Ralph's row component

A simple double entry accounting system is defined by $Ay = x$ where $y$ is a vector of transaction amounts $[y_1\ y_2\ y_3]^T$, $x$ is a vector of changes in account balances $[2\ -3\ 1]^T$, and

$$A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix},$$

an incidence matrix.

Suggested:

A. singular value decomposition and pseudoinverse

1. Find the singular value decomposition of $A$ such that $A = UDV^T$ where $V$ is a matrix of orthonormal eigenvectors of $A^TA$, $D$ is a diagonal matrix containing the square root of the eigenvalues of $A^TA$, and $U = AVD^\dagger$ where $D^\dagger$ is a diagonal matrix containing the inverse of the nonzero elements plus the zero elements of $D$.

2. Find the pseudo-inverse of $A$ such that $A^\dagger = VD^\dagger U^T$, $AA^\dagger A = A$, and $A^\dagger AA^\dagger = A^\dagger$.

3. Find the consistent solution for y that lies entirely in the rows of A, the row component, $y_{row} = A^\dagger x$. $Ay_{row} = x$, $AN^T = 0$, and $Ny_{row} = 0$. (Hint: find the nullspace of $A$, $N$.)

4. Verify $y_{row} = P_A\, y_p = A_0^T\, (A_0\, A_0^T)^{-1}\, A_0 y_p$ where $A_0$ is the $A$ matrix after dropping the last row, and $P_A = A^\dagger A = A_0^T\, (A_0\, A_0^T)^{-1}\, A_0$ where $y_p$ is any solution of $Ay = x$. (Hint: form a spanning tree to find $y_p$.)

5. Verify $y_{row} = (I - P_N)y_p = (I - N^T\, (N\, N^T)^{-1}\, N)y_p$ and $I - P_N = P_A$.

B. *QR* decomposition

*QR* decomposition of an *m* x *n* matrix *A* (in this example $A_0^T$) constructs orthonormal columns in *Q* and a square, invertible upper triangular matrix *R* by a Gram-Schmidt, Gaussian elimination-like series of steps.

Gram-Schmidt *QR* algorithm:
Let *a* represent the first column of $A_0^T$. Normalize *a* to form $a_1 = a/(a^Ta)^{(1/2)}$.
Let *a* represent the second column of $A_0^T$. Use Gram-Schmidt to make it orthogonal to $a_1$, $a = (I - a_1a_1^T)a$. Normalize *a* to form $a_2 = a/(a^Ta)^{(1/2)}$.
Next, let *a* represent the third column of $A_0^T$. Use Gram-Schmidt to make it orthogonal to $a_1$ and $a_2$, $a = (I - a_1a_1^T - a_2a_2^T)\ a$ (orthogonalization of the third column can be applied one step-at-a-time since $a_1$ and $a_2$ are already orthogonal). Normalize *a* to form $a_3 = a/(a^Ta)^{(1/2)}$.
Repeat for all *n* columns.
Form *Q* from $[a_1\ a_2\ \ldots\ a_n]$.
Construct $R = Q^TA_0^T$.

Householder *QR* algorithm:
Ralph seeks an operation where the first n rows of $H_n\ldots H_1A_0^T$ yield *R* and the first *n* rows of $H_n\ldots H_1$ yields $Q^T$ where $Q^TQ = I$ and $H_i$ ($i$ = 1, …, n) operates on column *i* of $H_{i-1}\ldots H_0A_0^T$ with $H_0 = I$.
Let $a_1$ represent the first column of $A_0^T$ and $e_i$ be a same length vector of zeros except for a one in position *i*.
Construct $v = a_1 + (a_1^Ta_1)^{(1/2)}\ e_1$.
Now, $H_1 = I - 2\ vv^T/(v^Tv)$.
In the next step, $a_2$ is the second column of $H_1A_0^T$ with zeros above the main diagonal position (in this case, the first row is set to zero).
Repeat the steps above by setting
$v = a_2 + (a_2^Ta_2)^{(1/2)}\ e_2$ and $H_2 = I - 2\ vv^T/(v^Tv)$.
$H_2H_1A_0^T$ creates the first two columns of *R*. The steps are repeated for all columns of $A_0^T$.
Some refinements make the algorithm computationally fast and stable for solving projections, etc.

Suggested:

1. Find the *QR* decomposition of $A_0^T = QR$ where *Q* is a rectangular matrix composed of orthonormal columns and *R* is a square invertible upper triangular matrix such that $A_0^TR^{-1} = Q$.

2. Find the row component of *y* via *QR*. Compare this result with the solution for $y_{row}$ from part A. (Hint: $A_0y = x_0$ where $x_0$ is formed by dropping the last element of *x* and $A_0 = R^TQ^T$. Hence, $R^TQ^Ty = x_0$ or $R^TQ^Ty_{row} = x_0$ so that

$$Q(R^T)^{-1}\ R^TQ^Ty_{row} = y_{row} = Q(R^T)^{-1}\ x_0.)$$