Ralph's stabilizer code

Quantum error-correcting codes can be written as stabilizer codes. These codes mirror classical linear codes, can be compactly represented by a set of generators, and standardize quantum error-correction. Here, we focus on CSS (Calderbank, Shor, and Steane) codes.These codes are capable of correcting multiple bit flips and phase flips (for sufficiently large codes). We illustrate CSS codes for the quantum analog to classical Hamming codes.

A classical Hamming code for correcting a single error has generator matrix

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

and parity check matrix

$$H = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

CSS codewords

CSS codes involve two code words for the quantum Hamming code. Let C_1 be the generator matrix for the Hamming code above and $C_2 = C_1^{\perp}$, its dual code, so that $C_2 \subset C_1$. C_1 is a $[n = 7, k_1 = 4]$ code, while C_2 is a $[n = 7, k_2 = 3]$ code. The dimension of all possible encodings is $N = 2^{k_1 - k_2}$. We choose code words $x_0 + x_1 \notin C_2$ which implies $\langle x_0 + C_2 | \, x_1 + C_2 \rangle = 0$. It is instructive to enumerate all linear combinations of C_2 mod 2 for identifying two code words for this code. This set is

$$\{0000000, 0001111, 0110011, 1010101, 0111100, 1011010, 1100110, 1101001\}$$

and leads to the quantum state or logical code (in normal form after swapping qubits one and four, then three and four, and finally swapping qubits six and seven)

$$|0_L\rangle = \frac{1}{\sqrt{8}}\{|0000000\rangle + |0011110\rangle + |0101011\rangle + |0110101\rangle$$
$$+ |1000111\rangle + |1011001\rangle + |1101100\rangle + |1110010\rangle\}$$

Our other code word is orthogonal to $|0_L\rangle$ so a simple choice is begin with $|1111111\rangle$ and add the elements of C_2 mod 2 (again, after swapping positions) or, equivalently, bit flip each qubit in $|0_L\rangle$. This produces

$$|1_L\rangle = \frac{1}{\sqrt{8}}\{|1111111\rangle + |1100001\rangle + |1010100\rangle + |1001010\rangle$$

$$+ |0111000\rangle + |0100110\rangle + |0010011\rangle + |0001101\rangle\}$$

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The idea of stabilizers is succinctly demonstrated by reference to an entangled state $|\beta_{00}\rangle = \frac{1}{\sqrt{2}}[|00\rangle + |11\rangle]$ and its stabilizers X_1X_2 and Z_1Z_2. Not only is the state unchanged by these operators but this is the unique state (up to phase) for which these operators are the stabilizers. Two properties are evident from these stabilizers: they commute (by matrix multiplication) and $-I$ is not part of the set. These are the properties of generators for the CSS code where the generators are a subset of the Pauli operators, $\pm I, \pm X, \pm Z, \pm Y, \pm iI, \pm iX, \pm iZ, \pm iY$.

Generators

The following generators form the set for the quantum analog to the $[7, 4]$ Hamming code.

$$g_1 = XIIIXXX = X_1X_5X_6X_7$$
$$g_2 = IXIXIXX = X_2X_4X_6X_7$$
$$g_3 = IIXXXXI = X_3X_4X_5X_6$$
$$g_4 = ZIZZIIZ = Z_1Z_3Z_4Z_7$$
$$g_5 = IZZIZIZ = Z_2Z_3Z_5Z_7$$
$$g_6 = ZZZIIZI = Z_1Z_2Z_3Z_6$$

The first three generators, g_1, g_2, g_3, allow detection and correction of any single phase flip and the last three generators, g_4, g_5, g_6, allow detection and correction of any single bit flip that might creep into the encoding. Further, analogous to the classical code the location of the error is identified by meaurement results (eigenvalues) equal to ± 1 (these are unitary operators) where $+1$ corresponds to I and -1 corresponds to Z. Hence, for example, $\langle\psi| g_4 |\psi\rangle = -1, \langle\psi| g_5 |\psi\rangle = 1, \langle\psi| g_6 |\psi\rangle = -1$ implies the first qubit is bit flipped where $|\psi\rangle$ is the encoding and is easily remedied by applying X_1 to the state. Since single qubit bit flip and/or phase flip errors involve eigenstates that reside in one of the two orthogonal subspaces, projections to reveal the syndrome do not change the state and can be performed sequentially.[1] Errors in other qubits are addressed in analogous fashion.

Quantum circuits and syndrome measurement

Quantum circuits are quick and flexible representations of quantum codes. Below are two pairs of useful circuits. The first equivalent pair employs observable X (measurement basis $|+\rangle, |-\rangle$) while the second equivalent pair employs observable Z (measurement basis $|0\rangle, |1\rangle$).

[1] If the bit flip and phase flip occur in the same qubit, with probability one-half the sign of the eigenstate is the negative of the original eigenstate.

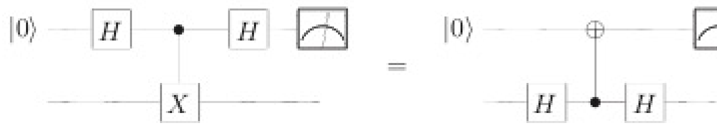

Figure 1. X observable equivalent quantum circuits

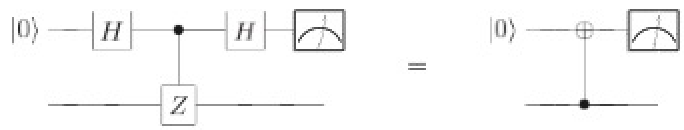

Figure 2. Z observable equivalent quantum circuits

CSS codes are reversible, now we address how this corresponds to measurement which changes the state. CSS codes employ ancilla for measurement (and implicit measurement of the code). Ancilla are constructed by tensoring sufficient $|0\rangle\, s$ to the state or quantum code and then applying controlled-NOT and/or controlled-Z unitary operators. We present two equivalent quantum circuits to illustrate the encoding and syndrome measurement.

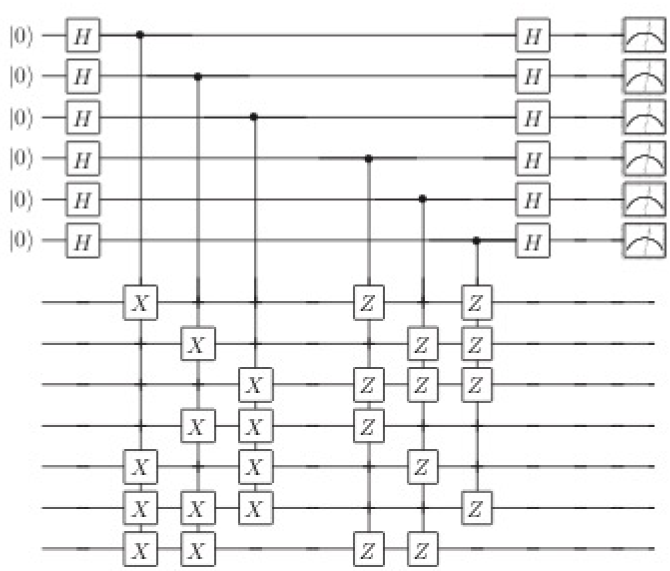

Figure 3. CSS quantum circuit

For the above quantum circuit, the top six qubits are the six-qubit ancilla all set to $|0\rangle$. A Hadamard operator is applied to each of these. this creates the eigenstate $\frac{1}{8}\left[|1\rangle + |2\rangle + \cdots + |64\rangle\right]$ where $|1\rangle = |000000\rangle$ or a vector with one followed by 63 zeroes, $|2\rangle = |000001\rangle$ or a vector 0,1 followed by 62 zeroes, and so on with $|64\rangle = |111111\rangle$ or a vector of 63 zeroes followed by one. The next step in the circuit is controlled-NOT where the ancilla qubits are the control and the eigenstate is the target (notice, this entangles the ancilla and target code). The first three ancilla employ controlled-NOT as indicated by the generators and the last three apply controlled-Z, again, as indicated by the generators. Finally, another set of Hadamard operators are applied to the ancilla followed by measurement of the ancilla (utilizing observable Z) to produce the syndrome. If all six measurement results are $+1$ then no adjustments of the code are needed; otherwise, utilize the syndrome as identifed by the generators to correct the code.

Syndrome measurement may be more readily visualized by the alternative (but equivalent) quantum circuit.

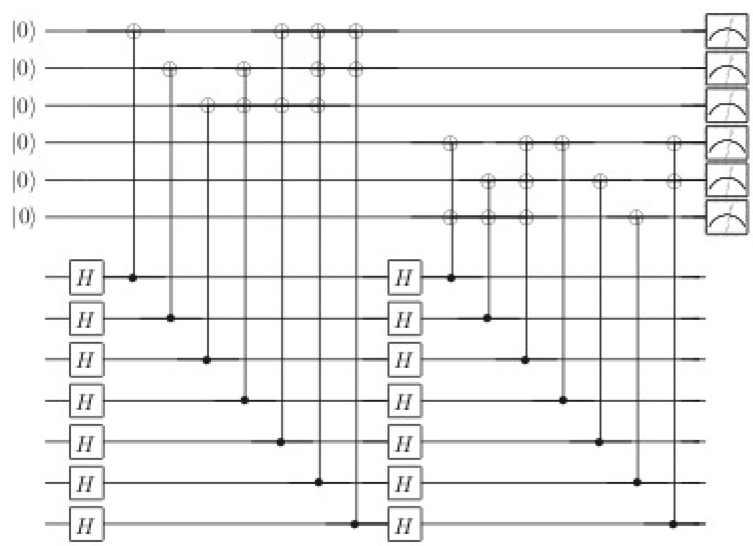

Figure 4. equivalent CSS quantum circuit

This circuit involves only controlled-NOT following the Hadamard operators where the control qubit is from the eigenstate or quantum code word and the target is the ancilla. Bit flip measurement of the ancilla involves qubits four through six producing +1 or -1 for each qubit.[2] If the first qubit, for instance, yields measurement results -1,+1,-1 then the first qubit is bit flipped and X_1 is applied to the eigenstate and the ancilla disposed. Then, Hadamard operators are applied to the "bit-flip corrected" eigenstate and the circuit is exercised again (with new ancilla if not in state $|o\rangle^{\otimes 6}$). If the syndrome is +1,-1,-1 then the second qubit is phase flipped and X_2 is applied followed by Hadamard operators applied to each qubit of the eigenstate. This produces a legal (corrected) code word and the ancilla is dismissed again.

[2]Analogously, phase filp measurement involves qubits one through three.

Suggested:

1. Show the X (Z) observable pairs of quantum circuits in figure 1 (figure 2) are equivalent. Explore X (Z) observable measurement for figure 1 (figure 2).

2. Create a code word, say, $|0_L\rangle$ (or alternatively, use the first CSS quantum circuit in figure 3 to create $|0_L\rangle$ from $|0\rangle^{\otimes 7}$) and verify the quantum circuit and syndrome.

3. Suppose a bit flip error occurs in the first qubit and a phase flip error occurs in the second qubit of the code word $|0_L\rangle$, use the generators to check for single phase and bit flip errors. Correct any detected errors.