(0\right) & \\
+0\left(12\right)=2 & \textit{suggested } ATE\left(p=0.4\right) \\
0ATE\left(T=2,p=0.6\right) & \\
+\frac{1}{2}ATE\left(T=3,p=0.6\right) & \\
+\frac{1}{2}ATE\left(T=4,p=0.6\right) & \\
=0\left(4\right)+\frac{1}{2}\left(0\right) & \\
+\frac{1}{2}\left(12\right)=6 & \textit{suggested } ATE\left(p=0.6\right)
\end{array}
\tag{3p}
$$

$$
\begin{array}{ll}
1ATE\left(T=2,p=0.4\right) & \\
+0ATE\left(T=2,p=0.6\right) & \textit{ATE p-score matched sample} \\
=1\left(4\right)+0\left(NA\right)=4 & \textit{suggested ATE}\left(T=2\right) \\
\frac{1}{3}ATE\left(T=3,p=0.4\right) & \\
+\frac{2}{3}ATE\left(T=3,p=0.6\right) & \\
=\frac{1}{3}\left(0\right)+\frac{2}{3}\left(0\right)=0 & \textit{suggested ATE}\left(T=3\right) \\
0ATE\left(T=4,p=0.4\right) & \\
+1ATE\left(T=4,p=0.6\right) & \\
=0\left(NA\right)+1\left(12\right)=12 & \textit{suggested ATE}\left(T=4\right)
\end{array}
\tag{3T}
$$

$$
\begin{array}{ll}
\frac{1}{6}ATE\left(T=2\right) & \\
+\frac{1}{2}ATE\left(T=3\right) & \textit{ATE p-score matched sample} \\
+\frac{1}{3}ATE\left(T=4\right) & \\
=\frac{1}{6}\left(4\right)+\frac{1}{2}\left(0\right)+\frac{1}{3}\left(12\right)=4\frac{2}{3} & \textit{suggested ATE}
\end{array}
\tag{3}
$$

and

$$
\begin{array}{ll}
g_1\left(T=2,p=0.6\right) & \\
-g_0\left(T=2,p=0.6\right) & \textit{unmatched sample} \\
=2-\left(-2\right)=4 & \textit{suggested ATE}\left(T=2,p=0.6\right) \\
g_1\left(T=3,p=0.6\right) & \\
-g_0\left(T=3,p=0.6\right) & \\
=0-0=0 & \textit{suggested ATE}\left(T=3,p=0.6\right) \\
g_1\left(T=4,p=0.6\right) & \\
-g_0\left(T=4,p=0.6\right) & \\
=6-\left(-6\right)=12 & \textit{suggested ATE}\left(T=4,p=0.6\right) \\
g_1\left(T=2,p=0.4\right) & \\
-g_0\left(T=2,p=0.4\right) & \\
=2-\left(-2\right)=4 & \textit{suggested ATE}\left(T=2,p=0.4\right) \\
g_1\left(T=3,p=0.4\right) & \\
-g_0\left(T=3,p=0.4\right) & \\
=0-0=0 & \textit{suggested ATE}\left(T=3,p=0.4\right) \\
g_1\left(T=4,p=0.4\right) & \\
-g_0\left(T=4,p=0.4\right) & \\
=6-\left(-6\right)=12 & \textit{suggested ATE}\left(T=4,p=0.4\right)
\end{array}
\tag{3Tp}
$$

$$
\begin{array}{cc}
\frac{1}{2}ATE\left(T=2,p=0.4\right) & \\
+\frac{1}{2}ATE\left(T=3,p=0.4\right) & \textit{unmatched sample} \\
+0ATE\left(T=4,p=0.4\right) & \\
=\frac{1}{2}\left(4\right)+\frac{1}{2}\left(0\right) & \\
+0\left(12\right)=2 & \textit{suggested } ATE\left(p=0.4\right) \\
0ATE\left(T=2,p=0.6\right) & \\
+\frac{1}{2}ATE\left(T=3,p=0.6\right) & \\
+\frac{1}{2}ATE\left(T=4,p=0.6\right) & \\
=0\left(4\right)+\frac{1}{2}\left(0\right) & \\
+\frac{1}{2}\left(12\right)=6 & \textit{suggested } ATE\left(p=0.6\right)
\end{array}
\tag{3p}
$$

$$
\begin{array}{cc}
1ATE\left(T=2,p=0.4\right) & \\
+0ATE\left(T=2,p=0.6\right) & \textit{unmatched sample} \\
=1\left(4\right)+0\left(NA\right)=4 & \textit{suggested } ATE\left(T=2\right) \\
\frac{1}{3}ATE\left(T=3,p=0.4\right) & \\
+\frac{2}{3}ATE\left(T=3,p=0.6\right) & \\
=\frac{1}{3}\left(0\right)+\frac{2}{3}\left(0\right)=0 & \textit{suggested } ATE\left(T=3\right) \\
0ATE\left(T=4,p=0.4\right) & \\
+1ATE\left(T=4,p=0.6\right) & \\
=0\left(NA\right)+1\left(12\right)=12 & \textit{suggested } ATE\left(T=4\right)
\end{array}
\tag{3T}
$$

$$
\begin{array}{cc}
\frac{1}{6}ATE\left(T=2\right) & \\
+\frac{1}{2}ATE\left(T=3\right) & \textit{unmatched sample} \\
+\frac{1}{3}ATE\left(T=4\right) & \\
=\frac{1}{6}\left(4\right)+\frac{1}{2}\left(0\right)+\frac{1}{3}\left(12\right)=4\frac{2}{3} & \textit{suggested } ATE
\end{array}
\tag{3}
$$

As potential outcomes and treatment effects are nonlinear in covariate T, only the nonparametric design identifies average treatment effects conditional on T. Unconditional average treatment effects are identified for this DGP by all DR designs considered but nonlinearity in T leads to identification failure conditional on T for the linear outcome models. Pure linear outcome models without propensity-score matching fail to identify unconditional average treatment effects as well as conditional average treatment effects.

Example 7 (*DR* with mis-specified propensity-score) *Suppose the DGP is*

	D	$P(Z)$	$\widehat{P(Z)}$	T_1	Y_1	T_0	Y_0	Y	T
	1	0.6	0.6	3	3	3	−3	3	3
	1	0.6	0.6	4	4	4	−4	4	4
	1	0.6	0.4	3	3	3	−3	3	3
	1	0.6	0.6	4	4	4	−4	4	4
	1	0.6	0.4	3	3	3	−3	3	3
	1	0.6	0.6	4	4	4	−4	4	4
	1	0.4	0.4	2	2	2	−2	2	2
	1	0.4	0.4	3	3	3	−3	3	3
	0	0.6	0.4	3	3	3	−3	−3	3
	0	0.6	0.6	4	4	4	−4	−4	4
	0	0.4	0.4	2	2	2	−2	−2	2
	0	0.4	0.4	3	3	3	−3	−3	3
means	$\frac{2}{3}$	$\frac{8}{15}$	$\frac{5}{12}$	$3\frac{1}{6}$	$3\frac{1}{6}$	$3\frac{1}{6}$	$-3\frac{1}{6}$	$1\frac{1}{6}$	$3\frac{1}{6}$

where $\widehat{P(Z)}$ is the (mis-specified) estimated propensity-score. Propensity-score mis-specification can arise from omitted or extraneous covariates in Z or in the link function $P(Z)$. Conditional and unconditional average treatment effects for this DGP are

conditioning	$ATT(\cdot)$	$ATUT(\cdot)$	$ATE(\cdot)$
$\Im(p)=1$	$3\frac{1}{2}-\left(-3\frac{1}{2}\right)=7$	$3\frac{1}{2}-\left(-3\frac{1}{2}\right)=7$	$3\frac{1}{2}-\left(-3\frac{1}{2}\right)=7$
$\Im(p)=0$	$2\frac{1}{2}-\left(-2\frac{1}{2}\right)=5$	$2\frac{1}{2}-\left(-2\frac{1}{2}\right)=5$	$2\frac{1}{2}-\left(-2\frac{1}{2}\right)=5$
$T=2$	$2-(-2)=4$	$2-(-2)=4$	$2-(-2)=4$
$T=3$	$3-(-3)=6$	$3-(-3)=6$	$3-(-3)=6$
$T=4$	$4-(-4)=8$	$4-(-4)=8$	$4-(-4)=8$
$\Im(p)=1, T=2$	$NA-NA=NA$	$NA-NA=NA$	$NA-NA=NA$
$\Im(p)=1, T=3$	$3-(-3)=6$	$3-(-3)=6$	$3-(-3)=6$
$\Im(p)=1, T=4$	$4-(-4)=8$	$4-(-4)=8$	$4-(-4)=8$
$\Im(p)=0, T=2$	$2-(-2)=4$	$2-(-2)=4$	$2-(-2)=4$
$\Im(p)=0, T=3$	$3-(-3)=6$	$3-(-3)=6$	$3-(-3)=6$
$\Im(p)=0, T=4$	$NA-NA=NA$	$NA-NA=NA$	$NA-NA=NA$
none	$3\frac{1}{4}-\left(-3\frac{1}{4}\right)=6\frac{1}{2}$	$3-(-3)=6$	$3\frac{1}{6}-\left(-3\frac{1}{6}\right)=6\frac{1}{3}$

where $\Im(p)$ is based on the DGP, not estimated propensity-score. On the other hand, propensity-score matched samples are based on estimated propensity-score with the following proportions in the ATT sample

$$12:12:12:12$$

for

$$\begin{array}{ccc} \left(D=1,\widehat{P(Z)}=0.6\right) & : & \left(D=1,\widehat{P(Z)}=0.4\right): \\ \left(D=0,\widehat{P(Z)}=0.6\right) & : & \left(D=0,\widehat{P(Z)}=0.4\right) \end{array}$$

in the ATUT sample

$$4:12:4:12$$

for

$$\begin{array}{ccc} \left(D=1,\widehat{P(Z)}=0.6\right) & : & \left(D=1,\widehat{P(Z)}=0.4\right): \\ \left(D=0,\widehat{P(Z)}=0.6\right) & : & \left(D=0,\widehat{P(Z)}=0.4\right) \end{array}$$

and the ATE sample is $2:1$ *ATT sample to ATUT sample. Design one yields*

$$\begin{array}{cc} Y=-2.937+6.573D-0.119T+\varepsilon_{1a} & \textit{ATT p-score matched sample} \\ 6.573 & \textit{suggested ATT} \\ Y=-2\frac{4}{7}+6D-\frac{1}{7}T+\varepsilon_{1a} & \textit{ATUT p-score matched sample} \\ 6 & \textit{suggested ATUT} \\ Y=-2.796+6.382D-0.132T+\varepsilon_{1a} & \textit{ATE p-score matched sample} \\ 6.382 & \textit{suggested ATE} \\ Y=-3\frac{9}{11}+6\frac{2}{11}D+\frac{3}{11}T+\varepsilon_{1a} & \textit{unmatched sample} \\ 6\frac{2}{11} & \textit{suggested ATE} \end{array} \tag{1a}$$

and

$$
\begin{array}{cc}
Y = -2\frac{2}{3} + 5\frac{5}{12}D - 1\frac{1}{3}\Im(p) & \\
+2\frac{1}{3}\Im(p) \times D + \varepsilon_{1b} & \textit{ATT p-score matched sample} \\
5\frac{5}{12} & \textit{suggested ATT } (p = 0.4) \\
5\frac{5}{12} + 2\frac{1}{3} = 7\frac{3}{4} & \textit{suggested ATT } (p = 0.6) \\
5\frac{5}{12} + 2\frac{1}{3}\left(\frac{5}{12}\right) = 6\frac{7}{18} & \textit{suggested ATT} \\
Y = -2\frac{2}{3} + 5\frac{5}{12}D - 1\frac{1}{3}\Im(p) & \\
+2\frac{1}{3}\Im(p) \times D + \varepsilon_{1b} & \textit{ATUT p-score matched sample} \\
5\frac{5}{12} & \textit{suggested ATUT } (p = 0.4) \\
5\frac{5}{12} + 2\frac{1}{3} = 7\frac{3}{4} & \textit{suggested ATUT } (p = 0.6) \\
5\frac{5}{12} + 2\frac{1}{3}\left(\frac{5}{12}\right) = 6\frac{7}{18} & \textit{suggested ATUT} \\
Y = -2\frac{2}{3} + 5\frac{5}{12}D - 1\frac{1}{3}\Im(p) & \\
+2\frac{1}{3}\Im(p) \times D + \varepsilon_{1b} & \textit{ATE p-score matched sample} \\
5\frac{5}{12} & \textit{suggested ATE } (p = 0.4) \\
5\frac{5}{12} + 2\frac{1}{3} = 7\frac{3}{4} & \textit{suggested ATE } (p = 0.6) \\
5\frac{5}{12} + 2\frac{1}{3}\left(\frac{5}{12}\right) = 6\frac{7}{18} & \textit{suggested ATE} \\
Y = -2\frac{2}{3} + 5\frac{5}{12}D - 1\frac{1}{3}\Im(p) & \\
+2\frac{1}{3}\Im(p) \times D + \varepsilon_{1b} & \textit{unmatched sample} \\
5\frac{5}{12} & \textit{suggested ATE } (p = 0.4) \\
5\frac{5}{12} + 2\frac{1}{3} = 7\frac{3}{4} & \textit{suggested ATE } (p = 0.6) \\
5\frac{5}{12} + 2\frac{1}{3}\left(\frac{5}{12}\right) = 6\frac{7}{18} & \textit{suggested ATE}
\end{array}
\tag{1b}
$$

Design one performs poorly (fails to identify average treatment effects) since mean independence of outcome unconditionally and conditional on estimated

propensity-score is not satisfied by the DGP. Design two yields

$$
\begin{array}{c|l}
Y = 0 + 0D - 1T + 2T \times D & \\
+0\Im(p) + 0\Im(p) \times D & \\
+0T \times \Im(p) & \textit{ATT p-score matched sample} \\
+(ND)\,T \times \Im(p) \times D + \varepsilon_{2a} & \\
0 + 2\left(2\tfrac{3}{4}\right) = 5\tfrac{1}{2} & \textit{suggested ATT } (p = 0.4) \\
0 + 2\left(3\tfrac{3}{4}\right) + 0 + 0\left(3\tfrac{3}{4}\right) = 7\tfrac{1}{2} & \textit{suggested ATT } (p = 0.6) \\
0 + 2(2) + 0(0) + 0(2 \times 0) = 4 & \textit{suggested ATT } (T = 2) \\
0 + 2(3) + 0\left(\tfrac{1}{4}\right) + 0\left(3 \times \tfrac{1}{4}\right) = 6 & \textit{suggested ATT } (T = 3) \\
0 + 2(4) + 0(1) + 0(4 \times 1) = 8 & \textit{suggested ATT } (T = 4) \\
0 + 2(2) = 4 & \textit{suggested ATT } (p = 0.4, T = 2) \\
0 + 2(2) + 0 + 0(2) = 4 & \textit{suggested ATT } (p = 0.6, T = 2) \\
0 + 2(3) = 6 & \textit{suggested ATT } (p = 0.4, T = 3) \\
0 + 2(3) + 0 + 0(3) = 6 & \textit{suggested ATT } (p = 0.6, T = 3) \\
0 + 2(4) = 8 & \textit{suggested ATT } (p = 0.4, T = 4) \\
0 + 2(4) + 0 + 0(4) = 8 & \textit{suggested ATT } (p = 0.6, T = 4) \\
0 + 2\left(3\tfrac{1}{4}\right) + 0\left(\tfrac{1}{2}\right) + (ND)\left(1\tfrac{7}{8}\right) = 6\tfrac{1}{2} & \textit{suggested ATT}
\end{array}
\tag{2a}
$$

$$
\begin{array}{cc}
Y = 0 + 0D - 1T + 2T \times D & \\
+0\Im(p) + 0\Im(p) \times D & \\
+0T \times \Im(p) & \textit{ATUT p-score matched sample} \\
+(ND)\, T \times \Im(p) \times D + \varepsilon_{2a} & \\
0 + 2\left(2\tfrac{2}{3}\right) = 5\tfrac{1}{3} & \textit{suggested ATUT}\,(p = 0.4) \\
0 + 2\,(4) + 0 + 0\,(4) = 8 & \textit{suggested ATUT}\,(p = 0.6) \\
0 + 2\,(2) + 0\,(0) + 0\,(2 \times 0) = 4 & \textit{suggested ATUT}\,(T = 2) \\
0 + 2\,(3) + 0\,(0) + 0\,(3 \times 0) = 6 & \textit{suggested ATUT}\,(T = 3) \\
0 + 2\,(4) + 0\,(1) + 0\,(4 \times 1) = 8 & \textit{suggested ATUT}\,(T = 4) \\
0 + 2\,(2) = 4 & \textit{suggested ATUT}\begin{pmatrix} p = 0.4, \\ T = 2 \end{pmatrix} \\
0 + 2\,(2) + 0 + 0\,(2) = 4 & \textit{suggested ATUT}\begin{pmatrix} p = 0.6, \\ T = 2 \end{pmatrix} \\
0 + 2\,(3) = 6 & \textit{suggested ATUT}\begin{pmatrix} p = 0.4, \\ T = 3 \end{pmatrix} \\
0 + 2\,(3) + 0 + 0\,(3) = 6 & \textit{suggested ATUT}\begin{pmatrix} p = 0.6, \\ T = 3 \end{pmatrix} \\
0 + 2\,(4) = 8 & \textit{suggested ATUT}\begin{pmatrix} p = 0.4, \\ T = 4 \end{pmatrix} \\
0 + 2\,(4) + 0 + 0\,(4) = 8 & \textit{suggested ATUT}\begin{pmatrix} p = 0.6, \\ T = 4 \end{pmatrix} \\
0 + 2\,(3) + 0\left(\tfrac{1}{4}\right) + (ND)\,(1) = 6 & \textit{suggested ATUT}
\end{array}
\tag{2a}
$$

$$
\begin{array}{cc}
Y = 0 + 0D - 1T + 2T \times D & \\
+0\Im(p) + 0\Im(p) \times D & \\
+0T \times \Im(p) & ATE \text{ p-score matched sample} \\
+(ND)\, T \times \Im(p) \times D + \varepsilon_{2a} & \\
0 + 2\left(2\frac{5}{7}\right) = 5\frac{3}{7} & \text{suggested } ATE\,(p = 0.4) \\
0 + 2\left(3\frac{4}{5}\right) + 0 + 0\left(3\frac{4}{5}\right) = 7\frac{3}{5} & \text{suggested } ATE\,(p = 0.6) \\
0 + 2\,(2) + 0\,(0) + 0\,(2 \times 0) = 4 & \text{suggested } ATE\,(T = 2) \\
0 + 2\,(3) + 0\left(\frac{1}{6}\right) & \\
+0\left(3 \times \frac{1}{6}\right) = 6 & \text{suggested } ATE\,(T = 3) \\
0 + 2\,(4) + 0\,(1) + 0\,(4 \times 1) = 8 & \text{suggested } ATE\,(T = 4) \\
0 + 2\,(2) = 4 & \text{suggested } ATE \begin{pmatrix} p = 0.4, \\ T = 2 \end{pmatrix} \\
0 + 2\,(2) + 0 + 0\,(2) = 4 & \text{suggested } ATE \begin{pmatrix} p = 0.6, \\ T = 2 \end{pmatrix} \\
0 + 2\,(3) = 6 & \text{suggested } ATE \begin{pmatrix} p = 0.4, \\ T = 3 \end{pmatrix} \\
0 + 2\,(3) + 0 + 0\,(3) = 6 & \text{suggested } ATE \begin{pmatrix} p = 0.6, \\ T = 3 \end{pmatrix} \\
0 + 2\,(4) = 8 & \text{suggested } ATE \begin{pmatrix} p = 0.4, \\ T = 4 \end{pmatrix} \\
0 + 2\,(4) + 0 + 0\,(4) = 8 & \text{suggested } ATE \begin{pmatrix} p = 0.6, \\ T = 4 \end{pmatrix} \\
0 + 2\left(3\frac{1}{6}\right) + 0\left(\frac{5}{12}\right) & \\
+(ND)\left(1\frac{7}{12}\right) = 6\frac{1}{3} & \text{suggested } ATE
\end{array}
\tag{2a}
$$

$$
\begin{array}{cc}
Y = 0 + 0D - 1T + 2T \times D & \\
+0\Im(p) + 0\Im(p) \times D & \textit{unmatched sample} \\
+(ND)\,T \times \Im(p) \times D + \varepsilon_{2a} & \\
0 + 2\left(2\frac{5}{7}\right) = 5\frac{3}{7} & \textit{suggested } ATE\,(p = 0.4) \\
0 + 2\left(3\frac{4}{5}\right) + 0 + 0\left(3\frac{4}{5}\right) = 7\frac{3}{5} & \textit{suggested } ATE\,(p = 0.6) \\
0 + 2\,(2) + 0\,(0) + 0\,(2 \times 0) = 4 & \textit{suggested } ATE\,(T = 2) \\
0 + 2\,(3) + 0\left(\frac{1}{6}\right) + 0\left(3 \times \frac{1}{6}\right) = 6 & \textit{suggested } ATE\,(T = 3) \\
0 + 2\,(4) + 0\,(1) + 0\,(4 \times 1) = 8 & \textit{suggested } ATE\,(T = 4) \\
0 + 2\,(2) = 4 & \textit{suggested } ATE \begin{pmatrix} p = 0.4, \\ T = 2 \end{pmatrix} \\
0 + 2\,(2) + 0 + 0\,(2) = 4 & \textit{suggested } ATE \begin{pmatrix} p = 0.6, \\ T = 2 \end{pmatrix} \\
0 + 2\,(3) = 6 & \textit{suggested } ATE \begin{pmatrix} p = 0.4, \\ T = 3 \end{pmatrix} \\
0 + 2\,(3) + 0 + 0\,(3) = 6 & \textit{suggested } ATE \begin{pmatrix} p = 0.6, \\ T = 3 \end{pmatrix} \\
0 + 2\,(4) = 8 & \textit{suggested } ATE \begin{pmatrix} p = 0.4, \\ T = 4 \end{pmatrix} \\
0 + 2\,(4) + 0 + 0\,(4) = 8 & \textit{suggested } ATE \begin{pmatrix} p = 0.6, \\ T = 4 \end{pmatrix} \\
0 + 2\left(3\frac{1}{6}\right) + 0\left(\frac{5}{12}\right) + (ND)\left(1\frac{7}{12}\right) = 6\frac{1}{3} & \textit{suggested } ATE
\end{array}
\tag{2a}
$$

and

$$
\begin{array}{cc}
Y = 0 + 0D - 1T & \\
+2T \times D + \varepsilon_{2b} & \textit{ATT p-score matched sample} \\
0 + 2\,(2) = 4 & \textit{suggested ATT}\,(T = 2) \\
0 + 2\,(3) = 6 & \textit{suggested ATT}\,(T = 3) \\
0 + 2\,(4) = 8 & \textit{suggested ATT}\,(T = 4) \\
0 + 2\left(3\tfrac{1}{4}\right) = 6\tfrac{1}{2} & \textit{suggested ATT} \\
Y = 0 + 0D - 1T & \\
+2T \times D + \varepsilon_{2b} & \textit{ATUT p-score matched sample} \\
0 + 2\,(2) = 4 & \textit{suggested ATUT}\,(T = 2) \\
0 + 2\,(3) = 6 & \textit{suggested ATUT}\,(T = 3) \\
0 + 2\,(4) = 8 & \textit{suggested ATUT}\,(T = 4) \\
0 + 2\,(3) = 6 & \textit{suggested ATUT} \\
Y = 0 + 0D - 1T & \\
+2T \times D + \varepsilon_{2b} & \textit{ATE p-score matched sample} \\
0 + 2\,(2) = 4 & \textit{suggested ATE}\,(T = 2) \\
0 + 2\,(3) = 6 & \textit{suggested ATE}\,(T = 3) \\
0 + 2\,(4) = 8 & \textit{suggested ATE}\,(T = 4) \\
0 + 2\left(3\tfrac{1}{6}\right) = 6\tfrac{1}{3} & \textit{suggested ATE} \\
Y = 0 + 0D - 1T & \\
+2T \times D + \varepsilon_{2b} & \textit{unmatched sample} \\
0 + 2\,(2) = 4 & \textit{suggested ATE}\,(T = 2) \\
0 + 2\,(3) = 6 & \textit{suggested ATE}\,(T = 3) \\
0 + 2\,(4) = 8 & \textit{suggested ATE}\,(T = 4) \\
0 + 2\left(3\tfrac{1}{6}\right) = 6\tfrac{1}{3} & \textit{suggested ATE}
\end{array}
\tag{2b}
$$

Design 2 captures the spirit of DR strategies. Even though propensity-score (equivalently, $\Im(p)$) is mis-specified, the DGP is linear in T and mean independence conditional on T is satisfied, consequently design 2a and 2b identify average treatment effects unconditionally and conditional on T but not condi-

tional on $\Im(p)$. *Design 3 yields*

$$
\begin{array}{ll}
g_1(T=2,p=0.6) & \\
-g_0(T=2,p=0.6) & \textit{ATT p-score matched sample} \\
=2-(-2)=4 & \textit{suggested } ATT(T=2,p=0.6) \\
g_1(T=3,p=0.6) & \\
-g_0(T=3,p=0.6) & \\
=3-(-3)=6 & \textit{suggested } ATT(T=3,p=0.6) \\
g_1(T=4,p=0.6) & \\
-g_0(T=4,p=0.6) & \\
=4-(-4)=8 & \textit{suggested } ATT(T=4,p=0.6) \\
g_1(T=2,p=0.4) & \\
-g_0(T=2,p=0.4) & \\
=2-(-2)=4 & \textit{suggested } ATT(T=2,p=0.4) \\
g_1(T=3,p=0.4) & \\
-g_0(T=3,p=0.4) & \\
=3-(-3)=6 & \textit{suggested } ATT(T=3,p=0.4) \\
g_1(T=4,p=0.4) & \\
-g_0(T=4,p=0.4) & \\
=4-(-4)=8 & \textit{suggested } ATT(T=4,p=0.4)
\end{array}
\tag{3Tp}
$$

$$
\begin{array}{ll}
\frac{1}{4}ATT(T=2,p=0.4) & \\
+\frac{3}{4}ATT(T=3,p=0.4) & \textit{ATT p-score matched sample} \\
+0ATT(T=4,p=0.4) & \\
=\frac{1}{4}(4)+\frac{3}{4}(6) & \\
+0(8)=5\frac{1}{2} & \textit{suggested } ATT(p=0.4) \\
0ATT(T=2,p=0.6) & \\
+\frac{1}{4}ATT(T=3,p=0.6) & \\
+\frac{3}{4}ATT(T=4,p=0.6) & \\
=0(4)+\frac{1}{4}(6) & \\
+\frac{3}{4}(8)=7\frac{1}{2} & \textit{suggested } ATT(p=0.6)
\end{array}
\tag{3p}
$$

$$
\begin{array}{ll}
1ATT\left(T=2,p=0.4\right) & \\
+0ATT\left(T=2,p=0.6\right) & \textit{ATT p-score matched sample} \\
=1\left(4\right)+0\left(4\right)=4 & \textit{suggested ATT}\left(T=2\right) \\
\frac{3}{4}ATT\left(T=3,p=0.4\right) & \\
+\frac{1}{4}ATT\left(T=3,p=0.6\right) & \\
=\frac{3}{4}\left(6\right)+\frac{1}{4}\left(6\right)=6 & \textit{suggested ATT}\left(T=3\right) \\
0ATT\left(T=4,p=0.4\right) & \\
+1ATT\left(T=4,p=0.6\right) & \\
=0\left(8\right)+1\left(8\right)=8 & \textit{suggested ATT}\left(T=4\right)
\end{array}
\tag{3T}
$$

$$
\begin{array}{ll}
\frac{1}{8}ATT\left(T=2\right) & \\
+\frac{1}{2}ATT\left(T=3\right) & \textit{ATT p-score matched sample} \\
+\frac{3}{8}ATT\left(T=4\right) & \\
=\frac{1}{8}\left(4\right)+\frac{1}{2}\left(6\right)+\frac{3}{8}\left(8\right)=6\frac{1}{2} & \textit{suggested ATT}
\end{array}
\tag{3}
$$

$$
\begin{array}{ll}
g_1\left(T=2,p=0.6\right) & \\
-g_0\left(T=2,p=0.6\right) & \textit{ATUT p-score matched sample} \\
=2-\left(-2\right)=4 & \textit{suggested ATUT}\left(T=2,p=0.6\right) \\
g_1\left(T=3,p=0.6\right) & \\
-g_0\left(T=3,p=0.6\right) & \\
=3-\left(-3\right)=6 & \textit{suggested ATUT}\left(T=3,p=0.6\right) \\
g_1\left(T=4,p=0.6\right) & \\
-g_0\left(T=4,p=0.6\right) & \\
=4-\left(-4\right)=8 & \textit{suggested ATUT}\left(T=4,p=0.6\right) \\
g_1\left(T=2,p=0.4\right) & \\
-g_0\left(T=2,p=0.4\right) & \\
=2-\left(-2\right)=4 & \textit{suggested ATUT}\left(T=2,p=0.4\right) \\
g_1\left(T=3,p=0.4\right) & \\
-g_0\left(T=3,p=0.4\right) & \\
=3-\left(-3\right)=6 & \textit{suggested ATUT}\left(T=3,p=0.4\right) \\
g_1\left(T=4,p=0.4\right) & \\
-g_0\left(T=4,p=0.4\right) & \\
=4-\left(-4\right)=8 & \textit{suggested ATUT}\left(T=4,p=0.4\right)
\end{array}
\tag{3Tp}
$$

$$
\begin{array}{rl}
\frac{1}{3}ATUT\,(T=2,p=0.4) & \\
+\frac{2}{3}ATUT\,(T=3,p=0.4) & \textit{ATUT p-score matched sample} \\
+0ATUT\,(T=4,p=0.4) & \\
=\frac{1}{3}\,(4)+\frac{2}{3}\,(6) & \\
+0\,(8)=5\frac{1}{3} & \textit{suggested ATUT}\,(p=0.4) \\
0ATUT\,(T=2,p=0.6) & \\
+0ATUT\,(T=3,p=0.6) & \\
+1ATUT\,(T=4,p=0.6) & \\
=0\,(4)+0\,(6) & \\
+1\,(8)=8 & \textit{suggested ATUT}\,(p=0.6)
\end{array}
\tag{3p}
$$

$$
\begin{array}{rl}
1ATUT\,(T=2,p=0.4) & \\
+0ATUT\,(T=2,p=0.6) & \textit{ATUT p-score matched sample} \\
=1\,(4)+0\,(4)=4 & \textit{suggested ATUT}\,(T=2) \\
1ATUT\,(T=3,p=0.4) & \\
+0ATUT\,(T=3,p=0.6) & \\
=1\,(6)+0\,(6)=6 & \textit{suggested ATUT}\,(T=3) \\
0ATUT\,(T=4,p=0.4) & \\
+1ATUT\,(T=4,p=0.6) & \\
=0\,(8)+1\,(8)=8 & \textit{suggested ATUT}\,(T=4)
\end{array}
\tag{3T}
$$

$$
\begin{array}{rl}
\frac{1}{4}ATUT\,(T=2) & \\
+\frac{1}{2}ATUT\,(T=3) & \textit{ATUT p-score matched sample} \\
+\frac{1}{4}ATUT\,(T=4) & \\
=\frac{1}{4}\,(4)+\frac{1}{2}\,(6)+\frac{1}{4}\,(8)=6 & \textit{suggested ATUT}
\end{array}
\tag{3}
$$

$$
\begin{array}{ll}
g_1\left(T=2,p=0.6\right) & \\
-g_0\left(T=2,p=0.6\right) & \textit{ATE p-score matched sample} \\
=2-(-2)=4 & \textit{suggested } ATE\left(T=2,p=0.6\right) \\
g_1\left(T=3,p=0.6\right) & \\
-g_0\left(T=3,p=0.6\right) & \\
=3-(-3)=6 & \textit{suggested } ATE\left(T=3,p=0.6\right) \\
g_1\left(T=4,p=0.6\right) & \\
-g_0\left(T=4,p=0.6\right) & \\
=4-(-4)=8 & \textit{suggested } ATE\left(T=4,p=0.6\right) \\
g_1\left(T=2,p=0.4\right) & \\
-g_0\left(T=2,p=0.4\right) & \\
=2-(-2)=4 & \textit{suggested } ATE\left(T=2,p=0.4\right) \\
g_1\left(T=3,p=0.4\right) & \\
-g_0\left(T=3,p=0.4\right) & \\
=3-(-3)=6 & \textit{suggested } ATE\left(T=3,p=0.4\right) \\
g_1\left(T=4,p=0.4\right) & \\
-g_0\left(T=4,p=0.4\right) & \\
=4-(-4)=8 & \textit{suggested } ATE\left(T=4,p=0.4\right)
\end{array}
\tag{3Tp}
$$

$$
\begin{array}{ll}
\frac{2}{7}ATE\left(T=2,p=0.4\right) & \\
+\frac{5}{7}ATE\left(T=3,p=0.4\right) & \textit{ATE p-score matched sample} \\
+0ATE\left(T=4,p=0.4\right) & \\
=\frac{2}{7}\left(4\right)+\frac{5}{7}\left(6\right) & \\
+0\left(8\right)=5\frac{3}{7} & \textit{suggested } ATE\left(p=0.4\right) \\
0ATE\left(T=2,p=0.6\right) & \\
+\frac{1}{5}ATE\left(T=3,p=0.6\right) & \\
+\frac{4}{5}ATE\left(T=4,p=0.6\right) & \\
=0\left(4\right)+\frac{1}{5}\left(6\right) & \\
+\frac{4}{5}\left(8\right)=7\frac{3}{5} & \textit{suggested } ATE\left(p=0.6\right)
\end{array}
\tag{3p}
$$

$$
\begin{array}{ll}
1ATE\left(T=2,p=0.4\right) & \\
+0ATE\left(T=2,p=0.6\right) & \textit{ATE p-score matched sample} \\
=1\left(4\right)+0\left(4\right)=4 & \textit{suggested ATE}\left(T=2\right) \\
\frac{5}{6}ATE\left(T=3,p=0.4\right) & \\
+\frac{1}{6}ATE\left(T=3,p=0.6\right) & \\
=\frac{5}{6}\left(6\right)+\frac{1}{6}\left(6\right)=6 & \textit{suggested ATE}\left(T=3\right) \\
0ATE\left(T=4,p=0.4\right) & \\
+1ATE\left(T=4,p=0.6\right) & \\
=0\left(8\right)+1\left(8\right)=8 & \textit{suggested ATE}\left(T=4\right)
\end{array}
\tag{3T}
$$

$$
\begin{array}{ll}
\frac{1}{6}ATE\left(T=2\right) & \\
+\frac{1}{2}ATE\left(T=3\right) & \textit{ATE p-score matched sample} \\
+\frac{1}{3}ATE\left(T=4\right) & \\
=\frac{1}{6}\left(4\right)+\frac{1}{2}\left(6\right)+\frac{1}{3}\left(8\right)=6\frac{1}{3} & \textit{suggested ATE}
\end{array}
\tag{3}
$$

and

$$
\begin{array}{ll}
g_1\left(T=2,p=0.6\right) & \\
-g_0\left(T=2,p=0.6\right) & \textit{unmatched sample} \\
=2-\left(-2\right)=4 & \textit{suggested ATE}\left(T=2,p=0.6\right) \\
g_1\left(T=3,p=0.6\right) & \\
-g_0\left(T=3,p=0.6\right) & \\
=3-\left(-3\right)=6 & \textit{suggested ATE}\left(T=3,p=0.6\right) \\
g_1\left(T=4,p=0.6\right) & \\
-g_0\left(T=4,p=0.6\right) & \\
=4-\left(-4\right)=8 & \textit{suggested ATE}\left(T=4,p=0.6\right) \\
g_1\left(T=2,p=0.4\right) & \\
-g_0\left(T=2,p=0.4\right) & \\
=2-\left(-2\right)=4 & \textit{suggested ATE}\left(T=2,p=0.4\right) \\
g_1\left(T=3,p=0.4\right) & \\
-g_0\left(T=3,p=0.4\right) & \\
=3-\left(-3\right)=6 & \textit{suggested ATE}\left(T=3,p=0.4\right) \\
g_1\left(T=4,p=0.4\right) & \\
-g_0\left(T=4,p=0.4\right) & \\
=4-\left(-4\right)=8 & \textit{suggested ATE}\left(T=4,p=0.4\right)
\end{array}
\tag{3Tp}
$$

$$
\begin{array}{rl}
\frac{1}{3}ATE\left(T=2,p=0.4\right) & \\
+\frac{1}{3}ATE\left(T=3,p=0.4\right) & \textit{unmatched sample} \\
+\frac{1}{3}ATE\left(T=4,p=0.4\right) & \\
=\frac{1}{3}\left(2\right)+\frac{1}{3}\left(4\right) & \\
+\frac{1}{3}\left(2\right)=2\frac{2}{3} & \textit{suggested } ATE\left(p=0.4\right) \\
\frac{1}{3}ATE\left(T=2,p=0.6\right) & \\
+\frac{1}{3}ATE\left(T=3,p=0.6\right) & \\
+\frac{1}{3}ATE\left(T=4,p=0.6\right) & \\
=\frac{1}{3}\left(2\right)+\frac{1}{3}\left(4\right) & \\
+\frac{1}{3}\left(2\right)=2\frac{2}{3} & \textit{suggested } ATE\left(p=0.6\right)
\end{array}
\tag{3p}
$$

$$
\begin{array}{rl}
1ATE\left(T=2,p=0.4\right) & \\
+0ATE\left(T=2,p=0.6\right) & \textit{unmatched sample} \\
=1\left(4\right)+0\left(4\right)=4 & \textit{suggested } ATE\left(T=2\right) \\
\frac{5}{6}ATE\left(T=3,p=0.4\right) & \\
+\frac{1}{6}ATE\left(T=3,p=0.6\right) & \\
=\frac{5}{6}\left(6\right)+\frac{1}{6}\left(6\right)=6 & \textit{suggested } ATE\left(T=3\right) \\
0ATE\left(T=4,p=0.4\right) & \\
+1ATE\left(T=4,p=0.6\right) & \\
=0\left(8\right)+1\left(8\right)=8 & \textit{suggested } ATE\left(T=4\right)
\end{array}
\tag{3T}
$$

$$
\begin{array}{rl}
\frac{1}{6}ATE\left(T=2\right) & \\
+\frac{1}{2}ATE\left(T=3\right) & \textit{unmatched sample} \\
+\frac{1}{3}ATE\left(T=4\right) & \\
=\frac{1}{6}\left(4\right)+\frac{1}{2}\left(6\right)+\frac{1}{3}\left(8\right)=6\frac{1}{3} & \textit{suggested } ATE
\end{array}
\tag{3}
$$

As with design 2, nonparametric design 3 identifies average treatment effects conditional on T and unconditionally but not conditional on mis-specified propensity-score. Design 2 in particular demonstrates the efficacy of DR strategies.

Finally, we note that restricted designs (1a and 2b) may produce inconsistencies when average treatment effects conditional on covariates T vary with T. That is, we may find

$$\widehat{ATE}\left(T\right)\neq\Pr\left(D=1\mid T\right)\widehat{ATT}\left(T\right)+\Pr\left(D=0\mid T\right)\widehat{ATUT}\left(T\right),\quad\text{for some }T$$

or

$$\widehat{ATE} \neq \Pr(D=1)\,\widehat{ATT} + \Pr(D=0)\,\widehat{ATUT}$$

where

$$\widehat{ATE} = E_T\left[\widehat{ATE}(T)\right]$$

and

$$\begin{aligned} & \Pr(D=1)\,\widehat{ATT} + \Pr(D=0)\,\widehat{ATUT} \\ = \;& E_T\left[\Pr(D=1 \mid T)\,\widehat{ATT}(T)\right] + E_T\left[\Pr(D=0 \mid T)\,\widehat{ATUT}(T)\right]. \end{aligned}$$

where $\widehat{\cdot}$ refers to the design-suggested parameter. Designs 1a and 2b for example 4 and design 2b for example 6 exhibit this deficiency — both DGPs have heterogeneous potential outcomes.