9.5 Propensity score approaches

Suppose the data are conditionally mean independent

$$E[Y_1 \mid X, D] = E[Y_1 \mid X]$$

$$E[Y_0 \mid X, D] = E[Y_0 \mid X]$$

so treatment is ignorable,and common X support leads to nondegenerate propensity scores

$$0 < p(X) = \Pr(D = 1 \mid X) < 1 \text{ for all } X$$

then average treatment effect estimands are

$$\begin{aligned} ATE &= E\left[\frac{(D - p(X))\,Y}{p(X)\,(1 - p(X))}\right] \\ ATT &= E\left[\frac{(D - p(X))\,Y}{(1 - p(X))}\right] / \Pr(D = 1) \\ ATUT &= E\left[\frac{(D - p(X))\,Y}{p(X)}\right] / \Pr(D = 0) \end{aligned}$$

The econometric procedure is to first estimate the propensity for treatment or propensity score, $p(X)$, via some flexible model (e.g., nonparametric regression; see chapter 6), then *ATE*, *ATT*, and *ATUT* are consistently estimated via sample analogs to the above.

9.5.1 ATE and propensity score

$ATE = E\left[\frac{(D-p(X))Y}{p(X)(1-p(X))}\right]$ is identified as follows. Observed outcome is

$$Y = DY_1 + (1-D)\,Y_0$$

Substitution for Y and evaluation of the conditional expectation produces

$$\begin{aligned}
& E\left[(D-p(X))\,Y \mid X\right] \\
= \; & E\left[DDY_1 + D(1-D)\,Y_0 - p(X)\,DY_1 - p(X)(1-D)\,Y_0 \mid X\right] \\
= \; & E\left[DY_1 + 0 - p(X)\,DY_1 - p(X)(1-D)\,Y_0 \mid X\right]
\end{aligned}$$

Letting $m_j(X) \equiv E[Y_j \mid X]$ and recognizing

$$\begin{aligned}
p(X) &\equiv Pr(D=1 \mid X) \\
&= E[D \mid X]
\end{aligned}$$

gives

$$\begin{aligned}
& E\left[DY_1 - p(X)\,DY_1 - p(X)(1-D)\,Y_0 \mid X\right] \\
= \; & p(X)\,m_1(X) - p^2(X)\,m_1(X) - p(X)(1-p(X))\,m_0(X) \\
= \; & p(X)(1-p(X))\,(m_1(X) - m_0(X))
\end{aligned}$$

This leads to the conditional average treatment effect

$$\begin{aligned} E\left[\frac{p(X)(1-p(X))(m_1(X)-m_0(X))}{p(X)(1-p(X))} \mid X\right] &= m_1(X)-m_0(X) \\ &= E[Y_1-Y_0 \mid X] \end{aligned}$$

The final connection to the estimand is made by iterated expectations,

$$\begin{aligned} ATE &= E[Y_1-Y_0] \\ &= E_X[E[Y_1-Y_0 \mid X]] \end{aligned}$$

9.5.2 *ATT, ATUT, and propensity score*

Similar logic identifies the estimand for the average treatment effect on the treated

$$ATT = E\left[\frac{(D - p(X))\,Y}{(1 - p(X))}\right] / \Pr(D = 1)$$

Utilize

$$E[(D - p(X))\,Y \mid X] = p(X)\,(1 - p(X))\,(m_1(X) - m_0(X))$$

from the propensity score identification of *ATE*. Eliminating $(1 - p(X))$ and rewriting gives

$$\begin{aligned} & \frac{p(X)\,(1 - p(X))\,(m_1(X) - m_0(X))}{(1 - p(X))} \\ = \;& p(X)\,(m_1(X) - m_0(X)) \\ = \;& \Pr(D = 1 \mid X)\,(E[Y_1 \mid X] - E[Y_0 \mid X]) \end{aligned}$$

Conditional mean independence implies

$$\begin{aligned} & \Pr(D = 1 \mid X)\,(E[Y_1 \mid X] - E[Y_0 \mid X]) \\ = \;& \Pr(D = 1 \mid X)\,(E[Y_1 \mid D = 1, X] - E[Y_0 \mid D = 1, X]) \\ = \;& \Pr(D = 1 \mid X)\,E[Y_1 - Y_0 \mid D = 1, X] \end{aligned}$$

Then, by iterated expectations, we have

$$\begin{aligned} & E_X[\Pr(D = 1 \mid X)\,E[Y_1 - Y_0 \mid D = 1, X]] \\ = \;& \Pr(D = 1)\,E[Y_1 - Y_0 \mid D = 1] \end{aligned}$$

Putting it all together produces the estimand

$$\begin{aligned} ATT &= E_X\left[\frac{(D - p(X))\,Y}{(1 - p(X))}\right] / \Pr(D = 1) \\ &= E[Y_1 - Y_0 \mid D = 1] \end{aligned}$$

For the average treatment effect on the untreated estimand

$$ATUT = E\left[\frac{(D-p(X))\,Y}{p(X)}\right] / \Pr(D=0)$$

identification is analogous to that for *ATT*. Eliminating $p(X)$ from

$$E[(D-p(X))\,Y \mid X] = p(X)\,(1-p(X))\,(m_1(X)-m_0(X))$$

and rewriting gives

$$\begin{aligned} & \frac{p(X)\,(1-p(X))\,(m_1(X)-m_0(X))}{p(X)} \\ = & \ (1-p(X))\,(m_1(X)-m_0(X)) \\ = & \ \Pr(D=0 \mid X)\,(E[Y_1 \mid X]-E[Y_0 \mid X]) \end{aligned}$$

Conditional mean independence implies

$$\begin{aligned} & \Pr(D=0 \mid X)\,(E[Y_1 \mid X]-E[Y_0 \mid X]) \\ = & \ \Pr(D=0 \mid X)\,(E[Y_1 \mid D=0,X]-E[Y_0 \mid D=0,X]) \\ = & \ \Pr(D=0 \mid X)\,E[Y_1-Y_0 \mid D=0,X] \end{aligned}$$

Iterated expectations yields

$$\begin{aligned} & E_X[\Pr(D=0 \mid X)\,E[Y_1-Y_0 \mid D=0,X]] \\ = & \ \Pr(D=0)\,E[Y_1-Y_0 \mid D=0] \end{aligned}$$

Putting everything together produces the estimand

$$\begin{aligned} ATUT &= E\left[\frac{(D-p(X))\,Y}{p(X)}\right] / \Pr(D=0) \\ &= E[Y_1-Y_0 \mid D=0] \end{aligned}$$

Finally, the average treatment effects are connected as follows.

$$\begin{aligned}
ATE &= \Pr(D=1)\, ATT + \Pr(D=0)\, ATUT \\
&= \Pr(D=1)\, E\left[\frac{(D-p(X))\,Y}{(1-p(X))}\right] / \Pr(D=1) \\
&\quad + \Pr(D=0)\, E\left[\frac{(D-p(X))\,Y}{p(X)}\right] / \Pr(D=0) \\
&= E\left[\frac{(D-p(X))\,Y}{(1-p(X))}\right] + E\left[\frac{(D-p(X))\,Y}{p(X)}\right] \\
&= E_X\left[\Pr(D=1 \mid X)\,(E[Y_1 \mid X] - E[Y_0 \mid X])\right] \\
&\quad + E_X\left[\Pr(D=0 \mid X)\,(E[Y_1 \mid X] - E[Y_0 \mid X])\right] \\
&= \Pr(D=1)\, E[Y_1 - Y_0] + \Pr(D=0)\, E[Y_1 - Y_0] \\
&= E[Y_1 - Y_0]
\end{aligned}$$