```
(* a simple IV *)

Z = uz; X = a * Z + c * L + ux; Y = b * X + d * L + uy
```

$d L + uy + b (c L + ux + a uz)$

```
H = {Z, X, Y}; (V = (Outer[Times, H, H] // Expand) //.
    {L uz → 0, L ux → 0, L uy → 0, uz ux → 0, uz uy → 0, ux uy → 0}) // MatrixForm
```

$$\begin{pmatrix} uz^2 & a\,uz^2 & a\,b\,uz^2 \\ a\,uz^2 & c^2 L^2 + ux^2 + a^2 uz^2 & b\,c^2 L^2 + c\,d\,L^2 + b\,ux^2 + a^2 b\,uz^2 \\ a\,b\,uz^2 & b\,c^2 L^2 + c\,d\,L^2 + b\,ux^2 + a^2 b\,uz^2 & b^2 c^2 L^2 + 2\,b\,c\,d\,L^2 + d^2 L^2 + b^2 ux^2 + uy^2 + a^2 b^2 uz^2 \end{pmatrix}$$

```
(* ryx confounded *)

V[[{3}, {2}]].Inverse[V[[{2}, {2}]]] // Flatten
```

$$\left\{\frac{b\,c^2 L^2 + c\,d\,L^2 + b\,ux^2 + a^2 b\,uz^2}{c^2 L^2 + ux^2 + a^2 uz^2}\right\}$$

```
(* Bayes normal *)

ryz = V[[{3}, {1}]].Inverse[V[[{1}, {1}]]] // Flatten
```

$\{a\,b\}$

```
rxz = V[[{2}, {1}]].Inverse[V[[{1}, {1}]]] // Flatten
```

$\{a\}$

```
iv = ryz / rxz
```

$\{b\}$

```
(* biv = (Z'X)^(-1) (Z'Y) projection *)

Inverse[V[[{1}, {2}]]].V[[{1}, {3}]] // Simplify // Flatten
```

$\{b\}$

```
(* IV via projecting X onto Z then Y onto Xhat *)

xhat = (V[[{2}, {1}]].Inverse[V[[{1}, {1}]]] * Z // Flatten)[[1]]
```

$a\,uz$

```
H = {Z, xhat, Y}; (V = (Outer[Times, H, H] // Expand) //.
    {L uz → 0, L ux → 0, L uy → 0, uz ux → 0, uz uy → 0, ux uy → 0}) // MatrixForm
```

$$\begin{pmatrix} uz^2 & a\,uz^2 & a\,b\,uz^2 \\ a\,uz^2 & a^2 uz^2 & a^2 b\,uz^2 \\ a\,b\,uz^2 & a^2 b\,uz^2 & b^2 c^2 L^2 + 2\,b\,c\,d\,L^2 + d^2 L^2 + b^2 ux^2 + uy^2 + a^2 b^2 uz^2 \end{pmatrix}$$

```
V[[{3}, {2}]].Inverse[V[[{2}, {2}]]] // Flatten
```

$\{b\}$

```
(* IV via projecting X onto Z then Y onto
 X and leftnull component of X with respect to Z *)
```

```
res = X - xhat
```

$c L + ux$

```
H = {res, X, Y}; (V = (Outer[Times, H, H] // Expand) //.
    {L uz → 0, L ux → 0, L uy → 0, uz ux → 0, uz uy → 0, ux uy → 0}) // MatrixForm
```

$$\begin{pmatrix} c^2 L^2 + ux^2 & c^2 L^2 + ux^2 & b c^2 L^2 + c d L^2 + b ux^2 \\ c^2 L^2 + ux^2 & c^2 L^2 + ux^2 + a^2 uz^2 & b c^2 L^2 + c d L^2 + b ux^2 + a^2 b uz^2 \\ b c^2 L^2 + c d L^2 + b ux^2 & b c^2 L^2 + c d L^2 + b ux^2 + a^2 b uz^2 & b^2 c^2 L^2 + 2 b c d L^2 + d^2 L^2 + b^2 ux^2 + uy^2 + a^2 \end{pmatrix}$$

```
V[[{3}, {1, 2}]].Inverse[V[[{1, 2}, {1, 2}]]] // Simplify // Flatten
```

$$\left\{\frac{c d L^2}{c^2 L^2 + ux^2}, b\right\}$$

```
(* b conditional IV *)
```

```
W = e * Z + uw; Y = b * X + d * L + f * W + uy
```

$d L + uy + b (c L + ux + a uz) + f (uw + e uz)$

```
H = {Z, X, Y, W};
(V = (Outer[Times, H, H] // Expand) //. {L uz → 0, L ux → 0, L uy → 0, L uw → 0,
     uz ux → 0, uz uy → 0, uz uw → 0, ux uy → 0, ux uw → 0, uy uw → 0}) // MatrixForm
```

$$\begin{pmatrix} uz^2 & a uz^2 & a b u \\ a uz^2 & c^2 L^2 + ux^2 + a^2 uz^2 & b c^2 L^2 + c d L^2 + b \\ a b uz^2 + e f uz^2 & b c^2 L^2 + c d L^2 + b ux^2 + a^2 b uz^2 + a e f uz^2 & b^2 c^2 L^2 + 2 b c d L^2 + d^2 L^2 + f^2 uw^2 + b^2 \\ e uz^2 & a e uz^2 & f uw^2 + a b \end{pmatrix}$$

```
(* ryx confounded *)
```

```
V[[{3}, {2}]].Inverse[V[[{2}, {2}]]] // Flatten
```

$$\left\{\left(b c^2 L^2 + c d L^2 + b ux^2 + a^2 b uz^2 + a e f uz^2\right) / \left(c^2 L^2 + ux^2 + a^2 uz^2\right)\right\}$$

```
(* Bayes normal *)
```

```
ryzw = V[[{3}, {1, 4}]].Inverse[V[[{1, 4}, {1, 4}]]] // Flatten // Simplify
```

$\{a b, f\}$

```
rxzw = V[[{2}, {1, 4}]].Inverse[V[[{1, 4}, {1, 4}]]] // Flatten // Simplify
```

$\{a, 0\}$

```
iv = ryzw[[1]] / rxzw[[1]]
```

$b$

```
(* double residual regression Z given W *)
```

```
reszGw = (Z - V[[{1}, {4}]].Inverse[V[[{4}, {4}]]] * W // Flatten)[[1]] //.
   {uz^2 → Vuz, uw^2 → Vuw} // Together
```

$$\frac{\text{uz Vuw} - \text{e uw Vuz}}{\text{Vuw} + \text{e}^2\text{ Vuz}}$$

```
(X * reszGw // Expand) //. {L uz → 0, L ux → 0, L uy → 0, L uw → 0, uz ux → 0, uz uy → 0,
   uz uw → 0, ux uy → 0, ux uw → 0, uy uw → 0} //. {uz^2 → Vuz, uw^2 → Vuw}
```

$$\frac{\text{a Vuw Vuz}}{\text{Vuw} + \text{e}^2\text{ Vuz}}$$

```
(reszGw^2 // Expand) //. {L uz → 0, L ux → 0, L uy → 0, L uw → 0, uz ux → 0, uz uy → 0,
   uz uw → 0, ux uy → 0, ux uw → 0, uy uw → 0} //. {uz^2 → Vuz, uw^2 → Vuw} // Together
```

$$\frac{\text{Vuw Vuz}}{\text{Vuw} + \text{e}^2\text{ Vuz}}$$

```
rxzGw = %% / %
```

a

```
(Y * reszGw // Expand) //. {L uz → 0, L ux → 0, L uy → 0, L uw → 0, uz ux → 0, uz uy → 0,
   uz uw → 0, ux uy → 0, ux uw → 0, uy uw → 0} //. {uz^2 → Vuz, uw^2 → Vuw}
```

$$\frac{\text{a b Vuw Vuz}}{\text{Vuw} + \text{e}^2\text{ Vuz}}$$

```
(reszGw^2 // Expand) //. {L uz → 0, L ux → 0, L uy → 0, L uw → 0, uz ux → 0, uz uy → 0,
   uz uw → 0, ux uy → 0, ux uw → 0, uy uw → 0} //. {uz^2 → Vuz, uw^2 → Vuw} // Together
```

$$\frac{\text{Vuw Vuz}}{\text{Vuw} + \text{e}^2\text{ Vuz}}$$

```
ryzGw = %% / %
```

a b

```
ryzGw / rxzGw
```

b

```
(* biv = (Z'X)^(-1) (Z'Y) *)

Inverse[V[[{1, 4}, {2, 4}]]].V[[{1, 4}, {3}]] // Simplify // Flatten
```

{b, f}

```
(* IV via projecting X onto Z,W then Y onto Xhat,W *)

xhat = (V[[{2}, {1, 4}]].Inverse[V[[{1, 4}, {1, 4}]]].{Z, W} // Flatten) // Simplify
```

{a uz}

```
xhat = xhat[[1]]
```

a uz

```
H = {W, xhat, Y};
(V = (Outer[Times, H, H] // Expand) //. {L uz → 0, L ux → 0, L uy → 0, L uw → 0,
     uz ux → 0, uz uy → 0, uz uw → 0, ux uy → 0, ux uw → 0, uy uw → 0}) // MatrixForm
```

$$\begin{pmatrix} uw^2 + e^2\, uz^2 & a\, e\, uz^2 & f\, uw^2 + a\, b\, e\, uz^2 + e^2\, f\, u \\ a\, e\, uz^2 & a^2\, uz^2 & a^2\, b\, uz^2 + a\, e\, f\, uz^2 \\ f\, uw^2 + a\, b\, e\, uz^2 + e^2\, f\, uz^2 & a^2\, b\, uz^2 + a\, e\, f\, uz^2 & b^2\, c^2\, L^2 + 2\, b\, c\, d\, L^2 + d^2\, L^2 + f^2\, uw^2 + b^2\, ux^2 + uy^2 + a^2 \end{pmatrix}$$

```
V[[{3}, {1, 2}]].Inverse[V[[{1, 2}, {1, 2}]]] // Flatten // Simplify
```

{f, b}

```
(* IV via projecting X onto Z then Y
 onto X and leftnull of X with respect to Z *)
```

```
res = X - xhat
```

$c\, L + ux$

```
H = {res, X, Y, W};
(V = (Outer[Times, H, H] // Expand) //. {L uz → 0, L ux → 0, L uy → 0, L uw → 0,
     uz ux → 0, uz uy → 0, uz uw → 0, ux uy → 0, ux uw → 0, uy uw → 0}) // MatrixForm
```

$$\begin{pmatrix} c^2\, L^2 + ux^2 & c^2\, L^2 + ux^2 & b \\ c^2\, L^2 + ux^2 & c^2\, L^2 + ux^2 + a^2\, uz^2 & b\, c^2\, L^2 + c\, d \\ b\, c^2\, L^2 + c\, d\, L^2 + b\, ux^2 & b\, c^2\, L^2 + c\, d\, L^2 + b\, ux^2 + a^2\, b\, uz^2 + a\, e\, f\, uz^2 & b^2\, c^2\, L^2 + 2\, b\, c\, d\, L^2 + d^2\, L^2 + f^2\, uw \\ 0 & a\, e\, uz^2 & f\, uw \end{pmatrix}$$

```
V[[{3}, {1, 2, 4}]].Inverse[V[[{1, 2, 4}, {1, 2, 4}]]] // Simplify // Flatten
```

$$\left\{\frac{c\, d\, L^2}{c^2\, L^2 + ux^2},\ b,\ f\right\}$$

```
(* b conditional IV; suppose W→Z (reversed) *)
```

```
W = uw; Z = e * W + uz; X = a * Z + c * L + ux; Y = b * X + d * L + f * W + uy
```

$d\, L + f\, uw + uy + b\, (c\, L + ux + a\, (e\, uw + uz))$

```
H = {Z, X, Y, W};
(V = (Outer[Times, H, H] // Expand) //. {L uz → 0, L ux → 0, L uy → 0, L uw → 0,
     uz ux → 0, uz uy → 0, uz uw → 0, ux uy → 0, ux uw → 0, uy uw → 0}) // MatrixForm
```

$$\begin{pmatrix} e^2\, uw^2 + uz^2 & a\, e^2\, uw^2 + a\, uz^2 & \\ a\, e^2\, uw^2 + a\, uz^2 & c^2\, L^2 + a^2\, e^2\, uw^2 + ux^2 + a^2\, uz^2 & \\ a\, b\, e^2\, uw^2 + e\, f\, uw^2 + a\, b\, uz^2 & b\, c^2\, L^2 + c\, d\, L^2 + a^2\, b\, e^2\, uw^2 + a\, e\, f\, uw^2 + b\, ux^2 + a^2\, b\, uz^2 & b^2\, c^2\, L^2 + 2\, b\, c \\ e\, uw^2 & a\, e\, uw^2 & \end{pmatrix}$$

```
(* ryx confounded *)
```

```
V[[{3}, {2}]].Inverse[V[[{2}, {2}]]] // Flatten
```

$$\left\{\left(b\, c^2\, L^2 + c\, d\, L^2 + a^2\, b\, e^2\, uw^2 + a\, e\, f\, uw^2 + b\, ux^2 + a^2\, b\, uz^2\right) \big/ \left(c^2\, L^2 + a^2\, e^2\, uw^2 + ux^2 + a^2\, uz^2\right)\right\}$$

```
(* Bayes normal *)
```

```
ryzw = V[[{3}, {1, 4}]].Inverse[V[[{1, 4}, {1, 4}]]] // Flatten // Simplify
```

{a b, f}

```
rxzw = V[[{2}, {1, 4}]].Inverse[V[[{1, 4}, {1, 4}]]] // Flatten // Simplify
```

{a, 0}

```
iv = ryzw[[1]] / rxzw[[1]]
```

b

```
(* biv = (Z'X)^(-1) (Z'Y) *)
Inverse[V[[{1, 4}, {2, 4}]]].V[[{1, 4}, {3}]] // Simplify // Flatten
```

{b, f}

```
(* IV via projecting X onto Z,W then Y onto Xhat,W *)
xhat = (V[[{2}, {1, 4}]].Inverse[V[[{1, 4}, {1, 4}]]].{Z, W} // Flatten) // Simplify
```

{a (e uw + uz)}

```
xhat = xhat[[1]]
```

a (e uw + uz)

```
H = {W, xhat, Y};
(V = (Outer[Times, H, H] // Expand) //. {L uz → 0, L ux → 0, L uy → 0, L uw → 0,
     uz ux → 0, uz uy → 0, uz uw → 0, ux uy → 0, ux uw → 0, uy uw → 0}) // MatrixForm
```

$$\begin{pmatrix} uw^2 & a\,e\,uw^2 & a\,b\,e\,uw^2 + f\,uw \\ a\,e\,uw^2 & a^2 e^2 uw^2 + a^2 uz^2 & a^2 b\,e^2 uw^2 + a\,e\,f\,uw^2 \\ a\,b\,e\,uw^2 + f\,uw^2 & a^2 b\,e^2 uw^2 + a\,e\,f\,uw^2 + a^2 b\,uz^2 & b^2 c^2 L^2 + 2\,b\,c\,d\,L^2 + d^2 L^2 + a^2 b^2 e^2 uw^2 + 2\,a\,b\,e\,f \end{pmatrix}$$

```
V[[{3}, {1, 2}]].Inverse[V[[{1, 2}, {1, 2}]]] // Flatten // Simplify
```

{f, b}

```
(* IV via projecting X onto Z then Y
 onto X and leftnull of X with respect to Z *)
res = X - xhat
```

c L + ux

```
H = {res, X, Y, W};
(V = (Outer[Times, H, H] // Expand) //. {L uz → 0, L ux → 0, L uy → 0, L uw → 0,
     uz ux → 0, uz uy → 0, uz uw → 0, ux uy → 0, ux uw → 0, uy uw → 0}) // MatrixForm
```

$$\begin{pmatrix} c^2 L^2 + ux^2 & c^2 L^2 + ux^2 & \\ c^2 L^2 + ux^2 & c^2 L^2 + a^2 e^2 uw^2 + ux^2 + a^2 uz^2 & b\,c^2 \\ b\,c^2 L^2 + c\,d\,L^2 + b\,ux^2 & b\,c^2 L^2 + c\,d\,L^2 + a^2 b\,e^2 uw^2 + a\,e\,f\,uw^2 + b\,ux^2 + a^2 b\,uz^2 & b^2 c^2 L^2 + 2\,b\,c\,d\,L^2 + \\ 0 & a\,e\,uw^2 & \end{pmatrix}$$

```
V[[{3}, {1, 2, 4}]].Inverse[V[[{1, 2, 4}, {1, 2, 4}]]] // Simplify // Flatten
```

$$\left\{\frac{c\,d\,L^2}{c^2 L^2 + ux^2}, b, f\right\}$$

```
(* c IV with additional confounders *)

Z = e * L2 + uz; X = a * Z + c * L + f * L2 + ux; Y = b * X + d * L + uy
```

$d\,L + uy + b\,(c\,L + f\,L2 + ux + a\,(e\,L2 + uz))$

```
H = {Z, X, Y};
(V = (Outer[Times, H, H] // Expand) //. {L uz → 0, L ux → 0, L uy → 0, L L2 → 0,
      uz ux → 0, uz uy → 0, uz L2 → 0, ux uy → 0, ux L2 → 0, uy L2 → 0}) // MatrixForm
```

$$\begin{pmatrix} e^2\,L2^2 + uz^2 & a\,e^2\,L2^2 + e\,f\,L2^2 + a\,uz^2 \\ a\,e^2\,L2^2 + e\,f\,L2^2 + a\,uz^2 & c^2\,L^2 + a^2\,e^2\,L2^2 + 2\,a\,e\,f\,L2^2 + f^2\,L2^2 + ux^2 + a^2\,uz^2 \\ a\,b\,e^2\,L2^2 + b\,e\,f\,L2^2 + a\,b\,uz^2 & b\,c^2\,L^2 + c\,d\,L^2 + a^2\,b\,e^2\,L2^2 + 2\,a\,b\,e\,f\,L2^2 + b\,f^2\,L2^2 + b\,ux^2 + a^2\,b\,uz^2 \end{pmatrix}$$

```
(* ryx confounded *)

V[[{3}, {2}]].Inverse[V[[{2}, {2}]]] // Flatten // Simplify
```

$$\{(c\,d\,L^2 + b\,(c^2\,L^2 + 2\,a\,e\,f\,L2^2 + f^2\,L2^2 + ux^2 + a^2\,(e^2\,L2^2 + uz^2)))\,/\,(c^2\,L^2 + 2\,a\,e\,f\,L2^2 + f^2\,L2^2 + ux^2 + a^2\,(e^2\,L2^2 + uz^2))\}$$

```
(* Bayes normal *)

ryz = V[[{3}, {1}]].Inverse[V[[{1}, {1}]]] // Flatten // Simplify
```

$$\left\{\frac{b\,(e\,f\,L2^2 + a\,(e^2\,L2^2 + uz^2))}{e^2\,L2^2 + uz^2}\right\}$$

```
rxz = V[[{2}, {1}]].Inverse[V[[{1}, {1}]]] // Flatten // Simplify
```

$$\left\{\frac{a\,e^2\,L2^2 + e\,f\,L2^2 + a\,uz^2}{e^2\,L2^2 + uz^2}\right\}$$

```
iv = ryz / rxz // Simplify
```

{b}

```
(* biv = (Z'X)^(-1) (Z'Y) *)

Inverse[V[[{1}, {2}]]].V[[{1}, {3}]] // Simplify // Flatten
```

{b}

```
(* IV via projecting X onto Z then Y onto Xhat *)

xhat = q * Z
```

q (e L2 + uz)

```
res = X - xhat // Simplify
```

c L + f L2 + ux + a (e L2 + uz) - q (e L2 + uz)

```
H = {res, xhat, Y};
(V = (Outer[Times, H, H] // Expand) //. {L uz → 0, L ux → 0, L uy → 0, L L2 → 0, uz ux → 0,
      uz uy → 0, uz L2 → 0, ux uy → 0, ux L2 → 0, uy L2 → 0} // Simplify) // MatrixForm
```

$$\begin{pmatrix} c^2 L^2 + f^2 L2^2 - 2 e f L2^2 q + e^2 L2^2 q^2 + ux^2 + q^2 uz^2 + a^2 (e^2 L2^2 + uz^2) - 2 a (-e f L2^2 + e^2 L2^2 q + q uz \\ q (e f L2^2 - e^2 L2^2 q - q uz^2 + a (e^2 L2^2 + uz^2)) \\ c d L^2 + b (c^2 L^2 + f^2 L2^2 - e f L2^2 q + ux^2 + a^2 (e^2 L2^2 + uz^2) - a (-2 e f L2^2 + e^2 L2^2 q + q uz^2)) \end{pmatrix}$$

```
iv = V[[{3}, {2}]].Inverse[V[[{2}, {2}]]] // Flatten // Simplify
```

$$\left\{\frac{b (e f L2^2 + a (e^2 L2^2 + uz^2))}{q (e^2 L2^2 + uz^2)}\right\}$$

```
iv /. q → rxz // Simplify // Flatten
```

{b}

```
(* IV via projecting X onto Z then Y
 onto X and leftnull of X with respect to Z *)
```

```
H = {res, X, Y};
(V = (Outer[Times, H, H] // Expand) //. {L uz → 0, L ux → 0, L uy → 0, L L2 → 0,
     uz ux → 0, uz uy → 0, uz L2 → 0, ux uy → 0, ux L2 → 0, uy L2 → 0}) // MatrixForm
```

$$\begin{pmatrix} c^2 L^2 + a^2 e^2 L2^2 + 2 a e f L2^2 + f^2 L2^2 - 2 a e^2 L2^2 q - 2 e f L2^2 q + e^2 L2^2 q^2 + ux^2 + a^2 uz^2 - 2 a q uz^2 + \\ c^2 L^2 + a^2 e^2 L2^2 + 2 a e f L2^2 + f^2 L2^2 - a e^2 L2^2 q - e f L2^2 q + ux^2 + a^2 uz^2 - a q uz^2 \\ b c^2 L^2 + c d L^2 + a^2 b e^2 L2^2 + 2 a b e f L2^2 + b f^2 L2^2 - a b e^2 L2^2 q - b e f L2^2 q + b ux^2 + a^2 b uz^2 - a b \end{pmatrix}$$

```
iv = V[[{3}, {1, 2}]].Inverse[V[[{1, 2}, {1, 2}]]] // Simplify // Flatten
```

$$\{(c d L^2 (e f L2^2 + a (e^2 L2^2 + uz^2))) / (q (e^2 L2^2 ux^2 + (f^2 L2^2 + ux^2) uz^2 + c^2 L^2 (e^2 L2^2 + uz^2))),$$
$$(-a c d L^2 (e^2 L2^2 + uz^2) + b c^2 L^2 q (e^2 L2^2 + uz^2) +$$
$$c d L^2 (-e f L2^2 + e^2 L2^2 q + q uz^2) + b q (e^2 L2^2 ux^2 + (f^2 L2^2 + ux^2) uz^2)) /$$
$$(q (e^2 L2^2 ux^2 + (f^2 L2^2 + ux^2) uz^2 + c^2 L^2 (e^2 L2^2 + uz^2)))\}$$

```
iv /. q → rxz // Simplify // Flatten
```

$$\{(c d L^2 (e^2 L2^2 + uz^2)) / (e^2 L2^2 ux^2 + (f^2 L2^2 + ux^2) uz^2 + c^2 L^2 (e^2 L2^2 + uz^2)), b\}$$

```
(* c' conditional IV (W) with additional confounders *)
```

```
Z = e * L2 + uz; W = g * Z + uw; X = a * Z + c * L + f * L2 + ux; Y = b * X + d * L + h * W + uy
```

d L + uy + b (c L + f L2 + ux + a (e L2 + uz)) + h (uw + g (e L2 + uz))

```
H = {Z, X, Y, W};
(V = (Outer[Times, H, H] // Expand) //. {L uz → 0, L ux → 0, L uy → 0, L L2 → 0,
     L uw → 0, uz ux → 0, uz uy → 0, uz L2 → 0, uz uw → 0, ux uy → 0,
     ux L2 → 0, ux uw → 0, uy L2 → 0,  uy uw → 0,  L2 uw → 0}) // MatrixForm
```

$$\begin{pmatrix} e^2 L2^2 + uz^2 & a e^2 L2^2 \\ a e^2 L2^2 + e f L2^2 + a uz^2 & c^2 L^2 + a^2 e^2 L2^2 + 2 a \\ a b e^2 L2^2 + b e f L2^2 + e^2 g h L2^2 + a b uz^2 + g h uz^2 & b c^2 L^2 + c d L^2 + a^2 b e^2 L2^2 + 2 a b e f L2^2 + b f^2 \\ e^2 g L2^2 + g uz^2 & a e^2 g L2^2 \end{pmatrix}$$

```
(* ryx confounded *)
V[[{3}, {2}]].Inverse[V[[{2}, {2}]]] // Flatten // Simplify
```

$$\left\{\left(c\,d\,L^2+g\,h\left(e\,f\,L2^2+a\left(e^2\,L2^2+uz^2\right)\right)+b\left(c^2\,L^2+2\,a\,e\,f\,L2^2+f^2\,L2^2+ux^2+a^2\left(e^2\,L2^2+uz^2\right)\right)\right)\Big/\left(c^2\,L^2+2\,a\,e\,f\,L2^2+f^2\,L2^2+ux^2+a^2\left(e^2\,L2^2+uz^2\right)\right)\right\}$$

```
(* Bayes normal *)
ryzw = V[[{3}, {1, 4}]].Inverse[V[[{1, 4}, {1, 4}]]] // Flatten // Simplify
```

$$\left\{\frac{b\left(e\,f\,L2^2+a\left(e^2\,L2^2+uz^2\right)\right)}{e^2\,L2^2+uz^2},\ h\right\}$$

```
rxzw = V[[{2}, {1, 4}]].Inverse[V[[{1, 4}, {1, 4}]]] // Flatten // Simplify
```

$$\left\{\frac{a\,e^2\,L2^2+e\,f\,L2^2+a\,uz^2}{e^2\,L2^2+uz^2},\ 0\right\}$$

```
iv = ryzw[[1]] / rxzw[[1]] // Simplify
```

b

```
(* biv = (Z'X)^(-1)(Z'Y) *)
Inverse[V[[{1, 4}, {2, 4}]]].V[[{1, 4}, {3}]] // Simplify // Flatten
```

{b, h}

```
(* IV via projecting X onto Z,W then Y onto Xhat *)
xhat = q * Z
```

$q\,(e\,L2+uz)$

```
res = X - xhat // Simplify
```

$c\,L+f\,L2+ux+a\,(e\,L2+uz)-q\,(e\,L2+uz)$

```
H = {res, xhat, Y, W};
(V = (Outer[Times, H, H] // Expand) //. {L uz → 0, L ux → 0, L uy → 0, L L2 → 0,
     L uw → 0, uz ux → 0, uz uy → 0, uz L2 → 0, uz uw → 0, ux uy → 0,
     ux L2 → 0, ux uw → 0, uy L2 → 0, uy uw → 0, L2 uw → 0}) // MatrixForm
```

$$\begin{pmatrix} c^2\,L^2+a^2\,e^2\,L2^2+2\,a\,e\,f\,L2^2+f^2\,L2^2-2\,a\,e^2\,L2^2\,q-2\,e\,f\,L2^2\,q+e^2\,L2 \\ a\,e^2\,L2^2\,q+e\,f\,L2^2\,q-e^2\,L2^2\,q^2+a\,q\,uz^2 \\ b\,c^2\,L^2+c\,d\,L^2+a^2\,b\,e^2\,L2^2+2\,a\,b\,e\,f\,L2^2+b\,f^2\,L2^2+a\,e^2\,g\,h\,L2^2+e\,f\,g\,h\,L2^2-a\,b\,e^2\,L2^2\,q-b\,e\,f\,L \\ a\,e^2\,g\,L2^2+e\,f\,g\,L2^2-e^2\,g\,L2^2\,q+a\,g\,uz^2 \end{pmatrix}$$

```
iv = V[[{3}, {2, 4}]].Inverse[V[[{2, 4}, {2, 4}]]] // Flatten // Simplify
```

$$\left\{\frac{b\left(e\,f\,L2^2+a\left(e^2\,L2^2+uz^2\right)\right)}{q\left(e^2\,L2^2+uz^2\right)},\ h\right\}$$

```
iv /. q → rxzw[[1]] // Simplify
```

{b, h}

```
(* IV via projecting X onto Z then Y
 onto X and leftnull of X with respect to Z *)

H = {res, X, Y, W};
(V = (Outer[Times, H, H] // Expand) //. {L uz → 0, L ux → 0, L uy → 0, L L2 → 0,
     L uw → 0, uz ux → 0, uz uy → 0, uz L2 → 0, uz uw → 0, ux uy → 0,
     ux L2 → 0, ux uw → 0, uy L2 → 0, uy uw → 0, L2 uw → 0}) // MatrixForm
```

$$\begin{pmatrix} c^2 L^2 + a^2 e^2 L2^2 + 2 a e f L2^2 + f^2 L2^2 - 2 a e^2 L2^2 q - 2 e f L2^2 q + e^2 L2 \\ c^2 L^2 + a^2 e^2 L2^2 + 2 a e f L2^2 + f^2 L2^2 - a e^2 L2^2 q - e f L2^2 \\ b c^2 L^2 + c d L^2 + a^2 b e^2 L2^2 + 2 a b e f L2^2 + b f^2 L2^2 + a e^2 g h L2^2 + e f g h L2^2 - a b e^2 L2^2 q - b e f L \\ a e^2 g L2^2 + e f g L2^2 - e^2 g L2^2 q + a g uz^2 \end{pmatrix}$$

```
iv = V[[{3}, {1, 2, 4}]].Inverse[V[[{1, 2, 4}, {1, 2, 4}]]] // Simplify // Flatten
```

$$\left\{\left(c\, d\, L^2 \left(e\, f\, L2^2 + a \left(e^2 L2^2 + uz^2\right)\right)\right) / \left(q \left(e^2 L2^2 ux^2 + \left(f^2 L2^2 + ux^2\right) uz^2 + c^2 L^2 \left(e^2 L2^2 + uz^2\right)\right)\right),\right.$$
$$\left(-a\, c\, d\, L^2 \left(e^2 L2^2 + uz^2\right) + b\, c^2 L^2 q \left(e^2 L2^2 + uz^2\right) + c\, d\, L^2 \left(-e\, f\, L2^2 + e^2 L2^2 q + q\, uz^2\right) + b\, q \left(e^2 L2^2 ux^2 + \left(f^2 L2^2 + ux^2\right) uz^2\right)\right) /$$
$$\left.\left(q \left(e^2 L2^2 ux^2 + \left(f^2 L2^2 + ux^2\right) uz^2 + c^2 L^2 \left(e^2 L2^2 + uz^2\right)\right)\right), h\right\}$$

```
iv /. q → rxzw[[1]] // Simplify
```

$$\left\{\left(c\, d\, L^2 \left(e^2 L2^2 + uz^2\right)\right) / \left(e^2 L2^2 ux^2 + \left(f^2 L2^2 + ux^2\right) uz^2 + c^2 L^2 \left(e^2 L2^2 + uz^2\right)\right), b, h\right\}$$


```
iv /. q → rxzw[[1]] // Simplify
```

$$\left\{\left(c\, d\, L^2 \left(e^2 L2^2 + uz^2\right)\right) / \left(e^2 L2^2 ux^2 + \left(f^2 L2^2 + ux^2\right) uz^2 + c^2 L^2 \left(e^2 L2^2 + uz^2\right)\right), b, h\right\}$$