Ralph's quantum error correction

Bit flip code

To guard against the fragility of qubits, Ralph encodes $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ as three entangled qubits $|\psi_3\rangle = \alpha|000\rangle + \beta|111\rangle$. However, the (perturbed) resultant state is $|\psi_3'\rangle = \alpha|010\rangle + \beta|101\rangle$.

Suggested:

1. Utilize the observable $E_X = 1 * P_0 + 2 * P_1 + 3 * P_2 + 4 * P_3$ where

$$P_0 = |000\rangle\langle 000| + |111\rangle\langle 111|$$

$$P_1 = X_1 P_0 X_1 = |100\rangle\langle 100| + |011\rangle\langle 011|$$
$$P_2 = X_2 P_0 X_2 = |010\rangle\langle 010| + |101\rangle\langle 101|$$
$$P_3 = X_3 P_0 X_3 = |001\rangle\langle 001| + |110\rangle\langle 110|$$

to find and correct any single qubit bit flip error. (hint: Any measurement causes the state to evolve described by projecting the state vector into the eigenstate(s) associated with the resultant measurement/eigenvalue. Error correction involves tracking this state evolution.)

2. Utilize the observables $Z_1 Z_2$ and $Z_2 Z_3$ to find and correct any single qubit bit flip error.

Phase flip code

Ralph encodes $|\psi\rangle$ again as three entangled qubits $|\psi_3\rangle = \alpha|000\rangle + \beta|111\rangle$ but then applies HHH. The (perhaps perturbed) resultant state is

$$|\psi_3''\rangle = \frac{\alpha+\beta}{2\sqrt{2}}(|000\rangle + |011\rangle) + \frac{\alpha-\beta}{2\sqrt{2}}(|001\rangle + |010\rangle)$$
$$-\left[\frac{\alpha-\beta}{2\sqrt{2}}(|100\rangle + |111\rangle) + \frac{\alpha+\beta}{2\sqrt{2}}(|101\rangle + |110\rangle)\right]$$

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Suggested:

3. Utilize the observable $E_Z = HHHE_X HHH$ to find and correct any single qubit phase flip error.

Shor code

Ralph knows the nine qubit ($2^9 = 512$ element vectors) Shor code detects and corrects any arbitrary single qubit i error where the error can be described by $E_i = c_0 I_i + c_1 X_i + c_2 Z_i + c_3 X_i Z_i$ (not necessarily normalized). The nine qubit Shor detector drawn from $|\psi\rangle$ is

$$|\psi_s\rangle = \alpha\,|\psi_0\rangle + \beta\,|\psi_1\rangle$$

where $|\psi_0\rangle = \frac{1}{2\sqrt{2}}\left(|000\rangle + |111\rangle\right)\left(|000\rangle + |111\rangle\right)\left(|000\rangle + |111\rangle\right)$
and $|\psi_1\rangle = \frac{1}{2\sqrt{2}}\left(|000\rangle - |111\rangle\right)\left(|000\rangle - |111\rangle\right)\left(|000\rangle - |111\rangle\right)$.

Suggested:

4. Show $HXH = Z$ and $HZH = X$.

5. Suppose $E_1 = X_1 Z_1$. Why would successive usage of phase flip and bit flip encoding fail to detect this error combination. Hint: phase flip encoding is $HHH\,|\psi_3\rangle$, phase flip (by assumption) gives $Z_1 HHH\,|\psi_3\rangle$, bit flip encoding reverses the phase flip encoding $HHHZ_1HHH\,|\psi_3\rangle$ but produces a bit flip (by assumption) $X_1HHHZ_1HHH\,|\psi_3\rangle$. Compare with $|\psi_3\rangle$ and utilize (4) to reconcile.

6. The projection into an eigenstate associated with an observable is the key to the nine qubit Shor code.[1] For example, the eigenvalues for Z_1Z_2 and $X_1X_2X_3$ are ± 1. What are the eigenstates for these two observables and which eigenstates are associated with $+1$ or -1? If we're feeling ambitious we could apply the Shor code to $E_1 = X_1 Z_1$.

[1] The Shor code employs ten observables: $Z_1Z_2, Z_2Z_3, Z_3Z_4, Z_4Z_5, Z_5Z_6, Z_6Z_7, Z_7Z_8, Z_8Z_9$, $X_1X_2X_3X_4X_5X_6$, and $X_4X_5X_6X_7X_8X_9$.