```
Z1 = uz1; X = b * Z1 + c * L1 + ux; Z2 = d * L1 + e * L2 + uz2; Y = a * Z1 + g * X + f * L2 + uy
```

$$f\,L2 + uy + a\,uz1 + g\,(c\,L1 + ux + b\,uz1)$$

```
H = {Z1, Z2, X, Y};
(V = (Outer[Times, H, H] // Expand) //. {L1 L2 → 0, L1 ux → 0, L1 uy → 0, L1 uz1 → 0,
      L1 uz2 → 0, L2 ux → 0, L2 uy → 0, L2 uz1 → 0, L2 uz2 → 0, ux uy → 0, ux uz1 → 0,
      ux uz2 → 0, uy uz1 → 0, uy uz2 → 0, uz1 uz2 → 0} // Simplify) // MatrixForm
```

$$\begin{pmatrix} uz1^2 & 0 & b\,uz1^2 & \\ 0 & d^2\,L1^2 + e^2\,L2^2 + uz2^2 & c\,d\,L1^2 & \\ b\,uz1^2 & c\,d\,L1^2 & c^2\,L1^2 + ux^2 + b^2\,uz1^2 & c^2\,g\,L1^2 + \\ (a + b\,g)\,uz1^2 & c\,d\,g\,L1^2 + e\,f\,L2^2 & c^2\,g\,L1^2 + a\,b\,uz1^2 + g\,(ux^2 + b^2\,uz1^2) & c^2\,g^2\,L1^2 + f^2\,L2^2 + g^2\,u \end{pmatrix}$$

```
Dimensions[V]
```

$$\{4, 4\}$$

```
(* Y onto Z1 unconfounded *)

ryz1 = V[[{4}, {1}]].Inverse[V[[{1}, {1}]]] // Simplify // Flatten
```

$$\{a + b\,g\}$$

```
(* Y onto X confounded *)

ryx = V[[{4}, {3}]].Inverse[V[[{3}, {3}]]] // Simplify // Flatten
```

$$\left\{\frac{c^2\,g\,L1^2 + g\,ux^2 + a\,b\,uz1^2 + b^2\,g\,uz1^2}{c^2\,L1^2 + ux^2 + b^2\,uz1^2}\right\}$$

```
(* Y onto X,Z1,Z2 confounded *)

ryxgz1z2 = V[[{4}, {1, 2, 3}]].Inverse[V[[{1, 2, 3}, {1, 2, 3}]]] // Simplify // Flatten
```

$$\left\{\frac{b\,c\,d\,e\,f\,L1^2\,L2^2 + a\,\left(c^2\,L1^2\,\left(e^2\,L2^2 + uz2^2\right) + ux^2\,\left(d^2\,L1^2 + e^2\,L2^2 + uz2^2\right)\right)}{c^2\,L1^2\,\left(e^2\,L2^2 + uz2^2\right) + ux^2\,\left(d^2\,L1^2 + e^2\,L2^2 + uz2^2\right)},\right.$$
$$\frac{e\,f\,L2^2\,\left(c^2\,L1^2 + ux^2\right)}{c^2\,L1^2\,\left(e^2\,L2^2 + uz2^2\right) + ux^2\,\left(d^2\,L1^2 + e^2\,L2^2 + uz2^2\right)},$$
$$\left.\frac{-c\,d\,e\,f\,L1^2\,L2^2 + c^2\,g\,L1^2\,\left(e^2\,L2^2 + uz2^2\right) + g\,ux^2\,\left(d^2\,L1^2 + e^2\,L2^2 + uz2^2\right)}{c^2\,L1^2\,\left(e^2\,L2^2 + uz2^2\right) + ux^2\,\left(d^2\,L1^2 + e^2\,L2^2 + uz2^2\right)}\right\}$$

```
(* double residual regression for X given Z1,Z2 *)

rygz1z2 = V[[{4}, {1, 2}]].Inverse[V[[{1, 2}, {1, 2}]]] // Simplify // Flatten
```

$$\left\{a + b\,g,\ \frac{c\,d\,g\,L1^2 + e\,f\,L2^2}{d^2\,L1^2 + e^2\,L2^2 + uz2^2}\right\}$$

```
resy = Y - rygz1z2.{Z1, Z2} //. {L1^2 → VL1, L2^2 → VL2, uz2^2 → Vuz2} // Simplify
```

$$f\,L2 + uy + a\,uz1 - (a + b\,g)\,uz1 + g\,(c\,L1 + ux + b\,uz1) - \frac{(d\,L1 + e\,L2 + uz2)\,\left(c\,d\,g\,VL1 + e\,f\,VL2\right)}{d^2\,VL1 + e^2\,VL2 + Vuz2}$$

```
rxgz1z2 = V[[{3}, {1, 2}]].Inverse[V[[{1, 2}, {1, 2}]]] // Simplify // Flatten
```

$$\left\{b,\ \frac{c\,d\,L1^2}{d^2\,L1^2+e^2\,L2^2+uz2^2}\right\}$$

```
resx = X - rxgz1z2.{Z1, Z2} //. {L1^2 → VL1, L2^2 → VL2, uz2^2 → Vuz2} // Simplify
```

$$c\,L1+ux-\frac{c\,d\,(d\,L1+e\,L2+uz2)\,VL1}{d^2\,VL1+e^2\,VL2+Vuz2}$$

```
(resy * resx // Expand) //. {L1 L2 → 0, L1 ux → 0, L1 uy → 0,
   L1 uz1 → 0, L1 uz2 → 0, L2 ux → 0, L2 uy → 0, L2 uz1 → 0, L2 uz2 → 0, ux uy → 0,
   ux uz1 → 0, ux uz2 → 0, uy uz1 → 0, uy uz2 → 0, uz1 uz2 → 0} // Simplify
```

$$\frac{1}{\left(d^2\,VL1+e^2\,VL2+Vuz2\right)^2}\Big(g\,ux^2\left(d^2\,VL1+e^2\,VL2+Vuz2\right)^2-c\,d\,e\,f\left(d^2\,L2^2\,VL1^2-uz2^2\,VL1\,VL2+e^2\,L1^2\,VL2^2+L2^2\,VL1\,Vuz2+L1^2\,VL2\,Vuz2\right)+c^2\,g\left(d^2\left(e^2\,L2^2+uz2^2\right)VL1^2+L1^2\left(e^2\,VL2+Vuz2\right)^2\right)\Big)$$

```
(resx^2 // Expand) //. {L1 L2 → 0, L1 ux → 0, L1 uy → 0, L1 uz1 → 0,
   L1 uz2 → 0, L2 ux → 0, L2 uy → 0, L2 uz1 → 0, L2 uz2 → 0, ux uy → 0,
   ux uz1 → 0, ux uz2 → 0, uy uz1 → 0, uy uz2 → 0, uz1 uz2 → 0} // Simplify
```

$$\left(ux^2\left(d^2\,VL1+e^2\,VL2+Vuz2\right)^2+c^2\left(d^2\left(e^2\,L2^2+uz2^2\right)VL1^2+L1^2\left(e^2\,VL2+Vuz2\right)^2\right)\right)\Big/\left(d^2\,VL1+e^2\,VL2+Vuz2\right)^2$$

```
resyx = %% / % //. {L1^2 → VL1, L2^2 → VL2, uz2^2 → Vuz2} // Simplify
```

$$\left(d^2\,g\,ux^2\,VL1-c\,d\,e\,f\,VL1\,VL2+g\left(ux^2+c^2\,VL1\right)\left(e^2\,VL2+Vuz2\right)\right)\Big/\left(d^2\,ux^2\,VL1+\left(ux^2+c^2\,VL1\right)\left(e^2\,VL2+Vuz2\right)\right)$$

```
rxGz1z2 = (-c d e f L1^2 L2^2 + c^2 g L1^2 (e^2 L2^2 + uz2^2) + g ux^2 (d^2 L1^2 + e^2 L2^2 + uz2^2)) /
      (c^2 L1^2 (e^2 L2^2 + uz2^2) + ux^2 (d^2 L1^2 + e^2 L2^2 + uz2^2)) //.
    {L1^2 → VL1, L2^2 → VL2, uz2^2 → Vuz2} // Simplify
```

$$\left(d^2\,g\,ux^2\,VL1-c\,d\,e\,f\,VL1\,VL2+g\left(ux^2+c^2\,VL1\right)\left(e^2\,VL2+Vuz2\right)\right)\Big/\left(d^2\,ux^2\,VL1+\left(ux^2+c^2\,VL1\right)\left(e^2\,VL2+Vuz2\right)\right)$$

```
rxGz1z2 - resyx // FullSimplify
```

0

```
(* double residual regression for X given Z1 *)
```

```
rygz1 = V[[{4}, {1}]].Inverse[V[[{1}, {1}]]] // Simplify // Flatten
```

$$\{a+b\,g\}$$

```
resygz1 = Y - rygz1 * Z1 // Simplify
```

$$\{c\,g\,L1+f\,L2+g\,ux+uy\}$$

```
rxgz1 = V[[{3}, {1}]].Inverse[V[[{1}, {1}]]] // Simplify // Flatten
```

$$\{b\}$$

```
resxgz1 = X - rxgz1 * Z1 // Simplify
```

$$\{c\ L1 + ux\}$$

```
(resygz1 * resxgz1 // Expand) //. {L1 L2 → 0, L1 ux → 0, L1 uy → 0,
   L1 uz1 → 0, L1 uz2 → 0, L2 ux → 0, L2 uy → 0, L2 uz1 → 0, L2 uz2 → 0, ux uy → 0,
   ux uz1 → 0, ux uz2 → 0, uy uz1 → 0, uy uz2 → 0, uz1 uz2 → 0} // Simplify
```

$$\left\{g\ \left(c^2\ L1^2 + ux^2\right)\right\}$$

```
(resxgz1^2 // Expand) //. {L1 L2 → 0, L1 ux → 0, L1 uy → 0, L1 uz1 → 0,
   L1 uz2 → 0, L2 ux → 0, L2 uy → 0, L2 uz1 → 0, L2 uz2 → 0, ux uy → 0,
   ux uz1 → 0, ux uz2 → 0, uy uz1 → 0, uy uz2 → 0, uz1 uz2 → 0} // Simplify
```

$$\left\{c^2\ L1^2 + ux^2\right\}$$

```
%% / % // Simplify
```

$$\{g\}$$

```
(* Y onto X,Z1 unconfounded *)
```

```
ryxgz1 = V[[{4}, {1, 3}]].Inverse[V[[{1, 3}, {1, 3}]]] // Simplify // Flatten
```

$$\{a, g\}$$

```
(* [1] X and Z1 *)
```

```
(* [2] ryx.z1 = g *)
```

```
ryxgz1z2 /. g → 0
```

$$\Big\{\left(b\ c\ d\ e\ f\ L1^2\ L2^2 + a\ \left(c^2\ L1^2\ \left(e^2\ L2^2 + uz2^2\right) + ux^2\ \left(d^2\ L1^2 + e^2\ L2^2 + uz2^2\right)\right)\right) \Big/ \left(c^2\ L1^2\ \left(e^2\ L2^2 + uz2^2\right) + ux^2\ \left(d^2\ L1^2 + e^2\ L2^2 + uz2^2\right)\right),$$
$$\left(e\ f\ L2^2\ \left(c^2\ L1^2 + ux^2\right)\right) \Big/ \left(c^2\ L1^2\ \left(e^2\ L2^2 + uz2^2\right) + ux^2\ \left(d^2\ L1^2 + e^2\ L2^2 + uz2^2\right)\right),$$
$$-\left(\left(c\ d\ e\ f\ L1^2\ L2^2\right) \Big/ \left(c^2\ L1^2\ \left(e^2\ L2^2 + uz2^2\right) + ux^2\ \left(d^2\ L1^2 + e^2\ L2^2 + uz2^2\right)\right)\right)\Big\}$$

```
(* [3] ryxgz1z2 is confounded;
even if g=0 (no causal effect of x on y) still biased *)
```

```
(* [4] Z1 ind Z2 given null; Z1 not ind Z2 given Y, X or Y&X *)
```

```
V[[{1}, {2, 3}]].Inverse[V[[{2, 3}, {2, 3}]]] // Simplify // Flatten
```

$$\Big\{-\left(\left(b\ c\ d\ L1^2\ uz1^2\right) \Big/ \left(c^2\ L1^2\ \left(e^2\ L2^2 + uz2^2\right) + \left(ux^2 + b^2\ uz1^2\right)\ \left(d^2\ L1^2 + e^2\ L2^2 + uz2^2\right)\right)\right),$$
$$\left(b\ uz1^2\ \left(d^2\ L1^2 + e^2\ L2^2 + uz2^2\right)\right) \Big/ \left(c^2\ L1^2\ \left(e^2\ L2^2 + uz2^2\right) + \left(ux^2 + b^2\ uz1^2\right)\ \left(d^2\ L1^2 + e^2\ L2^2 + uz2^2\right)\right)\Big\}$$

```
V[[{1}, {2, 4}]].Inverse[V[[{2, 4}, {2, 4}]]] // Simplify // Flatten
```

$$\Big\{\big((a+b\,g)\,(-c\,d\,g\,L1^2-e\,f\,L2^2)\,uz1^2\big)\Big/\Big(-(c\,d\,g\,L1^2+e\,f\,L2^2)^2+(c^2\,g^2\,L1^2+f^2\,L2^2+g^2\,ux^2+uy^2+a^2\,uz1^2+2\,a\,b\,g\,uz1^2+b^2\,g^2\,uz1^2)\,(d^2\,L1^2+e^2\,L2^2+uz2^2)\Big),$$
$$\big((a+b\,g)\,uz1^2\,(d^2\,L1^2+e^2\,L2^2+uz2^2)\big)\Big/\Big(-(c\,d\,g\,L1^2+e\,f\,L2^2)^2+(c^2\,g^2\,L1^2+f^2\,L2^2+g^2\,ux^2+uy^2+a^2\,uz1^2+2\,a\,b\,g\,uz1^2+b^2\,g^2\,uz1^2)\,(d^2\,L1^2+e^2\,L2^2+uz2^2)\Big)\Big\}$$

```
V[[{1}, {2, 3, 4}]].Inverse[V[[{2, 3, 4}, {2, 3, 4}]]] // Simplify // Flatten
```

$$\Big\{-\Big(\big(\big(a\,e\,f\,L2^2\,(c^2\,L1^2+ux^2)+b\,c\,d\,L1^2\,(f^2\,L2^2+uy^2)\big)\,uz1^2\big)\Big/$$
$$\big(e^2\,L2^2\,ux^2\,uy^2+2\,a\,b\,c\,d\,e\,f\,L1^2\,L2^2\,uz1^2+a^2\,e^2\,L2^2\,ux^2\,uz1^2+b^2\,e^2\,L2^2\,uy^2\,uz1^2+$$
$$d^2\,L1^2\,\big(b^2\,uy^2\,uz1^2+ux^2\,(uy^2+a^2\,uz1^2)+f^2\,L2^2\,(ux^2+b^2\,uz1^2)\big)+$$
$$f^2\,L2^2\,ux^2\,uz2^2+ux^2\,uy^2\,uz2^2+b^2\,f^2\,L2^2\,uz1^2\,uz2^2+a^2\,ux^2\,uz1^2\,uz2^2+$$
$$b^2\,uy^2\,uz1^2\,uz2^2+c^2\,L1^2\,\big(e^2\,L2^2\,(uy^2+a^2\,uz1^2)+(f^2\,L2^2+uy^2+a^2\,uz1^2)\,uz2^2\big)\big)\Big),$$
$$\Big(uz1^2\,\Big(b\,\big(-c\,d\,e\,f\,g\,L1^2\,L2^2+e^2\,L2^2\,uy^2+d^2\,L1^2\,(f^2\,L2^2+uy^2)+(f^2\,L2^2+uy^2)\,uz2^2\big)-$$
$$a\,\big(-c\,d\,e\,f\,L1^2\,L2^2+c^2\,g\,L1^2\,(e^2\,L2^2+uz2^2)+g\,ux^2\,(d^2\,L1^2+e^2\,L2^2+uz2^2)\big)\Big)\Big)\Big/$$
$$\big(e^2\,L2^2\,ux^2\,uy^2+2\,a\,b\,c\,d\,e\,f\,L1^2\,L2^2\,uz1^2+a^2\,e^2\,L2^2\,ux^2\,uz1^2+b^2\,e^2\,L2^2\,uy^2\,uz1^2+$$
$$d^2\,L1^2\,\big(b^2\,uy^2\,uz1^2+ux^2\,(uy^2+a^2\,uz1^2)+f^2\,L2^2\,(ux^2+b^2\,uz1^2)\big)+$$
$$f^2\,L2^2\,ux^2\,uz2^2+ux^2\,uy^2\,uz2^2+b^2\,f^2\,L2^2\,uz1^2\,uz2^2+a^2\,ux^2\,uz1^2\,uz2^2+$$
$$b^2\,uy^2\,uz1^2\,uz2^2+c^2\,L1^2\,\big(e^2\,L2^2\,(uy^2+a^2\,uz1^2)+(f^2\,L2^2+uy^2+a^2\,uz1^2)\,uz2^2\big)\big),$$
$$\Big(uz1^2\,\Big(b\,c\,d\,e\,f\,L1^2\,L2^2+a\,\big(c^2\,L1^2\,(e^2\,L2^2+uz2^2)+ux^2\,(d^2\,L1^2+e^2\,L2^2+uz2^2)\big)\Big)\Big)\Big/$$
$$\big(e^2\,L2^2\,ux^2\,uy^2+2\,a\,b\,c\,d\,e\,f\,L1^2\,L2^2\,uz1^2+a^2\,e^2\,L2^2\,ux^2\,uz1^2+b^2\,e^2\,L2^2\,uy^2\,uz1^2+$$
$$d^2\,L1^2\,\big(b^2\,uy^2\,uz1^2+ux^2\,(uy^2+a^2\,uz1^2)+f^2\,L2^2\,(ux^2+b^2\,uz1^2)\big)+$$
$$f^2\,L2^2\,ux^2\,uz2^2+ux^2\,uy^2\,uz2^2+b^2\,f^2\,L2^2\,uz1^2\,uz2^2+a^2\,ux^2\,uz1^2\,uz2^2+$$
$$b^2\,uy^2\,uz1^2\,uz2^2+c^2\,L1^2\,\big(e^2\,L2^2\,(uy^2+a^2\,uz1^2)+(f^2\,L2^2+uy^2+a^2\,uz1^2)\,uz2^2\big)\big)\Big\}$$

```
(* [5] same as [4] and X ind Y given Z1 if g =
 0 but X not ind Y given Z2 or Z1&Z2 if g = 0 *)
V[[{4}, {2, 3}]].Inverse[V[[{2, 3}, {2, 3}]]] // Simplify // Flatten
```

$$\Big\{\big(c^2\,e\,f\,L1^2\,L2^2-a\,b\,c\,d\,L1^2\,uz1^2+e\,f\,L2^2\,(ux^2+b^2\,uz1^2)\big)\Big/$$
$$\big(c^2\,L1^2\,(e^2\,L2^2+uz2^2)+(ux^2+b^2\,uz1^2)\,(d^2\,L1^2+e^2\,L2^2+uz2^2)\big),$$
$$\big(-c\,d\,e\,f\,L1^2\,L2^2+c^2\,g\,L1^2\,(e^2\,L2^2+uz2^2)+$$
$$(a\,b\,uz1^2+g\,(ux^2+b^2\,uz1^2))\,(d^2\,L1^2+e^2\,L2^2+uz2^2)\big)\Big/$$
$$\big(c^2\,L1^2\,(e^2\,L2^2+uz2^2)+(ux^2+b^2\,uz1^2)\,(d^2\,L1^2+e^2\,L2^2+uz2^2)\big)\Big\}$$

```
(* [6] revisits direction of X,
Z1 relation since Z0 ind Z1 if Z1→X unless condition on X but Z0 not ind Z1 if X→
  Z1 unless condition on X ([4] already addreses this);
main point:** if there is a bow between X and Y or Z1 not observed,
Z0 allows identification of X→
 Y causal effect ** (do vanishing tetrads discover this latent variable L3? no);
ryz0 = gh; rxz0 = h; ryz0/rxz0 = g IV estimand *)

Z0 = uz0; Z1 = uz1; X = b * Z1 + c * L1 + h * Z0 + i * L3 + ux;
Z2 = d * L1 + e * L2 + uz2; Y = a * Z1 + g * X + f * L2 + j * L3 + uy
```

$$f\,L2+j\,L3+uy+a\,uz1+g\,(c\,L1+i\,L3+ux+h\,uz0+b\,uz1)$$

```
H = {Z1, Z2, Z0, X, Y};
(V = (Outer[Times, H, H] // Expand) //. {L1 L2 → 0, L1 ux → 0, L1 uy → 0, L1 uz1 → 0,
      L1 uz2 → 0, L1 uz0 → 0, L1 L3 → 0, L3 L2 → 0, L3 ux → 0, L3 uy → 0, L3 uz1 → 0,
      L3 uz2 → 0, L3 uz0 → 0, L2 ux → 0, L2 uy → 0, L2 uz1 → 0, L2 uz2 → 0, L2 uz0 → 0,
      ux uy → 0, ux uz1 → 0, ux uz2 → 0, ux uz0 → 0, uy uz1 → 0, uy uz2 → 0,
      uy uz0 → 0, uz1 uz2 → 0, uz1 uz0 → 0, uz2 uz0 → 0} // Simplify) // MatrixForm
```

$$\begin{pmatrix} uz1^2 & 0 & 0 & b\,uz1^2 \\ 0 & d^2\,L1^2 + e^2\,L2^2 + uz2^2 & 0 & c\,d\,L1^2 \\ 0 & 0 & uz0^2 & h\,uz0^2 \\ b\,uz1^2 & c\,d\,L1^2 & h\,uz0^2 & c^2\,L1^2 + i^2\,L3^2 + ux^2 + h^2\,uz0^2 + b^2\,uz1^2 \\ (a + b\,g)\,uz1^2 & c\,d\,g\,L1^2 + e\,f\,L2^2 & g\,h\,uz0^2 & c^2\,g\,L1^2 + i\,j\,L3^2 + a\,b\,uz1^2 + g\,(i^2\,L3^2 + ux^2 + h^2\,uz0 \end{pmatrix}$$

```
ryz0 = V[[{5}, {3}]].Inverse[V[[{3}, {3}]]] // Simplify // Flatten
```

$\{g\,h\}$

```
rxz0 = V[[{4}, {3}]].Inverse[V[[{3}, {3}]]] // Simplify // Flatten
```

$\{h\}$

```
ryz0 / rxz0
```

$\{g\}$

```
(* (Z'X)^(-1) (Z'Y) *)

Inverse[V[[{3}, {4}]]].V[[{3}, {5}]] // Simplify // Flatten
```

$\{g\}$

```
(* ryx confounded *)

ryx = V[[{5}, {4}]].Inverse[V[[{4}, {4}]]] // Simplify // Flatten
```

$$\left\{\left(c^2\,g\,L1^2 + i\,j\,L3^2 + a\,b\,uz1^2 + g\left(i^2\,L3^2 + ux^2 + h^2\,uz0^2 + b^2\,uz1^2\right)\right) \Big/ \left(c^2\,L1^2 + i^2\,L3^2 + ux^2 + h^2\,uz0^2 + b^2\,uz1^2\right)\right\}$$