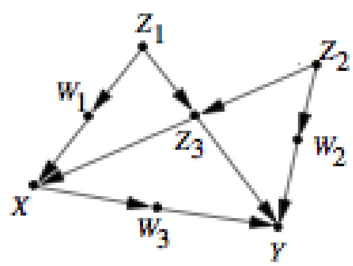
Fundamental structural causal modeling (SCM) questions

For concreteness, consider two models for discussion (but without specifying their content). The models are nonparametric, structural models with endogenous variables $Y, X, Z_1, Z_2, Z_3, W_1, W_2, W_3$, mutually independent, unobserved exogenous variables $U, U\prime, U_1, U_1', U_2, U_2', U_3, U_3'$, and arbitrary, unknown functions $f, f_1, f_2, f_3, g, g_1, g_2, g_3$.

Model 1

$$\begin{array}{cc} Y = f\left(W_3, Z_3, W_2, U\right) & X = g\left(W_1, Z_3, U'\right) \\ W_3 = g_3\left(X, U_3'\right) & W_1 = g_1\left(Z_1, U_1'\right) \\ Z_3 = f_3\left(Z_1, Z_2, U_3\right) & Z_1 = f_1\left(U_1\right) \\ W_2 = g_2\left(Z_2, U_2'\right) & Z_2 = f_2\left(U_2\right) \end{array}$$

The DAG below is equivalent to the algebraic representation.



Model 1b is the same except the direction between X and Z_3 is reversed, $X \rightarrow Z_3$. In the familiar, special linear model case, we have Model 2

$$\begin{array}{cc} Y = aW_2 + bZ_3 + cW_3 + U & X = t_1 W_1 + t_2 Z_3 + U' \\ W_3 = c_3 X + U_3' & W_1 = a_1 Z_1 + U_1' \\ Z_3 = a_3 Z_1 + b_3 Z_2 + U_3 & Z_1 = U_1 \\ W_2 = c_2 Z_2 + U_2' & Z_2 = U_2 \end{array}$$

The orthogonality conditions render these equations regressional. However, if some of the endogenous variables are not measurable, then we can illustrate non-regressional models. We explore the following 8 fundamental questions.

1. Testable implication (misspecification tests)

   a. What are the testable implications of assumptions embedded in model 1?

   b. Assume only X,Y,Z_3, and W_3 are measured, are there any testable implications?

   c. The same as (b), except only X,Y, and Z_3 are measured?

d. The same as (b), except all but Z_3 are measured?

e. What statistical tests distinguish between models 1 and 1b?

f. Identify the regression coefficient for the test devised in e.

2. Equivalent models

The conditions for edge replacement are (i) Rule 1: An arrow $X \to Y$ can be replaced by a bidirected arrow $X \leftrightarrow Y$ only if every neighbor or parent of X is inseparable from Y. (By neighbor we mean any node connected to X by a bidirected arrow). (ii) Rule 2: An arrow $X \to Y$ can be reversed by $X \leftarrow Y$ only if every neighbor or parent of Y (excluding X) is inseparable from X every neighbor or parent of X is inseparable from Y. Further, in a Markovian model (a DAG with jointly independent errors) arrows are reversible if and only if every parent of X is also a parent of Y.

a. Which arrows in the DAG can be reversed without being detected by any statistical test?

b. Is there an equivalent model (statistically indistinguishable) in which variable Z_3 can be made a mediator between X and Y (arrow $Z_3 \to X$ is reversed)?

3. Identification

a. Suppose we wish to identify the average causal effect of X on Y

$$ACE = \Pr\left[Y \mid do\left(X=1\right)\right] - \Pr\left[Y \mid do\left(X=0\right)\right]$$

Which subsets of variables need to be adjusted to obtain an unbiased estimate of ACE?

b. Is there a single variable, if adjusted would allow an unbiased estimate of ACE?

c. If there is a choice between subsets of variables $\{Z_3, Z_1\}$ or $\{Z_3, Z_2\}$, which would be preferred?

4. Instrumental variables

Definition 1 (Instrument) *A variable Z is an instrument for the $X \to Y$ relationship if there exists a set of measurement $S = s$, unaffected by X, such that the following graphical criteria holds. (i) $\left(Z \perp\!\!\!\perp Y \mid S\right)_{G_{\overline{X}}}$ and (ii) $\left(Z \not\perp\!\!\!\perp X \mid S\right)_{G}$ where $\not\perp\!\!\!\perp$ is read not independent.*

Definition 2 (linear-IV estimand)

$$b \equiv \frac{\partial}{\partial x} E\left[Y \mid \widehat{x}\right] = \frac{E\left[Y \mid z, s\right]}{E\left[X \mid z, s\right]} = \frac{r_{YZ \cdot S}}{r_{XZ \cdot S}}$$

where $r_{AB \cdot C}$ is the regression coefficient for A regressed on B conditional on C.

Definition 3 (Generalized conditional instrument) *Z is an instrument for the parameter c in the $X \rightarrow Y$ relationship conditional on variable set W if the following conditions are satisfied:* (1) *W contains only non-descendents of Y;* (2) *W d-separates Z from Y in the subgraph G_c, obtained by removing edge $X \rightarrow Y$ from G;* (3) *W does not d-separate Z from X in G_c.*

Definition 4 (Generalized conditional IV estimand)

$$c \equiv \frac{Cov\left[Y, Z \mid w\right]}{Cov\left[X, Z \mid w\right]} = \frac{r_{YZ \cdot W}}{r_{XZ \cdot W}}$$

where $r_{AB \cdot C}$ is the regression coefficient for A regressed on B conditional on C.

a. Is there an instrumental variable for the $Z_3 \rightarrow Y$ relationship? If so, what is the (linear) IV estimand for parameter b in model 2?

b. Is there an instrumental variable for the $X \rightarrow Y$ relationship? If so, what is the (linear) IV estimand for the product $c_3 c$ in model 2?

5. Mediation

The total effect of $X \rightarrow Y$ is $\Pr\left(Y \mid \widehat{x}\right)$. The average (natural or pure) total effect can be decomposed into average direct and indirect effects. The direct effect is the sensitivity of Y to changes in X holding all other factors constant. For the example, this implies there is no direct effect for $X \rightarrow Y$ as it is mediated by W_3 and holding W_3 constant disconnects X from Y.

Indirect effects are only defined for averages. The average direct effect is

$$DE_{x,x'}\left(Y\right) = E\left[Y\left(x', Z\left(x\right)\right) - E\left[Y\left(x\right)\right]\right]$$

where Z represents all parents of Y excluding X and $Y\left(x', Z\left(x\right)\right)$ represents the value Y would take under action $X = x\prime$ and simultaneously setting Z to the value obtained when setting $X = x$. The average indirect effect is

$$IE_{x,x'}\left(Y\right) = E\left[Y\left(x, Z\left(x'\right)\right) - E\left[Y\left(x\right)\right]\right]$$

and is interpreted as the value Y would obtain holding X constant at x while changing the value of Z to whatever it would obtain if X is set to $x\prime$. Their sum is the average total effect

$$\begin{aligned} TE_{x,x'}\left(Y\right) &= DE_{x,x'}\left(Y\right) + IE_{x,x'}\left(Y\right) \\ &= E\left[Y\left(x'\right) - Y\left(x\right)\right] \end{aligned}$$

In the simple case of unconfounded mediators, the natural direct and indirect effects are estimable via two regressions called the mediation formula.

$$\begin{aligned} DE_{x,x'}(Y) &= \sum_z \{E[Y \mid x', z] - E[Y \mid x, z]\} P(z \mid x) \\ IE_{x,x'}(Y) &= \sum_z E[Y \mid x, z] \{P(z \mid x') - P(z \mid x)\} \end{aligned}$$

a. What variables must be measured if we wish to estimate the direct effect of Z_3 on Y?

Graphically, the expected natural direct effect is identified if there exists four set of variables, W_0, W_1, W_2, W_3 such that

(i) $\left(Y \perp\!\!\!\perp Z \mid W_0\right)_{G_{\underline{XZ}}}$

(ii) $\left(Y \perp\!\!\!\perp X \mid W_0, W_1\right)_{G_{\underline{X}\overline{Z}}}$

(iii) $\left(Y \perp\!\!\!\perp Z \mid X, W_0, W_1, W_2\right)_{G_{\underline{Z}}}$

(iv) $\left(Z \perp\!\!\!\perp X \mid W_0, W_3\right)_{G_{\underline{X}}}$

(v) W_0, W_1, W_3 contain no descendant of X and W_2 contains no descendant of Z

and its estimand for $X \to Y$ in Markovian models is

$$\begin{aligned} & NDE(x, x^*; Y) \\ = & \sum_s \sum_z \{E[Y \mid x, z] - E[Y \mid x^*, z]\} \Pr(z \mid x^*, s) \Pr(s) \end{aligned}$$

b. What variables must be measured if we wish to estimate the indirect effect of Z_3 on Y, mediated by X?

The expected natural indirect effect is identified analogously to the expected natural direct effect but with x and x^* reversed. That is, the average natural indirect effect is identified if there exists a set W, nondescendants of X or Z, such that

$$Y_{x^*z} \perp\!\!\!\perp Z_x \mid W \quad \text{for all } z, x$$

and its estimand is

$$\begin{aligned} & NIE(x, x^*; Y) \\ = & \sum_w \sum_z E[Y_{x^*z} \mid w] \{\Pr(Z_x = z \mid w) - \Pr(Z_{x^*} = z \mid w)\} \Pr(w) \end{aligned}$$

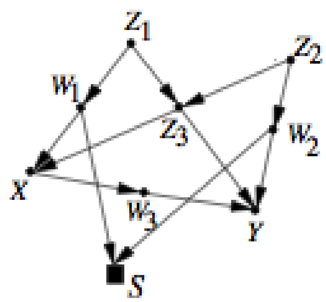
c. What is the estimand for the indirect effect in (b), if all variables are binary?

6. Sampling selection bias

Suppose we wish to estimate the conditional expectation $E[Y \mid x = x]$ and samples are preferentially selected to the data set depending on a set V_s of variables.[1]

a. Let $V_s = \{W_1, W_2\}$, what set, T, of variables need be measured to correct for selection bias? (Assuming we can estimate $\Pr(T = t)$ from external sources e.g., census data.)

Consider the augmented graph G_S below where node or variable $S = 1$ for inclusion in the sample and 0 for exclusion.



b. In general, for which sets, V_s, would selection bias be correctable.

c. Repeat (a) and (b) assuming that our aim is to estimate the causal effect of X on Y.

7. Linear digressions (consider the linear model, Model 2)

a. Name three testable implications of this model

b. Suppose X, Y, and W_3 are the only variables that can be observed. Which parameters can be identified from the data?

c. If we regress Z_1 on all other variables in the model, which regression coefficient will be zero?

d. If we regress Z_1 on all the other variables in the model and then remove Z_3 from the regressor set, which coefficient will not change?

[1] For a more complete discussion see Pearl, 2012, "A solution to a class of selction-bias problems."

e. ("Robustness" – a more general version of d.) Model 2 implies that certain regression coefficients will remain invariant when an additional variable is added as a regressor. Identify five such coefficients with their added regressors.

8. Counterfactual reasoning

The back-door condition leads to conditional independence. For the causal effect $X \to Y$, the back-door condition involves identifying a conditioning set Z such that

1. we block all spurious paths from X to Y,
2. we leave all directed paths unperturbed,
3. we create no new spurious paths.

a. Find a set S of endogenous variables such that X would be independent of the counterfactual Y_x conditioned on S.

b. Determine if X is independent of the counterfactual Y_x conditioned on all the other endogenous variables.

c. Determine if X is independent of the counterfactual $W_{3,x}$ conditioned on all the other endogenous variables.

d. Determine if the counterfactual relationship $\Pr(Y_x|X = x\prime)$ is identifiable, assuming that only X, Y , and W_3 are observed.