3

accounting as a communication channel

The topic of this chapter is how the double entry system acts as a communication channel. Using the linear algebra representation from chapter 2, that is,

$$Ay = x$$

the central question is easy to state: How much of the information in the vector, y, gets through the double entry matrix, A, to the financial statement vector, x? Once the concept of the row space of a matrix is introduced, the question has a straightforward answer: Only the component of y residing in the row space of A gets through the channel.

Computation of the row component of y can be done a number of ways; the chapter contains five methods. It is not entirely obvious that all methods will always yield the same answer, so some connections are made in that regard.

3.1 the row space of A

The first example will be a simple one: only three journal entries. Cash is paid out for an expense and for an asset. Some of the asset is then amortized (moved to expense).

Example 3.1

expense	4	
cash		4
asset	8	
cash		8

expense	5	
asset		5

If these were the only journal the entries, the resulting financial statements (making use of negative numbers) are

income statement	
expense	9

balance sheet

	ending	beginning		ending	beginning
cash	-12	0			
asset	3	0	retained earnings	-9	0
total assets	-9	0	total equities	-9	0

It is useful to access the linear algebra representation for this example.

$$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 5 \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

The financial statement vector x is

$$x = \begin{bmatrix} \text{cash} \\ \text{expense} \\ \text{asset} \end{bmatrix} = \begin{bmatrix} -12 \\ 9 \\ 3 \end{bmatrix}$$

And we have the linear algebra representation

$$Ay = x$$

To deal with the question of how much of y gets through A to x, the concept of orthogonality is important.

Definition 3.1 *Two vectors are orthogonal if their vector product is zero.*

Visually, two vectors are orthogonal if they are perpendicular or at right angles. In the (two dimensional) plane, vectors $\begin{bmatrix} 1 & 0 \end{bmatrix}^T$ and $\begin{bmatrix} 0 & 1 \end{bmatrix}^T$ are obviously at right angles, and, just as obviously, the vector product is zero. Another example is $\begin{bmatrix} 1 & 1 \end{bmatrix}^T$ and $\begin{bmatrix} 1 & -1 \end{bmatrix}^T$. The algebraic orthogonality condition for vectors is true in any dimensional space, not just the plane. That's what allows us to think about vectors at right angles even in high dimensions.

Back to the example. Without worrying about where the numbers come from for now, notice y can be decomposed into the following two orthogonal components.

$$y = \begin{bmatrix} 4 \\ 8 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ 2 \end{bmatrix} - 3 \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

It is convenient to have names for the two components. Let them be y_{row} and y_N. The reasons for these particular names will become apparent momentarily.

$$y_{\text{row}} = \begin{bmatrix} 7 \\ 5 \\ 2 \end{bmatrix} \qquad y_N = 3 \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

It is easy to check for orthogonality.

$$y_N^T y_{\text{row}} = 3 \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}^T \begin{bmatrix} 7 \\ 5 \\ 2 \end{bmatrix} = 3\,(7 - 5 - 2) = 0$$

Vector orthogonality is an important idea, but perhaps more important is the concept of orthogonality of vector spaces. Here we are interested in two vector spaces, and both of them arise from the matrix A: the row space of A, and the null space of A.

Definition 3.2 *The row space of A consists of all the vectors which are linear combinations of the rows of A.*

y_{row} is one vector (among many) in the rows of A, hence the name.

$$y_{\text{row}} = \begin{bmatrix} 7 \\ 5 \\ 2 \end{bmatrix} = -5 \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

The first vector on the right hand side is the first row of A, and the second is the second row of A. (There are many other ways to form y_{row} from the rows of A.)

Another important space of the matrix A is the null space.

Definition 3.3 *The null space of A consists of all the vectors y_N satisfying $Ay_N =$* 0.

Here we have

$$y_N = 3 \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

So it is easy to check that y_N is in the null space.

$$Ay_N = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} 3 = 3 \begin{bmatrix} -1+1 \\ 1-1 \\ -1+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So far we have decomposed y into a row and null component. We have not, as yet, actually computed the components: that is done in succeeding sections of the chapter. But once we have the orthogonal decomposition, we can answer the question about the communication channel. As it turns out, y_{row} goes through the channel, y_N does not.

To see that only y_{row} gets through, suppose all we have is the financial statement vector, x. Then y_{row} is unique, because there is only one solution to

$$\begin{aligned} Ay &= x \\ y^T y_N &= 0 \end{aligned}$$

and it is y_{row}. For our example we have three independent equations and three unknowns. There are two independent T-account equations (the third one is not independent because of double entry), and the orthogonality equation to solve for the three elements of y.

On the other hand, x tells us nothing about y_N. Since $Ay_N = 0$, y_N disappears from $Ay = x$.

$$Ay = A\left(y_{\text{row}} + y_N\right) = Ay_{\text{row}} + Ay_N = Ay_{\text{row}} = x$$

Given x, y_N can be anything in the null space of A.

So the financial statements tell us everything about y_{row} and nothing about y_N. That is the sense in which the row component is all that gets through the channel. And "row component through the channel" will be a recurring theme in later chapters.

One more thing: we can specify *how much* gets through. Because y_{row} and y_N are orthogonal (perpendicular), they obey the theorem of Pythagoras: the sum of the square of the two sides of a right triangle equals the square of the hypotenuse. The "square" of a vector is the vector product of the vector with itself.

$$\begin{aligned} y_{\text{row}}^T y_{\text{row}} &= \begin{bmatrix} 7 \\ 5 \\ 2 \end{bmatrix}^T \begin{bmatrix} 7 \\ 5 \\ 2 \end{bmatrix} = 7^2 + 5^2 + 2^2 = 78 \\ y_N^T y_N &= \begin{bmatrix} -3 \\ 3 \\ 3 \end{bmatrix}^T \begin{bmatrix} -3 \\ 3 \\ 3 \end{bmatrix} = 3^2 + 3^2 + 3^2 = 27 \\ y^T y &= \begin{bmatrix} 4 \\ 8 \\ 5 \end{bmatrix}^T \begin{bmatrix} 4 \\ 8 \\ 5 \end{bmatrix} = 4^2 + 8^2 + 5^2 = 105 \end{aligned}$$

Note the Pythagorean result:

$$\begin{aligned} y_{\text{row}}^T y_{\text{row}} + y_N^T y_N &= y^T y \\ 78 + 27 &= 105 \end{aligned}$$

The fraction getting through, then, is

$$\frac{y_{\text{row}}^T y_{\text{row}}}{y^T y} = \frac{78}{105} \approx .75$$

In this section we derived the main idea of this, and some future, chapters: only the row component gets through. We did not, however, actually compute the row component, y_{row}^T, or the null component, y_N. They were simply stated, and their properties noted and checked. The next several sections of the chapter offers a variety of ways to compute the components. The methods we will use most often are those that solve for the null component first.

3.2 expanded setup

We'll use a slightly expanded example to help demonstrate a number of methods for computing y_{row} from a given set of financial statements.

Example 3.2

balance sheet

	ending	*beginning*		*ending*	*beginning*
cash	*40*	*20*	*accrued liabilities*	*45*	*30*
acc'ts rec.	*35*	*30*	*capital stock*	*40*	*40*
inventory	*50*	*30*	*retained earnings*	*40*	*10*
total assets	*125*	*80*	*total equities*	*125*	*80*

income statement

sales	*120*
cost of goods sold	*60*
gen'l & admin. expenses	*30*
income	*30*

The journal entries (except for closing entries) are presented absent amounts.

accounts receivable	y_1	
sales		y_1
cash	y_2	
accounts receivables		y_2
accrued liabilities	y_3	
cash		y_3
g & a expense	y_4	
accounts receivable		y_4

g & a expense	y_5	
accrued liabilities		y_5
inventory	y_6	
accrued liabilities		y_6
cost of goods sold	y_7	
inventory		y_7

Some of the journal entries might deserve comment.

- y_1: Sales are often made on account, and the cash will be collected later. Accounts receivable is the resulting asset.
- y_2: Cash is collected for accounts receivable.
- y_4: On occasion not all accounts will turn out to be collectible; firms sometimes go bankrupt or otherwise disappear. In that case the receivable asset is reduced, the offsetting debit is often to an income statement account called something like bad debt expense. Here the debit is included in general and administrative expenses.
- y_5 and y_6: Things like labor, raw materials, and supplies are often acquired on account; a liability is thereby created.

Alternative representations for the financial statements are the directed graph and the incidence matrix. The directed graph is in figure 3.1, with cash on the left, balance sheet accounts in the middle column, and income statement accounts on the right. Attached to the account nodes are the (changes in) account balances.

The 6×7 incidence matrix, A, is presented below, along with the account balance vector, x.

x		y_1	y_2	y_3	y_4	y_5	y_6	y_7
20	cash	0	1	-1	0	0	0	0
5	acc'ts rec.	1	-1	0	-1	0	0	0
20	inventory	0	0	0	0	0	1	-1
-15	acc. liab.	0	0	1	0	-1	-1	0
-120	sales	-1	0	0	0	0	0	0
60	cgs	0	0	0	0	0	0	1
30	g&a	0	0	0	1	1	0	0

Our task is to solve for the transaction amounts, y_1 through y_7 which satisfy $Ay = x$. Some of the y values are straightforward. For example, from the directed graph it is immediate that $y_1 = 120$, $y_6 = 80$, and $y_7 = 60$. Those arrow values are the only way to achieve the node balances in sales, inventory, and cgs. The others, however, do not have unique answers.

Definition 3.4 *A loop exists in a directed graph if it is possible to travel from one node, touch others, and return to the original node without traveling on an* *arrow more than once. It is all right to travel backward on the arrow; that is, the debit/credit direction of the arrow does not matter when determining a loop.*

Arrows y_2, y_3, y_4, and y_5 constitute a loop in the directed graph. Some possible solutions are

y_2	115	0	105
y_3	95	-20	85
y_4	0	115	10
y_5	30	-85	20

(Check to see these alternate solutions all result in the appropriate node balances.)

We will calculate a unique answer to the problem, $Ay = x$, generating a y vector with certain, perhaps desirable properties. Some of the properties of the answer are

- It is the "shortest" possible y vector which generates the given statements, that is, which solves $Ay = x$.
- It resides entirely within the rows of A, and, indeed, is the only solution to $Ay = x$ which does so. That is, it can be constructed by taking a linear combination of the rows of A. This "residing in the rows" property supplies a convenient name for the vector: y_{row}.
- It is, in the sense discussed previously, the only component of y which gets through the double entry channel. In other words, y_{row} contains all the information available in the financial statements about the transaction amounts.

We cover several methods; the first solves directly for the shortest y vector.

3.3 quadratic programming

The idea is to find the shortest vector satisfying $Ay = x$. Length is defined as the sum of the squared elements of y; squaring eliminates the problem of how negative elements affect the length. It is simple enough to state the problem, and, indeed, it is a simple matter for a computer to solve the problem so stated. So, for the first time through, we will let a computer do the work.

$$\begin{aligned} minimize\ y^T y &= \sum y_i^2 \\ subject\ to\ Ay = x & \end{aligned}$$

$Ay = x$ is a system of seven linear equations - one for each account - with seven unknowns - one for each transaction. Because of the balancing property (or, equivalently, because A is an incidence matrix), there are only six *independent* equations, and, hence, rather than a unique solution to the system, there are several possible solutions.

Most spreadsheets have an optimizer capable of solving the problem; Excel has a pretty good one called Solver. It doesn't take long to type in the data. If we use the Excel function *sumproduct* to accomplish vector multiplication, we don't have to type in the zeros in the A matrix.

Whatever computer routine we use, the solution appears, called y_{row}.

$$y_{row} = \begin{bmatrix} 120 & 55 & 35 & 60 & -30 & 80 & 60 \end{bmatrix}^T$$

There are a couple of things to notice about the solution.

- For the transactions not in a loop (y_1, y_6, and y_7), the solution is consistent with what is apparent from the directed graph: $y_1 = 120$, $y_6 = 80$, and $y_7 = 60$.
- There is a negative element: $y_5 = -30$. Since we set up the transactions to flow through the accounting system in basically one direction, the existence, in our preferred solution, of a transaction going in the other direction might cause some discomfort. There is no need to stifle the discomfort at this stage.

Example 3.3 *Reconsider the set-up from example 3.1 to compute the y_{row} we already know. Use a computer optimizer like Solver in Excel Recall*

$$x = \begin{bmatrix} -12 \\ 9 \\ 3 \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

It is a fairly simple matter for a computer optimizer like Solver to compute y_{row} using a quadratic program. See exercise 3.9 for a pencil and paper approach.

3.4 regression

3.4.1 computing y_{row} with regression

The first method minimized the sum of the squares of the y vector. Regression also minimizes the sum of squares; this method exploits the connection, and gets us closer to a paper and pencil method to solve the problem. The first step is to find any solution to $Ay = x$; call it y_p for y particular. Then project y_p into the rows of A, that is, run a regression.

Spanning trees are useful for a variety of things, one of which is to find a particular solution, y_p.

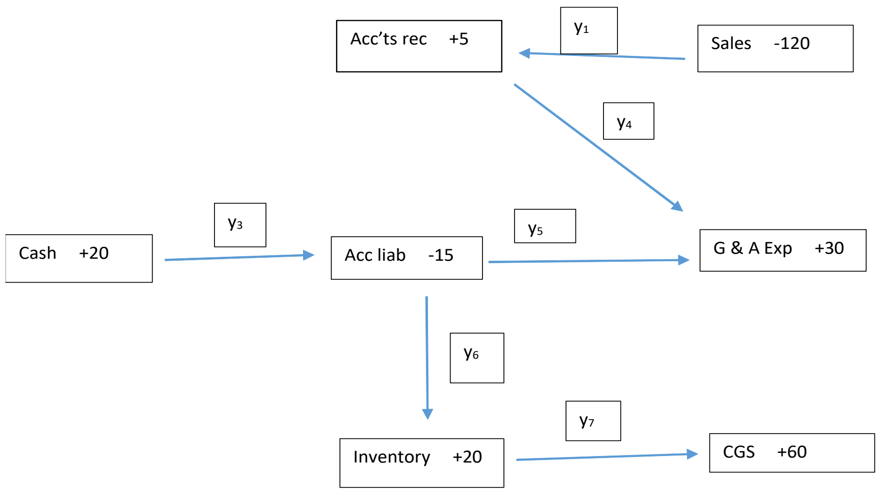

Figure 3.2

Example 3.2 Spanning tree with $y_2 = 0$

Definition 3.5 *A spanning tree is a directed graph with two properties.*

- It spans. That is, every node can be reached from every other node by tracing a path on the arrows. (It’s okay to go backward on an arrow.)

- It is a tree. That is, the arrows don’t loop. (Most, I think all, trees in nature have this property.)

To form a spanning tree from a directed graph containing loops, erase an arrow from each loop, and set the erased $y_i = 0$. Figure 3.2 contains a spanning tree for example 3.1 with $y_2 = 0$.

It’s easy to compute the y vector from a spanning tree: add up the amounts of the node on one end of the arrow (if it’s the tail, change the sign of the sum). Since there are no loops, there is no ambiguity about which end of the arrow a node is connected to. For this spanning tree

$$y_p = \begin{bmatrix} 120 & 0 & -20 & 115 & -85 & 80 & 60 \end{bmatrix}^T$$

Now run a regression; Excel is pretty good at this, as well. The dependent variable is y_p and the independent variables (regressors) are the rows of A.

dependent variable	independent variables						omit one column
120	0	1	0	0	-1	0	0
0	1	-1	0	0	0	0	0
-20	-1	0	0	1	0	0	0
115	0	-1	0	0	0	0	1
-85	0	0	0	-1	0	0	1
80	0	0	1	-1	0	0	0
60	0	0	-1	0	0	1	0

The independent variables are recognized as the rows of A in columns: A^T. It is important to eliminate one of the columns before running the regression: Excel complains when the independent variables are not independent of each other. It doesn't matter which column is eliminated; y_{row} is the same, and that should be checked.

Here is (part of) a sample regression output when the last column is eliminated.

coefficients on independent variables	predicted variable	residuals
-5	120	0
-60	55	-55
110	35	-55
30	60	55
-180	-30	-55
170	80	0
	60	0

The predicted variable is seen to be y_{row}. The coefficients are the weights on the rows of A used to generate y_{row}. The first element of y_{row}, for example, is the coefficient vector times the first row of A^T (first column of A omitting the last element).

$$\begin{bmatrix} 0 & 1 & 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} -5 \\ -60 \\ 110 \\ 30 \\ -180 \\ 170 \end{bmatrix} = -60 - (-180) = 120$$

The other elements of y_{row} are generated in a similar fashion. As y_{row} is constructed as a weighted combination of the rows of A, it is said that y_{row} resides in the rows of A, hence the name y_{row}.

Before leaving the regression output notice the residual vector - this is the difference between the predicted variable, y_{row}, and the dependent variable, y_p. The residual vector actually looks pretty simple, and, indeed, a direct calculation of the residuals is the basis of the next method. But before proceeding to the next method, there is more to say about regressions.

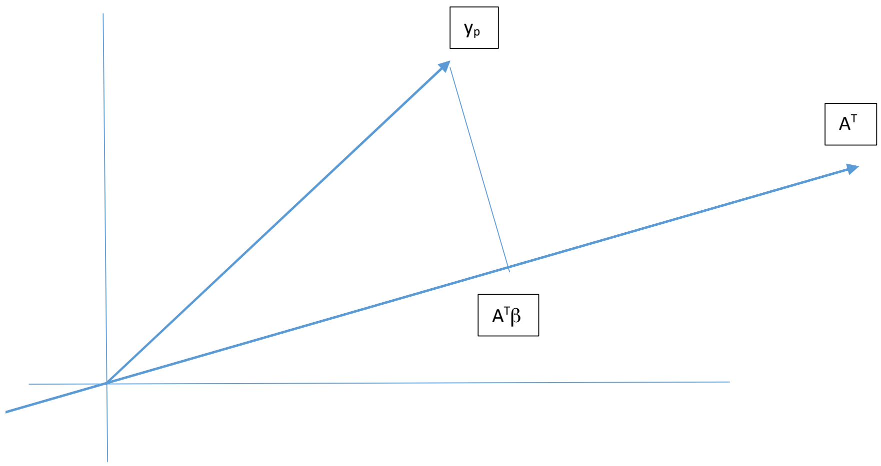

Figure 3.3

Choose β $y_p - A^T\beta$ is minimized

3.4.2 some more about regressions

Geometrically, the regression is finding the closest point in the rows of A (in this case, the columns of A^T) to y_p. Roughly speaking, the way to do that is construct a line from y_p to A^T that is perpendicular (orthogonal) to A^T. Geometrically, perpendicular and orthogonal have the same meaning; two lines are perpendicular if they form a right angle. The geometric idea is in figure 3.3.

It's the same idea as the shortest way to get from the interior of a room to a wall is to walk perpendicular to the wall. That simple idea is enough to allow us to write down the basic equation of regression, called the orthogonality conditions. All the regression calculations follow from these conditions.

In Figure 3.3 β is the vector of coefficients on the columns of A^T (rows of A). The regression routine chooses the coefficients so as to make the difference vector, $y_p - A^T\beta$ orthogonal to all the columns of A^T. This requires an algebraic interpretation of orthogonality in terms of vectors: we already know that two vectors are orthogonal if their vector product is zero.

For the regression problem, the vectors we are interested in setting orthogonal to each other are the difference vector $y_p - A^T\beta$ and any vector in A^T, itself. The orthogonality condition is so important, we write it as a theorem.

Theorem 3.1 *The coefficient vector β which minimizes the squared distance from vector y_p to the rows of matrix A solves the orthogonality condition: $A(y_p - A^T\beta) = 0$.*

Once the orthogonality conditions are written down, we can do the regression calculations just like the computer routine does. To solve for β rearrange the orthogonality condition.

$$AA^T\beta = Ay_p$$

This is a parsimonious representation of seven (six independent) equations with seven unknowns.

$$Ay_p = x = \begin{bmatrix} 20 \\ 5 \\ 20 \\ -15 \\ -120 \\ 60 \\ 30 \end{bmatrix}$$

(As Ay_p is equal to x for any y_p, we can see the orthogonality condition is the same for any choice of y_p. In other words, it doesn't matter which particular solution we choose.)

Since

$$A^T = \begin{bmatrix} 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 2 & -1 & 0 & -1 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 2 & -1 & 0 & -1 & 0 \\ -1 & 0 & -1 & 3 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 2 \end{bmatrix}$$

The matrix equation, $AA^T\beta = Ay_p = x$, can be written as 7 linear equations.

$$\begin{aligned} 2\beta_1 - \beta_2 - \beta_4 &= 20 \\ -\beta_1 + 3\beta_2 - \beta_5 - \beta_7 &= 5 \\ 2\beta_3 - \beta_4 - \beta_6 &= 20 \\ -\beta_1 - \beta_3 + 3\beta_4 - \beta_7 &= -15 \\ -\beta_2 + \beta_5 &= -120 \\ -\beta_3 + \beta_6 &= 60 \\ -\beta_2 - \beta_4 + 2\beta_7 &= 30 \end{aligned}$$

It is not beyond our wit to solve the system, but it is tedious enough so we are grateful for computer aid. But we can verify the regression output solves the system. That is, the row coefficients are

$$\beta = \begin{bmatrix} -5 & -60 & 110 & 30 & -180 & 170 \end{bmatrix}^T$$

with $\beta_7 = 0$ for the omitted row. Our next task is to put the problem in a form that is paper and pencil doable. Before doing so, however, let's do the projection exercise on the simpler problem from example 3.1

Example 3.4 *As there are only two independent rows in A, we can use the first two, and redefine A accordingly.*

$$A = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

And we can use

$$y_p = \begin{bmatrix} 4 \\ 8 \\ 5 \end{bmatrix}$$

So

$$Ay_p = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 8 \\ 5 \end{bmatrix} = \begin{bmatrix} -12 \\ 9 \end{bmatrix}$$

(Any y_p satisfying $Ay = x$ will yield the elements of x.)

$$AA^T = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

The orthogonality conditions can be written:

$$\begin{aligned} AA^T\beta &= Ay_p \\ \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\beta &= \begin{bmatrix} -12 \\ 9 \end{bmatrix} \end{aligned}$$

The two orthogonality equations written separately:

$$\begin{aligned} 2\beta_1 - \beta_2 &= -12 \\ -\beta_1 + 2\beta_2 &= 9 \end{aligned}$$

Solving two linear equations in two unknowns is not difficult. Here the solution is

$$\begin{aligned} \beta_1 &= -5 \\ \beta_2 &= 2 \end{aligned}$$

So

$$y_{\text{row}} = -5 \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ 2 \end{bmatrix}$$

And, we knew that.

3.5 projection into the nullspace

The method of this section can often be accomplished with a paper and pencil. The idea is to project a particular solution to $Ay = x$ into what is called the nullspace, effectively calculating the residual vector $y_p - y_{row}$ from the previous regression. Notice from the regression output the residual vector looks a bit simpler than y_{row}. What we will be doing is to divide any y_p solution into two orthogonal parts, y_{row} and the null component, denoted y_N.

$$\begin{aligned} y_p &= y_{\text{row}} + y_N \\ \text{where } (y_{\text{row}})^T y_N &= 0 \end{aligned}$$

Recall the nullspace of a matrix A consists of all the vectors that are orthogonal to the rows of A.

Typically, calculating the nullspace of a matrix is a little bit complicated, but for an incidence matrix, there is really nothing to it: simply read off the loops from the associated directed graph. Recall from figure 3.1, the loop consists of y_2, y_3, $-y_5$, and y_4, so the null vector is

$$N = \begin{bmatrix} 0 & 1 & 1 & -1 & 1 & 0 & 0 \end{bmatrix}^T$$

, simply place a one in the position of the arrow in the loop, zero for arrows not in the loop. The direction around the loop is important, so if the arrow is traversed from head to tail, as y_4 is in the example, the sign is negative.

It is a remarkable property of incidence matrices that the nullspace consists of vectors with all positive and negative ones and zeros. Also, it doesn't matter which way the loop is traversed, a counter-clockwise null vector of

$$\begin{bmatrix} 0 & -1 & -1 & 1 & -1 & 0 & 0 \end{bmatrix}^T$$

would work just as well, as can be verified as the procedure unfolds.

It is easy to verify that the loop supplies the null vector, since $AN = 0$.

$$\begin{bmatrix} 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

When projecting y_p into N, the result is y_N which we shall calculate as N times a regression coefficient.

$$y_N = N\beta$$

The geometric picture is depicted in figure 3.4. The difference vector $y_p - y_N$ is orthogonal to the nullspace, hence it is in the row space, so $y_{row} = y_p - y_N$.

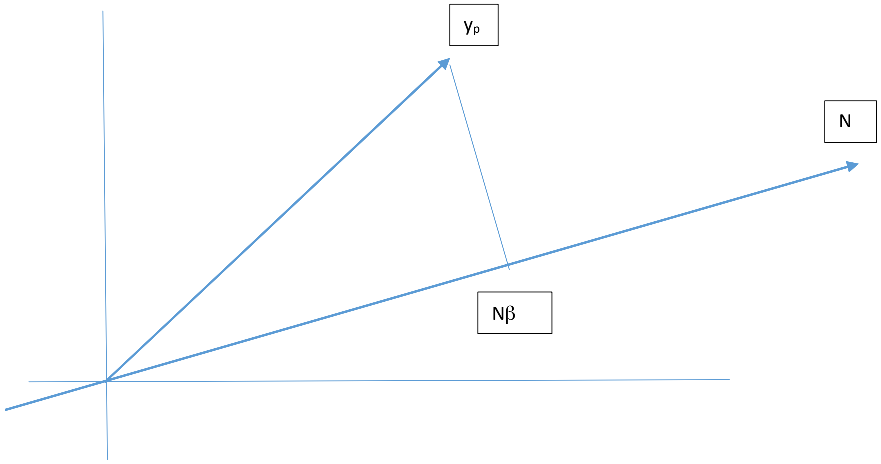

Figure 3.4

Projecting y_p into the nullspace

To calculate the regression coefficient β, use the same y_p as before (any solution to $Ay = x$ will do).

	y_1	y_2	y_3	y_4	y_5	y_6	y_7
y_p	120	0	-20	115	-85	80	60
N^T	0	1	1	-1	1	0	0

The orthogonality condition requires the difference vector, $y_p - y_N$ to be orthogonal to the N vector, so the orthogonality condition is the same as before with A replaced by N as in figure 3.4.

$$\begin{aligned} & N^T(y_p - y_N) \\ = \; & N^T(y_p - N\beta) = 0 \end{aligned}$$

$$N^T N\beta = N^T y_p$$

$N^T N$ and $N^T y_p$ are computed as vector products.

$$\begin{aligned} N^T N &= 4 \\ N^T y_p &= -20 - 115 - 85 = -220 \end{aligned}$$

Hence,

$$\begin{aligned} 4\beta &= -220 \\ \beta &= -55 \end{aligned}$$

Completing the table.

	y_1	y_2	y_3	y_4	y_5	y_6	y_7
y_p	120	0	-20	115	-85	80	60
N	0	1	1	-1	1	0	0
$N\beta$	0	-55	-55	55	-55	0	0
$y_{row} = y_p - N\beta$	120	55	35	60	-30	80	60

y_{row} from this paper and pencil calculation is the same as from the previous methods.

Example 3.5 *For a simpler example reconsider once again the set-up in example 3.1 We have*

$$N = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

which we can derive by following around the loop in the directed graph representation in figure 3.5. We can use

$$y_p = \begin{bmatrix} 4 \\ 8 \\ 5 \end{bmatrix}$$

So we have

$$N^T N = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}^T \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = 3$$

$$N^T y_p = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}^T \begin{bmatrix} 4 \\ 8 \\ 5 \end{bmatrix} = 4 - 8 - 5 = -9$$

So the orthogonality conditions are

$$\begin{aligned} N^T N\beta &= N^T y_p \\ 3\beta &= -9 \\ \beta &= -3 \end{aligned}$$

And

$$\begin{aligned} y_N &= N\beta = -3 \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \\ y_{\text{row}} &= y_p - y_N =^T \begin{bmatrix} 4 \\ 8 \\ 5 \end{bmatrix} - \begin{bmatrix} -3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ 2 \end{bmatrix} \end{aligned}$$

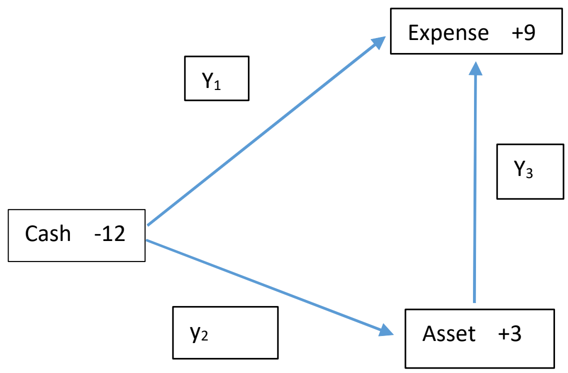

Figure 3.5

Directed graph for example 3.4

3.6 average spanning tree

Calculating the average spanning tree is another method, one which offers a paper and pencil solution for relatively small problems. That is, derive a y_p for every possible spanning tree, add up, and divide by the number of spanning trees. For the ongoing example there are four spanning trees. The spanning tree with $y_2 = 0$, call it spanning tree 1, was presented in Figure 3.2. The other three spanning trees, omitting y_3, y_4, and y_5 in turn, are formed in similar fashion, and the resulting solutions are presented in the following table.

	y_1	y_2	y_3	y_4	y_5	y_6	y_7
spanning tree 1	120	0	-20	115	-85	80	60
spanning tree 2	120	20	0	95	-65	80	60
spanning tree 3	120	115	95	0	30	80	60
spanning tree 4	120	85	65	30	0	80	60
sum	480	220	140	240	-120	320	240
$y_{row} = sum/4$	120	55	35	60	-30	80	60

And it is seen that the average spanning tree equals y_{row} from the earlier methods.

For even slightly larger problems, it is not always easy to confidently list all the spanning trees. There is a quite remarkable theorem, called the matrix tree theorem, which provides a simple and useful expression for the number of spanning trees.

Theorem 3.2 *For any directed graph the number of spanning trees is given by the determinant of $N^T N$, where N is the matrix with the nullspace vectors in the columns.*[1]

For the example the single nullspace vector is $\begin{bmatrix} 0 & 1 & 1 & -1 & 1 & 0 & 0 \end{bmatrix}^T$ so $N^T N$ is 4, which is also the determinant. The determinant of a two by two matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $ad - bc$. The formula for the determinant of larger matrices gets a little bit complicated; in any event, spreadsheets can calculate them numerically.

Example 3.6 *Referring to figure 3.5, the three spanning trees for the simpler example are*

$$y = \begin{bmatrix} 0 \\ 12 \\ 9 \end{bmatrix}, \begin{bmatrix} 12 \\ 0 \\ -3 \end{bmatrix}, \textit{and} \begin{bmatrix} 9 \\ 3 \\ 0 \end{bmatrix}.$$

The average spanning tree amounts are

$$\frac{1}{3} \begin{matrix} 0+12+9 \\ 12+0+3 \\ 9-3+0 \end{matrix} = \begin{bmatrix} 7 \\ 5 \\ 3 \end{bmatrix}$$

3.7 augmented A matrix

Finally, another method is to add enough equations to $Ay = x$ so the system has a solution, and, further, the solution is y_{row}. Here we can use the extra equation(s) implied by the nullspace relationships. We have seen that y_{row} is orthogonal to the nullspace. In the example the nullspace consisted of only one vector $\begin{bmatrix} 0 & 1 & 1 & -1 & 1 & 0 & 0 \end{bmatrix}^T$ The extra equation is the vector product of the nullspace vector with y is equal to zero which is the last row of augmented A and the last element of augmented x.

$$\begin{bmatrix} 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \end{bmatrix} = \begin{bmatrix} 20 \\ 5 \\ 20 \\ -15 \\ -120 \\ 30 \\ 60 \\ 0 \end{bmatrix}$$

[1] The matrix tree theorem is typically stated and proved in terms of the incidence matrix which eliminates confusion when there are no loops. See, for example, Harris, Hirst, and Mossinghoff, page 29.

There are now seven independent linear equations which can be used to solve for the seven elements of y_{row}, as the null row adds another independent equation. (Although there are 8 equations, only 7 are independent, as any particular T-account can be constructed from the others.) Solving seven linear equations is a little bit tedious, but it is easy to verify (or use a spreadsheet to find) that the system is solved by

$$y_{\text{row}} = \begin{bmatrix} 120 & 55 & 35 & 60 & -30 & 80 & 60 \end{bmatrix}^T$$

And we have five different methods to find y_{row}, but there are still some things to learn and connections to make, one of which is the fundamental theorem of linear algebra.

Example 3.7 *For the simpler example the augmented equations are*

$$\begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -12 \\ 9 \\ 3 \\ 0 \end{bmatrix}$$

The unique solution to all 4 equations is

$$y_{\text{row}} = \begin{bmatrix} 7 \\ 5 \\ 2 \end{bmatrix}$$

3.8 the fundamental theorem of linear algebra

As far as the ongoing numerical example is concerned, all the foregoing methods for finding y_{row} arrive at the same answer. It is, indeed, true for all examples. However, the example approach does not demonstrate the equivalence of all the methods for all possible problems. In this section we show the equivalence for two of the methods: quadratic programming and regression. The demonstration uses the fundamental theorem of linear algebra.

For the example we have been doing, the row space of A (where A has m rows and n columns) is not "complete" in the sense that not all vectors with n elements (where n is the number of journal entries) can be formed using weighted combinations of the rows. For example, the chosen y_p's could not be so formed, because a regression had a residual vector $y_p - A^T\beta$. So, in cases like this, it is said the row space does not *span* the set, or *space*, of n element vectors.

According to the fundamental theorem of linear algebra the nullspace completes the row space, and they are said to be *complements*. That is, the combination of the two spaces spans the space of n element vectors. Furthermore, as all the vectors in the nullspace are orthogonal to all the vectors in the row space, they are said to be orthogonal complements. All matrices have the same property as stated in the theorem.

Theorem 3.3 *The row space and nullspace of any matrix are orthogonal complements.*[2]

The theorem allows any vector composed of n elements to be decomposed into orthogonal components, and that's often an instructive thing to do. Any solution to $Ay = x$ can be written as $y = y_{\text{row}} + Nk$, where k is a vector of weights on the nullvector(s) N. The decomposition is always possible by the fundamental theorem. Now we can revisit the first method to see that y_{row} is always a solution to the quadratic program.

$$\begin{aligned} \min\ & y^T y \\ s.t.\ & Ay = x \end{aligned}$$

Substitute in the objective function.

$$\begin{aligned} y^T y &= (y_{\text{row}} + Nk)^T (y_{\text{row}} + Nk) \\ &= y_{\text{row}}^T y_{\text{row}} + 2(Nk)^T y_{\text{row}} + (Nk)^T (Nk) \end{aligned}$$

The crossproduct term $(Nk)^T y_{\text{row}}$ is zero by orthogonality. Hence, finding the minimum $y^T y$ is equivalent to finding the k vector that minimizes

$$y_{\text{row}}^T y_{\text{row}} + (Nk)^T (Nk)$$

Choosing k equal to zero obviously does the trick. Hence, we have shown y_{row}, derived as the solution to the projection problem, is also the solution to the original (method 1) quadratic programming problem.

A relative of the fundamental theorem, called Euler's theorem, is useful when looking for the loops in a directed graph. The number of loops is equivalent to the number of independent vectors in the null space. When the graph is extensive it is not always easy to find the independent loops; Euler's theorem tells us how many loops we are looking for.

Theorem 3.4 *Euler's theorem says the number of independent loops in a directed graph is equal to the number of arcs (journal entries) minus the number of nodes (accounts) plus one.*

From the fundamental theorem the total number of independent vectors in the row space of A plus the nullspace is n, where n is the number of columns (and journal entries).[3] The number of independent vectors in the row space is one

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[2]See Strang, page 138. The theorem is also true for the transpose of a matrix; the orthogonal complements are called the column space and the left nullspace.

[3]The number of independent vectors in the row space is called the "rank" of the matrix. It is kind of remarkable that the rank is the same whether looking at the matrix or its transpose. That is, the number of independent vectors in the row space is equal to the number of independent vectors in the column space.

less than the number of rows, m, as double entry implies one T-account must be determinable from all the others. Since the sum of the loops plus the sum of independent row vectors is n, we have

$$\begin{aligned}
\text{\# of loops } + (m-1) &= n \\
\text{\# of loops} &= n - m + 1 \\
&= \text{\# of journal entries - \# of accounts +1}
\end{aligned}$$

3.9 multiple loops

When there is more than one loop, calculation of y_{row} proceeds in the same way. To project into the nullspace, the orthogonality conditions are used. However, instead of just one equation, when there are two loops, there are two equations to solve for two regression coefficients.

Here is an example, the form of which will reappear in later sections. The (partial) financial statements have three assets and two expense accounts. The assets could be various prepayments or equipment, and the two expenses could be general expenses and cost of goods sold. Cash is another asset, so the example has 6 total accounts.

Example 3.8 *Here are partial financial statements.*

partial balance sheet

	ending balance	*beginning balance*
asset 1	*10*	*10*
asset 2	*15*	*15*
asset 3	*8*	*5*

partial income statement

expense 1	*6*
expense 2	*3*

Here are the only journal entries affecting the above accounts.

asset 1	y_1	
cash		y_1
asset 2	y_2	
cash		y_2
asset 3	y_3	
cash		y_3
expense 1	y_4	
asset 1		y_4

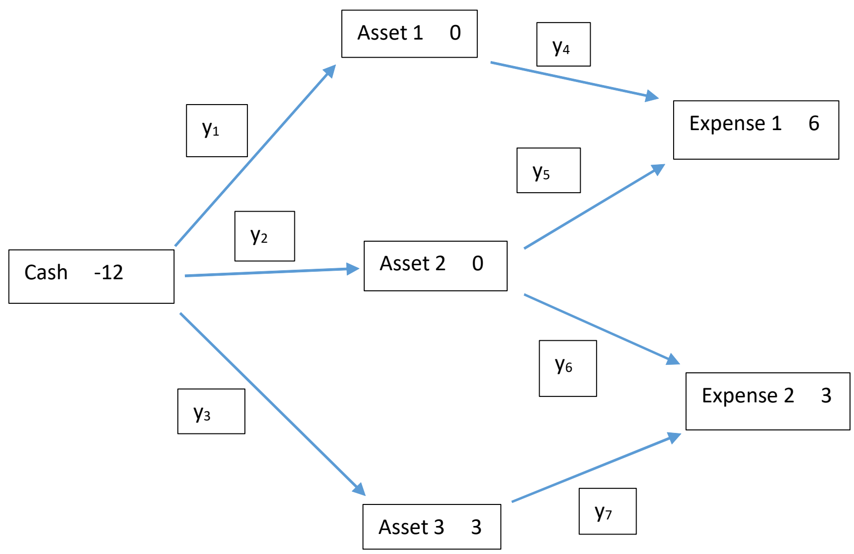

Figure 3.6

Example 3.8 in directed graph format

expense 1	y_5	
asset 2		y_5
expense 2	y_6	
asset 2		y_6
expense 2	y_7	
asset 3		y_7

Compute y_{row}.

To identify the loops (the nullspace vectors) construct the graph in figure 3.6. It is clear from inspection (and can also be verified by Euler's theorem) that there are two independent loops.[4] An algebraic characterization of the two loops,

[4]The number of loops is the number of journal entries (7) minus the number of accounts (6) plus one.

along with a sample y_p solution, is presented in the table.

	y_1	y_2	y_3	y_4	y_5	y_6	y_7
y_p	0	9	3	0	6	3	0
N^T	1	−1	0	1	−1	0	0
	0	1	−1	0	0	1	−1

The orthogonality condition requires the difference vector $y_p - N\beta$ be orthogonal to both columns of N.

$$N^T(y_p - N\beta) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The two zeros on the right are for the two vector products. Restating

$$N^T N\beta = N^T y_p$$

represents two linear equations with two unknowns, the regression coefficients $\beta = \begin{bmatrix} \beta_1 & \beta_2 \end{bmatrix}^T$. Computing the vector products

$$\begin{aligned} N^T N &= \begin{bmatrix} 1 & -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \\ 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} N^T y_p &= \begin{bmatrix} 1 & -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 9 \\ 3 \\ 0 \\ 6 \\ 3 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -9-6 \\ 9-3+3 \end{bmatrix} = \begin{bmatrix} -15 \\ 9 \end{bmatrix} \end{aligned}$$

Substituting into the orthogonality conditions

$$\begin{aligned} N^T N\beta &= N^T y_p \\ \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} &= \begin{bmatrix} -15 \\ 9 \end{bmatrix} \end{aligned}$$

The two linear equations written separately are

$$\begin{aligned} 4\beta_1 - \beta_2 &= -15 \\ -\beta_1 + 4\beta_2 &= 9 \end{aligned}$$

There are many ways to solve two linear equations in two unknowns. One way that does the trick is to multiply the first equation times 4, and then add the two equations together, thereby eliminating β_2. The solution

$$\begin{aligned} \beta_1 &= -3.4 \\ \beta_2 &= 1.4 \end{aligned}$$

The nullspace vector y_N is computed as $N\beta$..

$$\begin{aligned} y_N &= N\beta \\ \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \\ 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -3.4 \\ 1.4 \end{bmatrix} &= \begin{bmatrix} -3.4 \\ 3.4+1.4 \\ -1.4 \\ -3.4 \\ 3.4 \\ 1.4 \\ -1.4 \end{bmatrix} = \begin{bmatrix} -3.4 \\ 4.8 \\ -1.4 \\ -3.4 \\ 3.4 \\ 1.4 \\ -1.4 \end{bmatrix} \end{aligned}$$

And y_{row} is computed from

$$\begin{aligned} y_p &= y\,\text{row} + y_N \\ y_{\text{row}} &= y_p - y_N \\ &= \begin{bmatrix} 0 \\ 9 \\ 3 \\ 0 \\ 6 \\ 3 \\ 0 \end{bmatrix} - \begin{bmatrix} -3.4 \\ 4.8 \\ -1.4 \\ -3.4 \\ 3.4 \\ 1.4 \\ -1.4 \end{bmatrix} = \begin{bmatrix} 3.4 \\ 4.2 \\ 4.4 \\ 3.4 \\ 2.6 \\ 1.6 \\ 1.4 \end{bmatrix} \end{aligned}$$

The calculations are summarized in the table.

	y_1	y_2	y_3	y_4	y_5	y_6	y_7
y_p	0	9	3	0	6	3	0
N	1	-1	0	1	-1	0	0
	0	1	-1	0	0	1	-1
$N\beta$	-3.4	4.8	-1.4	-3.4	3.4	1.4	-1.4
$y_{row} = y_p - N\beta$	3.4	4.2	4.4	3.4	2.6	1.6	1.4

y_{row} could, and should, be checked by verifying

$$\begin{aligned} Ay_{\text{row}} &= x \\ N^T y_{\text{row}} &= 0 \end{aligned}$$

One way to verify both conditions is to check the directed graph in figure 3.7. The orthogonality condition can be verified by computing the sum of the directed amounts around each of the loops. The first loop, for example, is

$$3.4 + 3.4 - 2.6 - 4.2 = 0$$

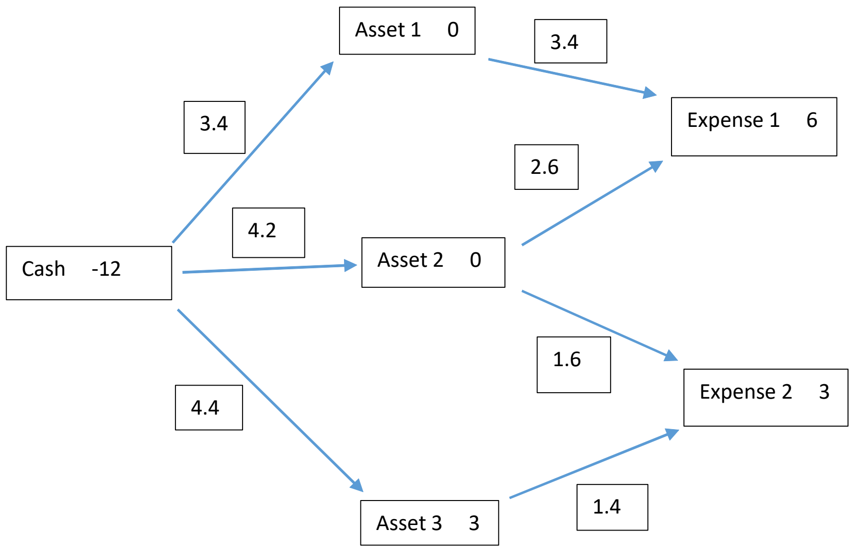

Figure 3.7

y_{row} for example 3.6

3.10 summary

The basic process in the chapter was to start with a journal entry vector, y, operate with a double entry accounting matrix, A, and generate a financial statement vector, x. The question is how much of the information in y reaches the vector x. And the answer is simple to state: the information in the row component in y, called y_{row}, gets through the channel to x, and it is all that gets to x.

The central part of the problem is the computation of y_{row}. We studied five solution techniques; it is a little bit remarkable that all five techniques yield the same solution. The fundamental theorem of linear algebra is instructive on this point. Each solution technique, in turn, adds more interpretation to the solution. As accounting is an information science, an information interpretation is useful: the unique solution, y_{row}, is all the information available in the financial statement vector, x, about the underlying journal entry amounts. Along the way the five solution techniques illustrate and invoke some theorems, notably fundamental theorems about the two basic activities of applied mathematics: optimization, estimation, and their interrelationships.

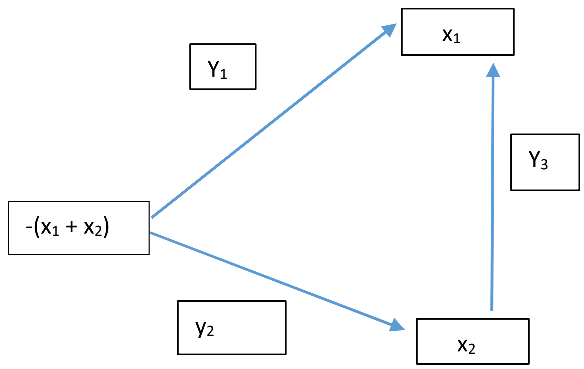

Figure 3.8

Exercise 3.1 directed graph

3.11 reference

Strang, Gilbert, *Linear Algebra and Its Applications*. Harcourt Brace Jovanovich, 1986.

3.12 exercises

Exercise 3.1 *Figure 3.8 presents a double entry example in directed graph form. It is equivalent to a cash payment, some of which goes to an asset, some to expense, and some of the asset is amortized over time.*

a. Suppose $x_1 = x_2 = 5$, and $y = \begin{bmatrix} 2 & 8 & 3 \end{bmatrix}^T$. Compute y_{row}. What fraction of y gets through to x?

b. Suppose $x_1 = 4$ and $x_2 = 6$, and $y = \begin{bmatrix} 2 & 8 & 2 \end{bmatrix}^T$. Compute y_{row}. What fraction of y gets through to x?

c. Use general x_1 and x_2. Compute y_{row} in terms of x.

Exercise 3.2 *Figure 3.9 presents a double entry example in directed graph form. It is generally equivalent to cash outlays to three cost pools which are then converted into two output products.*

a. Let $x = 5$ and $y = \begin{bmatrix} 2 & 5 & 3 & 2 & 3 & 2 & 3 \end{bmatrix}^T$. Compute y_{row} and the fraction of y reaching x (R^2).

b. Use general x. Compute y_{row} in terms of x.

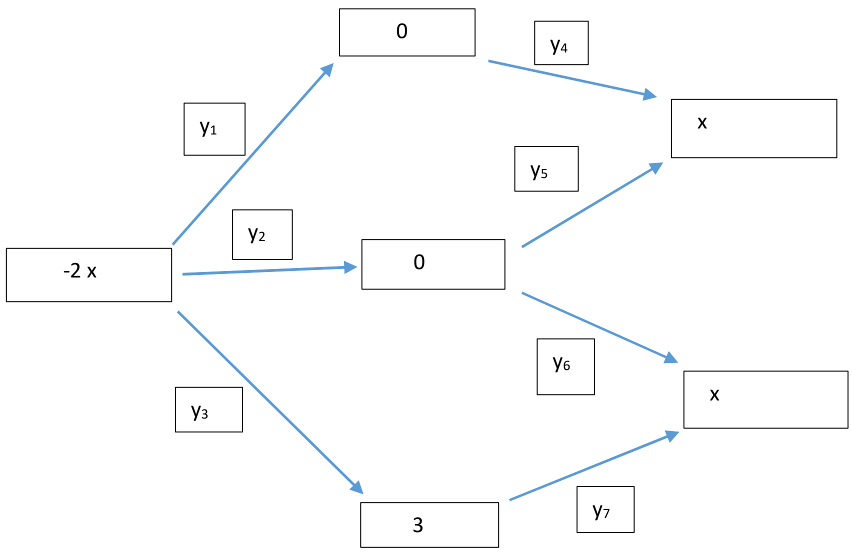

Figure 3.9

Exeercise 3.2 in directed graph format

Exercise 3.3 *Here are partial financial statements.*

partial balance sheet

	ending balance	*beginning balance*
asset 1	*100*	*100*
asset 2	*150*	*150*
asset 3	*115*	*70*

partial income statement

expense 1	*75*
expense 2	*105*

Here are the only journal entries affecting the above accounts.

asset 1	y_1	
cash		y_1
asset 2	y_2	
cash		y_2
asset 3	y_3	
cash		y_3
expense 1	y_4	
asset 1		y_4
expense 1	y_5	
asset 2		y_5
expense 2	y_6	
asset 2		y_6
expense 2	y_7	
asset 3		y_7

Compute $\mathrm{y}_{\mathrm{row}}$.

Exercise 3.4 *Here are financial statements.*

balance sheet

	ending	*beginning*		*ending*	*beginning*
cash	*4*	*2*	*payables*	*11*	*8*
receivables	*8*	*4*	*capital stock*	*10*	*10*
inventory	*8*	*9*	*retained earnings*	*9*	*4*
equipment	*10*	*7*			
total assets	*30*	*22*	*total equities*	*30*	*22*

income statement

sales	*10*
cost of goods sold	*2*
gen'l & admin. expenses	*3*
income	*5*

Here are the only journal entries affecting the above accounts.

receivables	y_1	
sales		y_1
cash	y_2	
receivables		y_2
payables	y_3	
cash		y_3
equipment	y_4	
cash		y_4
g&a expense	y_5	
payables		y_5
CGS	y_6	
inventory		y_6
inventory	y_7	
equipment		y_7
inventory	y_8	
payables		y_8

Compute y_{row}.

Exercise 3.5 *Here are partial financial statements.*

partial balance sheet

	ending balance	*beginning balance*
asset 1	*90*	*100*
asset 2	*180*	*150*

partial income statement

expense	*80*

Here are the only journal entries affecting the above accounts.

asset 1	y_1	
cash		y_1
asset 2	y_2	
cash		y_2
expense	y_3	
cash		y_3

expense	y_4	
asset 1		y_4
expense	y_5	
asset 2		y_5

Compute $\mathrm{y}_{\mathrm{row}}$.

The following three exercises are about pension accounting. The idea is that directed graphs are a useful tool for unraveling pension accounting, in particular, computing numbers like the pension investment made during the period, and pension payments made to pensioners.

Exercise 3.6 *Revisit exercise 2.1, particularly the directed graph depicting pension activity. Here is an example of the supplementary disclosure accompanying financial statements.*

	2007	*2006*
projected obligation	*$289.500*	*$265.000*
plan assets	*190.600*	*159.600*
prepaid/(accrued) pension cost	*$(98.900)*	*$(105.400)*
comprehensive income adjustments:		
unrecognized (gain)/loss	*23.728*	*29.940*
unrecognized prior service cost	*14.400*	*32.000*

Notice the supplementary disclosure allows computing the x vector. For example, the PBO node is -24.5 (increase in a credit balance). Also available is the pension cost: 44.312.

Construct a directed graph with all the accounts affected by the pension journal entries. Attach the change in the account balances (x) wherever possible.

Exercise 3.7 *This is a continuation of the previous problem. Additional supplementary disclosure includes the components of pension cost. This allows specifying some of the elements of the y vector.*

components of pension cost:	*2007*
service cost	*$16.000*
interest	*26.500*
(return)	*(22.000)*
unexpected gain/loss	*6.040*
amort. of prior service cost	*17.600*
amort. of unrecog. (gain)/loss	*0.172*
net pension cost	*$44.312*

Which y vector generates the pension disclosure amounts? Is there more than one solution to $Ay = x$?

Exercise 3.8 *Here is another example of supplementary financial disclosure for pensions.*

	2007	2006
projected obligation	*$60*	*$50*
plan assets	*38*	*30*
prepaid/(accrued) pension cost	*$(22)*	*$(20)*
comprehensive income adjustments:		
unrecognized (gain)/loss	*10*	*6*
unrecognized prior service cost	*8*	*12*

components of pension cost:	2007
service cost	20
interest	10
(return)	(15)
unexpected gain/loss	3
amort. of prior service cost	4
amort. of unrecog. (gain)/loss	2
net pension cost	$24

Compute a solution to $Ay = x$. Is there more than one solution?

Exercise 3.9 *Reconsider example 3.3. The quadratic program is (after eliminating a redundant constraint)*

$$\begin{aligned} & Min\ y_1^2 + y_2^2 + y_3^2 \\ s.t. y_1 + y_2 &= 12 \\ y_1 + y_3 &= 9 \end{aligned}$$

The method of Lagrange combines the objective with the left-hand side of the constraints into one expression.

$$\mathcal{L} = y_1^2 + y_2^2 + y_3^2 + \lambda_1 (y_1 + y_2) + \lambda_2 (y_1 + y_3)$$

The λ's are Lagrange multipliers (shadow prices on the constraints). The first order conditions for a local optimum are the partial derivatives of $\mathcal{L}$ are equal to zero.

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial y_1} &= 2y_1 + \lambda_1 + \lambda_2 = 0 \\ \frac{\partial \mathcal{L}}{\partial y_2} &= 2y_2 + \lambda_1 = 0 \\ \frac{\partial \mathcal{L}}{\partial y_1} &= 2y_3 + \lambda_2 = 0 \end{aligned}$$

With the two original constraints we have five linear equations in five unknowns. In this problem the multipliers are easily substituted out.

$$\begin{aligned} \lambda_1 &= -2y_2 \\ \lambda_2 &= -2y_3 \end{aligned}$$

And we are left with three relatively simple linear equations in three unknowns to get y_{row}.

$$\begin{aligned} 2y_1 - 2y_2 - 2y_3 &= 0 \\ y_1 + y_2 &= 12 \\ y_1 + y_3 &= 9 \end{aligned}$$

Exercise 3.10 *Decompose the vector* $y^T = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ *into 2 orthogonal components, one of which is a scalar multiple of* $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$. *What is the* R^2?

Exercise 3.11 *Decompose the vector* $y^T = \begin{bmatrix} 0 & 5 & 10 \end{bmatrix}$ *into 2 orthogonal components, one of which is in the row space of*

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

Exercise 3.12 *Redo the previous exercise with* $y^T = \begin{bmatrix} 5 & 0 & 10 \end{bmatrix}$.