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Introduction

Fundamentally, firms are vehicles of production. That is, a firm is an organization through which raw resources or inputs are transformed into valued products (goods and services). Managers are stewards over the production process. They are responsible for efficient production, that is, managers are responsible for identifying efficient technologies and acquiring low cost, high quality resources for their transformation into products. As production technologies and resource availability are dynamic and uncertain, much of what managers do involves experimentation and interpretation of evidence generated by the experiments. In the spirit of Demski's *Managerial Uses of Accounting Information*, we take it as given that (at least) a rudimentary understanding of experimental design and consistent evaluation of evidence is foundational for a well-prepared, responsible manager.

1.1 Study plan

Our study plan typically involves the following components:

1. experimentation to gather evidence.

2. evidence or experimental information is typically interpreted through a counterfactual or causal effect lens.

3. managers act by creating and amending production technologies and projects.

4. managers respond to information by rebalancing projects and searching for new projects.

5. maximization of expected utility objective is often simplified as maximization of long-run wealth via the Kelly criterion.

1.2 Experiments and information

Experiments generate data — often substantial amounts of data. Data from well-designed experiments are related or framed through systems of equations that facilitate interpretation of the data. Interpretation of the data usually takes the form of solving for a set of summary parameters. However, solutions to large systems of equations are rarely unique. It is much more common for there to be an abundance of solutions or no exact solution. In these pages, we'll primarily focus on the latter case where no exact solution exists. When no exact solution exists we frame outcomes (what we're attempting to explain/understand) in terms of an incomplete set of observables (regressors or covariates) combined with unobservables (model errors).

Unobservables are defined by probability distributions assigned to (so-called) random variables based on our background knowledge of the setting. Some argue random variables are objectively random (whatever that means) while others argue that the system is sufficiently complex that we abandon all hope of fully coping with its detail within an acceptable time frame (the information is conserved view of say, quantum mechanics). We'll simply refer to elements that the analyst/manager is unable to directly observe as the unobservables (or occasionally the latent variables) in the equations.

We begin our search for causal effects with simple, single parameter problems, that is, where outcomes of interest are related to what is known via a single parameter (usually a mean effect). Then, expand the discussion to multiple means ($ANOVA$ and $ANCOVA$) and richer contexts where explanatory variables as well as outcomes are endogenously determined.[1]

Next, we briefly discuss manager's project identification, selection, diversification, and rebalancing based on information collected (via experiments) with an eye on the Kelly criterion — long-run wealth maximization.

[1] Quantities or variables that are taken as given or are provided from outside the analysis are said to be exogenous. On the other hand, endogenous variables are determined within the analysis or model at hand.

1.3 Kelly criterion

Expected long-run wealth is maximized via the Kelly criterion. That is, maximize compound (or geometric mean) return on investment

$$\max_{w} \quad G[r] = \prod_{j=1}^{n} \left(\sum_{i=1}^{m} w_i r_{ij}\right)^{p_j}$$
$$s.t. \quad \sum_{i=1}^{m} w_i = 1$$

or equivalently maximize the arithmetic mean (expected value) of the natural logarithm of returns

$$\max_{w} \quad E[r] = \sum_{j=1}^{n} p_j \ln\left(\sum_{i=1}^{m} w_i r_{ij}\right)$$
$$s.t. \quad \sum_{i=1}^{m} w_i = 1$$

where w_i is portion of wealth invested in project i (this quantity may be negative which translates into borrowing against its future payoff, that is borrow the investment amount and return the payoff to the lender, also known as short-selling), r_{ij} is return (payoff on investment equal to one) on project i in state j (so that $\sum_{i=1}^{m} w_i r_{ij}$ is the return on the portfolio of projects in state j), and p_j is the probability the manager assigns to state j. Note, $G[r] = \exp(E[r])$ is a consistency check on the analysis.

Analysis of the problem (project selection/diversification) is greatly simplified by converting nominal projects into Arrow-Debreu investments (assets that payoff in exactly one state and zero in all other states). Let A denote a matrix of payoffs/returns (on normalized to unity investments) where the rows indicate the project and the columns indicate the states, v denote a vector of investment costs associated with the projects and y denote a vector of Arrow-Debreu (or state) values/prices.

$$Ay = v$$

If A is full rank ($n \times n$ and comprised of linearly independent rows and columns), then

$$y = A^{-1}v$$

and the elements in row j of A^{-1} identify the portfolio weights on the nominal projects for constructing the Arrow-Debreu investment that pays off in state j. As the investment cost for each of the n Arrow-Debreu portfolios implied by these weights is y_j the return on Arrow-Debreu state j portfolio is $\frac{1}{y_j}$ where y_j is positive otherwise arbitrage (profiting from zero investment and bearing no risk) opportunities exist. The optimal fraction of wealth, k, invested in each Arrow-Debreu portfolio of projects is

$$\max_{k} \quad G[r] = \prod_{j=1}^{n} \left(k_j \frac{1}{y_j}\right)^{p_j}$$
$$s.t. \quad \sum_{j=1}^{n} k_j = 1$$

or

$$\max_{k} \quad E[r] = \sum_{j=1}^{n} p_j \ln\left(k_j \frac{1}{y_j}\right)$$
$$s.t. \quad \sum_{j=1}^{n} k_j = 1$$

The first order conditions for the Lagrangian

$$\mathcal{L} = \sum_{j=1}^{n} p_j \ln\left(k_j \frac{1}{y_j}\right) - \lambda\left(\sum_{j=1}^{n} k - 1\right)$$

are

$$\frac{p_j}{k_j} - \lambda = 0, \quad \text{for all } j$$

Since $\sum k_j = 1 = \sum \frac{p_j}{\lambda} = \frac{1}{\lambda}$, $\lambda = 1$ and $k_j = p_j$. In other words, probability assignment to state j identifies the optimal fractional investment in state j.

Notice there are no negative investments in Arrow-Debreu project portfolios and the optimal weight doesn't depend on the payoff. Maximization of geometric mean or expected compound return doesn't include selecting a portfolio with zero (or negative) return in any state as that kills long-run wealth accumulation. One can never fully deplete the asset base with this investment policy as some fraction of wealth is invested in the state that pays off.

Importantly, the Kelly criterion connects to Shannon's noisy channel theorem by equating mutual information,[2] $I(info; y) = H(info) + H(y) - H(info, y)$, with the expected gain (in returns) due to information, $\Delta = E[r \mid \text{info}] - E[r]$, where $H(\cdot) = -\sum_{j=1}^{n} p_j \ln p_j$, or entropy. Let the infor-

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[2] Mutual information is usually defined as

$$\begin{aligned} I(X;Y) &= H(Y) - H(Y \mid X) \\ &= H(X) - H(X \mid Y) \end{aligned}$$

but the additivity axiom

$$\begin{aligned} H(X,Y) &= H(Y) + H(X \mid Y) \\ &= H(X) + H(Y \mid X) \end{aligned}$$

allows a form that if often computationally simpler. Substitute

$$H(X \mid Y) = H(X,Y) - H(Y)$$

into the expression for mutual information

$$\begin{aligned} I(X;Y) &= H(X) - (H(X,Y) - H(Y)) \\ &= H(X) + H(Y) - H(X,Y) \end{aligned}$$

mation signals be denoted by z_j and the states by s_i, then

$$
\begin{aligned}
H(info) &= -\sum_{j=1}^{n} \Pr(z_j) \ln \Pr(z_j) \\
H(y) &= -\sum_{i=1}^{n} \Pr(s_i) \ln \Pr(s_i) \\
H(info, y) &= -\sum_{i=1}^{n} \sum_{j=1}^{n} \Pr(s_i, z_j) \ln \Pr(s_i, z_j) \\
I(info; y) &= -\sum_{j=1}^{n} \Pr(z_j) \ln \Pr(z_j) - \sum_{i=1}^{n} \Pr(s_i) \ln \Pr(s_i) \\
&\quad + \sum_{i=1}^{n} \sum_{j=1}^{n} \Pr(s_i, z_j) \ln \Pr(s_i, z_j) \\
&= -\sum_{i=1}^{n} \sum_{j=1}^{n} \Pr(s_{i,} z_j) \ln \Pr(z_j) - \sum_{i=1}^{n} \Pr(s_i) \ln \Pr(s_i) \\
&\quad + \sum_{i=1}^{n} \sum_{j=1}^{n} \Pr(s_i, z_j) \ln \Pr(s_i, z_j) \\
&= -\sum_{i=1}^{n} \Pr(s_i) \ln \Pr(s_i) + \sum_{i=1}^{n} \sum_{j=1}^{n} \Pr(s_i, z_j) \ln \Pr(s_i, z_j) \\
&\quad - \sum_{i=1}^{n} \sum_{j=1}^{n} \Pr(s_{i,} z_j) \ln \Pr(z_j) \\
&= -\sum_{i=1}^{n} \Pr(s_i) \ln \Pr(s_i) + \sum_{i=1}^{n} \sum_{j=1}^{n} \Pr(s_i, z_j) \ln \frac{\Pr(s_i, z_j)}{\Pr(z_j)} \\
&= -\sum_{i=1}^{n} \Pr(s_i) \ln \Pr(s_i) + \sum_{i=1}^{n} \sum_{j=1}^{n} \Pr(s_i, z_j) \ln \Pr(s_i \mid z_j)
\end{aligned}
$$

while

$$\begin{aligned}
E[r \mid info] &= \sum_{j=1}^{n} \Pr(z_j) \sum_{i=1}^{n} \Pr(s_i \mid z_j) \ln \frac{\Pr(s_i \mid z_j)}{y_i} \\
&= \sum_{j=1}^{n} \sum_{i=1}^{n} \Pr(s_i, z_j) \ln \Pr(s_i \mid z_j) - \sum_{j=1}^{n} \sum_{i=1}^{n} \Pr(s_i, z_j) \ln y_i \\
&= \sum_{j=1}^{n} \sum_{i=1}^{n} \Pr(s_i, z_j) \ln \Pr(s_i \mid z_j) - \sum_{i=1}^{n} \Pr(s_i) \ln y_i \\
E[r] &= \sum_{i=1}^{n} \Pr(s_i) \ln \frac{\Pr(s_i)}{y_i} \\
&= \sum_{i=1}^{n} \Pr(s_i) \ln \Pr(s_i) - \sum_{i=1}^{n} \Pr(s_i) \ln y_i \\
\Delta &= E[r \mid info] - E[r] \\
&= \sum_{j=1}^{n} \sum_{i=1}^{n} \Pr(s_i, z_j) \ln \Pr(s_i \mid z_j) - \sum_{i=1}^{n} \Pr(s_i) \ln \Pr(s_i) \\
&= I(info; y)
\end{aligned}$$

This serves to almost immediately identify the value of information as illustrated below.

Contrast the Kelly criterion with maximization of simple returns. This approach calls for an arbitrarily large position on the state with the highest payoff and either no coverage of or borrowing against (or short selling) the other states. Such a strategy assures bankruptcy in the long-run.

Next, we consider some examples to explore the Kelly criterion.

Example 1 (base setting) *Suppose a manager believes he operates in a four state world and he identifies the following collection of projects (including a safe project) and corresponding payoffs/returns.*

$$A = \begin{bmatrix} & s_1 & s_2 & s_3 & s_4 \\ project_1 & 1 & 1 & 1 & 1 \\ project_2 & 1.1 & \frac{1}{1.1} & 1 & 1 \\ project_3 & 1 & 1 & 1.1 & \frac{1}{1.1} \\ project_4 & \sqrt{1.1} & \sqrt{1.1} & \sqrt{\frac{1}{1.1}} & \sqrt{\frac{1}{1.1}} \end{bmatrix}$$

$$v = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

and initial joint probability assignment based on (uninformative) state partitioning is

	s_1	s_2	s_3	s_4	$\Pr(info)$
left	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{2}$
right	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{2}$
$\Pr(s)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	

The Arrow-Debreu investment portfolios are formed as

$$A^{-1} = \begin{bmatrix} -10 & 5.238 & 0 & 4.994 \\ 0 & -5.238 & 0 & 5.494 \\ 0 & 0 & 5.238 & -4.994 \\ 11 & 0 & -5.238 & -5.494 \end{bmatrix}$$

and state prices are

$$y = A^{-1}v = \begin{bmatrix} 0.232 \\ 0.256 \\ 0.244 \\ 0.268 \end{bmatrix}$$

The expected (compound) return is $E[r] = 0.0014$ *or* $G[r] = \exp(E[r]) = 1.0014$. *We explore this from two perspectives. First, write the returns on each Arrow-Debreu portfolio.*

$$\begin{bmatrix} \frac{1}{0.232} & 0 & 0 & 0 \\ 0 & \frac{1}{0.256} & 0 & 0 \\ 0 & 0 & \frac{1}{0.244} & 0 \\ 0 & 0 & 0 & \frac{1}{0.268} \end{bmatrix} = \begin{bmatrix} 4.3025 & 0 & 0 & 0 \\ 0 & 3.9114 & 0 & 0 \\ 0 & 0 & 4.1027 & 0 \\ 0 & 0 & 0 & 3.7293 \end{bmatrix}$$

Since the Kelly criterion indicates the optimal weight k equals the state probability, $k = p = 0.25$ *for each Arrow-Debreu portfolio,*[3] *the expected returns are*

$$\begin{aligned} E[r] &= 0.25\log(0.25 \times 4.3025) + 0.25\log(0.25 \times 3.9114) \\ &\quad +0.25\log(0.25 \times 4.1027) + 0.25\log(0.25 \times 3.7293) \\ &= 0.0014 \end{aligned}$$

[3] Maximization of simple return calls for "going all-in" on state $s1$ and short-selling Arrow-Debreu portfolios covering other states. In other words, a strategy leading to bankruptcy with probability 0.75.

and the geometric mean of returns is

$$\begin{aligned}G[r] &= (0.25 \times 4.3025)^{0.25} \times (0.25 \times 3.9114)^{0.25} \\ &\quad \times (0.25 \times 4.1027)^{0.25} \times (0.25 \times 3.7293)^{0.25} \\ &= \exp(E[r]) = 1.0014\end{aligned}$$

Alternatively, find the portfolio weights, w, that maximize the expected compound return on the projects.

$$\begin{aligned}&\max_{w} \quad \prod_{i=1}^{4} \left(w^T r_i\right)^{p_i} \\ &s.t. \qquad \iota^T w = 1\end{aligned}$$

where r_i is a vector of project returns in state i (column i of matrix A), p_i is the probability assigned to state i, and ι is a vector of ones. The solution is

$$w = \begin{bmatrix} -0.5006 \\ 0.5122 \\ 0.4884 \\ 0.5000 \end{bmatrix}$$

$$\begin{aligned}\prod_{i=1}^{4} \left(w^T r_i\right)^{p_i} &= (1.0756)^{0.25} \times (0.9778)^{0.25} \times (1.0256)^{0.25} \times (0.9323)^{0.25} \\ &= G[r] = 1.0014\end{aligned}$$

and

$$\ln\left(\prod_{i=1}^{4} \left(w^T r_i\right)^{p_i}\right) = E[r] = 0.0014$$

Left/right signals (perhaps odd labels for this setting but bear with us) in this base setting are uninformative as is apparent as either signal leads to the same (optimal) portfolios as described above with no information. Hence, this information leads to no rebalancing. For completeness, we show mutual information is zero in this setting.

$$\begin{aligned}I(info; y) &= H(info) + H(y) - H(info, y) \\ &= \left(-\frac{1}{2}\ln\frac{1}{2} - \frac{1}{2}\ln\frac{1}{2}\right) \\ &\quad + \left(-\frac{1}{4}\ln\frac{1}{4} - \frac{1}{4}\ln\frac{1}{4} - \frac{1}{4}\ln\frac{1}{4} - \frac{1}{4}\ln\frac{1}{4}\right) \\ &\quad - \left(\begin{array}{c} -\frac{1}{8}\ln\frac{1}{8} - \frac{1}{8}\ln\frac{1}{8} - \frac{1}{8}\ln\frac{1}{8} - \frac{1}{8}\ln\frac{1}{8} \\ -\frac{1}{8}\ln\frac{1}{8} - \frac{1}{8}\ln\frac{1}{8} - \frac{1}{8}\ln\frac{1}{8} - \frac{1}{8}\ln\frac{1}{8} \end{array}\right) \\ &= \ln 2 + \ln 4 - \ln 8 = 0\end{aligned}$$

With each realization, uncertainty is (partially) resolved and the manager rebalances the portfolio of projects by expanding and/or contracting investment (recall the optimal investment strategy matches the investment in each state with the likelihood assigned to the state).

Example 2 (rebalancing) *Suppose the manager's operations lead to the following joint probability assignment*

	s_1	s_2	s_3	s_4	$\Pr(info)$
left	0.20	0.20	0.05	0.05	$\frac{1}{2}$
right	0.05	0.05	0.20	0.20	$\frac{1}{2}$
$\Pr(s)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	

Since the payoffs are the same as the base case, Arrow-Debreu portfolios are the same as the base case and only the weights are adjusted to reflect the updated state probability assignments given the information signal. Hence, if the left signal is observed $k_{left} = \begin{bmatrix} 0.40 \\ 0.40 \\ 0.10 \\ 0.10 \end{bmatrix}$ *with expected return*

$$\begin{aligned} E[r \mid left] &= 0.40 \log(0.40 \times 4.3025) + 0.40 \log(0.40 \times 3.9114) \\ &\quad + 0.10 \log(0.10 \times 4.1027) + 0.10 \log(0.10 \times 3.7293) \\ &= 0.2085 \end{aligned}$$

while if the right signal is observed $k_{right} = \begin{bmatrix} 0.10 \\ 0.10 \\ 0.40 \\ 0.40 \end{bmatrix}$ *with expected return*

$$\begin{aligned} E[r \mid right] &= 0.10 \log(0.10 \times 4.3025) + 0.10 \log(0.10 \times 3.9114) \\ &\quad + 0.40 \log(0.40 \times 4.1027) + 0.40 \log(0.40 \times 3.7293) \\ &= 0.1799 \end{aligned}$$

and the expected return conditional on acquiring the information is

$$\begin{aligned} E[r \mid info] &= \Pr(right)\, E[r \mid right] + \Pr(left)\, E[r \mid left] \\ &= 0.5 \times 0.2085 + 0.5 \times 0.1799 = 0.1941 \end{aligned}$$

or

$$\begin{aligned} G[r \mid info] &= \exp(E[r \mid info]) \\ &= G[r \mid left]^{\Pr(left)} \times G[r \mid right]^{\Pr(right)} \\ &= 1.2318^{0.5} \times 1.1971^{0.5} \\ &= 1.2143 \end{aligned}$$

The project portfolio weights conditional on the information are

$$w_{left} = \begin{bmatrix} -13.1077 \\ 0.8195 \\ 0.1953 \\ 13.0929 \end{bmatrix}$$

and

$$w_{right} = \begin{bmatrix} 12.1066 \\ 0.2049 \\ 0.7814 \\ -12.0929 \end{bmatrix}$$

with the same expected returns as derived above based on the Arrow-Debreu (state-based) portfolios. The expected gain from the information (relative to the uninformative base case) is a striking

$$\begin{aligned} E\left[gain \mid info\right] &= E\left[r \mid info\right] - E\left[r\right] \\ &= 0.1941 - 0.0014 = 0.1927 \end{aligned}$$

or geometric gain

$$\begin{aligned} G\left[gain \mid info\right] &= \frac{G\left[r \mid info\right]}{G\left[r\right]} = \frac{1.2143}{1.0014} \\ &= \exp\left(E\left[gain \mid info\right]\right) = \exp\left(0.1927\right) \\ &= 1.2126 \end{aligned}$$

Again, mutual information equals the expected gain (in returns) from the information

$$\begin{aligned} & I(info; y) \\ = \; & H(info) + H(y) - H(info, y) \\ = \; & \left(-\frac{1}{2}\ln\frac{1}{2} - \frac{1}{2}\ln\frac{1}{2}\right) \\ & + \left(-\frac{1}{4}\ln\frac{1}{4} - \frac{1}{4}\ln\frac{1}{4} - \frac{1}{4}\ln\frac{1}{4} - \frac{1}{4}\ln\frac{1}{4}\right) \\ & - \left(\begin{array}{c} -0.20\ln 0.20 - 0.20\ln 0.20 - 0.05\ln 0.05 - 0.05\ln 0.05 \\ -0.05\ln 0.05 - 0.05\ln 0.05 - 0.20\ln 0.20 - 0.20\ln 0.20 \end{array}\right) \\ = \; & \ln 2 + \ln 4 - 1.8867 \\ = \; & 0.6931 + 1.3863 - 1.8867 = 0.1927 \end{aligned}$$

Example 3 (highly informative rebalancing) *Suppose the manager's operations/experiments are highly informative leading to the following joint probability assignments*

	s_1	s_2	s_3	s_4	$\Pr(info)$
left	0.249	0.249	0.001	0.001	$\frac{1}{2}$
right	0.001	0.001	0.249	0.249	$\frac{1}{2}$
$\Pr(s)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	

Again, the Arrow-Debreu portfolios are the same as the base case and only the weights are adjusted to reflect the updated state probability assignments given the information signal. Hence, if the left signal is observed $k_{left} = \begin{bmatrix} 0.498 \\ 0.498 \\ 0.002 \\ 0.002 \end{bmatrix}$ *with expected return*

$$\begin{aligned} E[r \mid left] &= 0.498 \log(0.498 \times 4.3025) + 0.498 \log(0.498 \times 3.9114) \\ &\quad +0.002 \log(0.002 \times 4.1027) + 0.002 \log(0.002 \times 3.7293) \\ &= 0.6921 \end{aligned}$$

while if the right signal is observed $k_{right} = \begin{bmatrix} 0.002 \\ 0.002 \\ 0.498 \\ 0.498 \end{bmatrix}$ *with expected return*

$$\begin{aligned} E[r \mid right] &= 0.002 \log(0.002 \times 4.3025) + 0.002 \log(0.002 \times 3.9114) \\ &\quad +0.498 \log(0.498 \times 4.1027) + 0.498 \log(0.498 \times 3.7293) \\ &= 0.6449 \end{aligned}$$

and the expected return conditional on the information is

$$\begin{aligned} E[r \mid info] &= \Pr(right) E[r \mid right] + \Pr(left) E[r \mid left] \\ &= 0.5 \times 0.6921 + 0.5 \times 0.6449 = 0.6685 \end{aligned}$$

or

$$\begin{aligned} G[r \mid info] &= \exp(E[r \mid info]) \\ &= G[r \mid left]^{\Pr(left)} \times G[r \mid right]^{\Pr(right)} \\ &= 1.9980^{0.5} \times 1.9057^{0.5} \\ &= 1.9513 \end{aligned}$$

The project portfolio weights conditional on the information are

$$w_{left} = \begin{bmatrix} -21.3444 \\ 1.0203 \\ 0.0039 \\ 21.3202 \end{bmatrix}$$

and

$$w_{right} = \begin{bmatrix} 20.3433 \\ 0.0041 \\ 0.9728 \\ -20.3202 \end{bmatrix}$$

with the same expected returns as derived above based on the Arrow-Debreu (state-based) portfolios. The expected gain from the information (relative to the uninformative base case) is a striking

$$\begin{aligned} E[gain \mid info] &= E[r \mid info] - E[r] \\ &= 0.6685 - 0.0014 = 0.6671 \end{aligned}$$

or geometric gain

$$\begin{aligned} G[gain \mid info] &= \frac{G[r \mid info]}{G[r]} = \frac{1.9513}{1.0014} \\ &= \exp(E[gain \mid info]) = \exp(0.6671) \\ &= 1.9485 \end{aligned}$$

Again, mutual information equals the expected gain (in returns) from the information

$$\begin{aligned} & I(info; y) \\ = \; & H(info) + H(y) - H(info, y) \\ = \; & \left(-\frac{1}{2}\ln\frac{1}{2} - \frac{1}{2}\ln\frac{1}{2}\right) \\ & + \left(-\frac{1}{4}\ln\frac{1}{4} - \frac{1}{4}\ln\frac{1}{4} - \frac{1}{4}\ln\frac{1}{4} - \frac{1}{4}\ln\frac{1}{4}\right) \\ & - \left(\begin{array}{c} -0.249\ln 0.249 - 0.249\ln 0.249 - 0.001\ln 0.001 - 0.001\ln 0.001 \\ -0.001\ln 0.001 - 0.001\ln 0.001 - 0.249\ln 0.249 - 0.249\ln 0.249 \end{array}\right) \\ = \; & \ln 2 + \ln 4 - 1.4124 \\ = \; & 0.6931 + 1.3863 - 1.4124 = 0.6671 \end{aligned}$$

Suppose the manager is unable to pursue projects sufficient to span the states. We focus on the base case with uninformative left/right information. There appear to be two types of such cases that merit consideration. One case involves a composite or aggregation of states with the same payoffs to fulfill spanning (this occurs when two or more different sets of events occur but lead to the same project payoffs). Since this allows formation of Arrow-Debreu project portfolios everything goes through as in the preceding examples. The second case is more troublesome as no such aggregation or composite spanning of states is possible and Arrow-Debreu project portfolios cannot be formed. This leads to the use of brute force methods for solving the optimal (long-run wealth maximizing) portfolio construction.

Example 4 (composite spanning) *Suppose equally likely payoffs/returns are as follows.*

$$A = \begin{bmatrix} & s_1 & s_2 & s_3 & s_4 \\ project_1 & 1.1 & \frac{1}{1.1} & \frac{1}{1.1} & \frac{1}{1.1} \\ project_2 & \frac{1}{1.1} & 1.1 & \frac{1}{1.1} & \frac{1}{1.1} \\ project_3 & \frac{1}{\sqrt{1.1}} & \frac{1}{\sqrt{1.1}} & \sqrt{1.1} & \sqrt{1.1} \end{bmatrix}$$

where

$$v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The portfolio weights, w, that maximize the expected compound return on the projects are

$$w = \begin{bmatrix} -2.7896 \\ -2.7896 \\ 6.5792 \end{bmatrix}$$

$$\begin{aligned} \prod_{i=1}^{4} \left(w^T r_i\right)^{p_i} &= (0.6685)^{0.25} \times (0.6685)^{0.25} \times (1.8283)^{0.25} \times (1.8283)^{0.25} \\ &= G[r] = 1.1055 \end{aligned}$$

and

$$\ln\left(\prod_{i=1}^{4} \left(w^T r_i\right)^{p_i}\right) = E[r] = 0.1003$$

With only three projects and four states spanning fails so it's not possible to construct Arrow-Debreu portfolios that span the four states. However, in this case as states s_3 and s_4 have the same payoffs we can aggregate to form, say s_{34}, with the same payoffs as s_3 or s_4 but with probability $p_{34} = p_3 + p_4 = 0.5$. Arrow-Debreu 3-state portfolios are formed as

$$A^{-1} = \begin{bmatrix} 4.0228 & -1.2153 & -2.4334 \\ -1.2153 & -5.238 & -2.4334 \\ -2.5522 & -2.5522 & 5.3779 \end{bmatrix}$$

and state prices are

$$y = A^{-1}v = \begin{bmatrix} 0.3740 \\ 0.3740 \\ 0.2735 \end{bmatrix}$$

The returns on the Arrow-Debreu portfolios are

$$\begin{bmatrix} \frac{1}{0.3740} & 0 & 0 \\ 0 & \frac{1}{0.3740} & 0 \\ 0 & 0 & \frac{1}{0.2735} \end{bmatrix} = \begin{bmatrix} 2.6738 & 0 & 0 \\ 0 & 2.6738 & 0 \\ 0 & 0 & 3.6566 \end{bmatrix}$$

Since the Kelly criterion indicates the optimal weight k equals the state probability, $k_1 = k_2 = 0.25$ (for Arrow-Debreu portfolios one and two) and $k_{34} = 0.5$, the expected returns are

$$\begin{aligned} E[r] &= 0.25 \log(0.25 \times 2.6738) + 0.25 \log(0.25 \times 2.6738) \\ &\quad +0.5 \log(0.5 \times 3.6566) \\ &= 0.1003 \end{aligned}$$

and

$$\begin{aligned} G[r] &= (0.25 \times 2.6738)^{0.25} \times (0.25 \times 2.6738)^{0.25} \\ &\quad \times (0.5 \times 3.6566)^{0.5} \\ &= \exp(E[r]) = 1.1055 \end{aligned}$$

Left/right signals in this setting are uninformative. Hence, this information leads to no rebalancing and offers no value to the manager. To complete the picture, mutual information is zero in this setting is

$$\begin{aligned} I(info; y) &= H(info) + H(y) - H(info, y) \\ &= \left(-\frac{1}{2}\ln\frac{1}{2} - \frac{1}{2}\ln\frac{1}{2}\right) \\ &\quad + \left(-\frac{1}{4}\ln\frac{1}{4} - \frac{1}{4}\ln\frac{1}{4} - \frac{1}{2}\ln\frac{1}{2}\right) \\ &\quad - \left(\begin{array}{c} -\frac{1}{8}\ln\frac{1}{8} - \frac{1}{8}\ln\frac{1}{8} - \frac{1}{4}\ln\frac{1}{4} \\ -\frac{1}{8}\ln\frac{1}{8} - \frac{1}{8}\ln\frac{1}{8} - \frac{1}{4}\ln\frac{1}{4} \end{array}\right) \\ &= \ln 2 + \ln 4 - \ln 8 \\ &= 0.6931 + 1.0397 - 1.7328 = 0 \end{aligned}$$

The second case is more troublesome.

Example 5 (spanning fails) *Suppose equally likely payoffs/returns are as follows.*

$$A = \begin{bmatrix} & s_1 & s_2 & s_3 & s_4 \\ project_1 & 1.1 & \frac{1}{1.1} & \frac{1}{1.1} & \frac{1}{1.1} \\ project_2 & \frac{1}{1.1} & 1.1 & \frac{1}{1.1} & \frac{1}{1.1} \\ project_3 & \frac{1}{\sqrt{1.1}} & \frac{1}{\sqrt[4]{1.1}} & \sqrt{1.1} & \sqrt[4]{1.1} \end{bmatrix}$$

where

$$v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The portfolio weights, w, that maximize the expected compound return on the projects are

$$w = \begin{bmatrix} -3.5021 \\ -4.5981 \\ 9.1002 \end{bmatrix}$$

$$\begin{aligned} \prod_{i=1}^{4} \left(w^T r_i\right)^{p_i} &= (0.6443)^{0.25} \times (0.6443)^{0.25} \times (2.1806)^{0.25} \times (1.9558)^{0.25} \\ &= G[r] = 1.1535 \end{aligned}$$

and

$$\ln \left(\prod_{i=1}^{4} \left(w^T r_i\right)^{p_i} \right) = E[r] = 0.1428$$

As in the previous case, spanning fails and even though there exist nonnegative state prices

$$y = \begin{bmatrix} 0.3795 \\ 0.3795 \\ 0 \\ 2.6124 \end{bmatrix} + \alpha \begin{bmatrix} -0.0482 \\ -0.0482 \\ -0.6502 \\ 0.7567 \end{bmatrix}, \quad -0.3452 \leq \alpha \leq 0$$

we cannot construct Arrow-Debreu project portfolios. Therefore, we're left with brute force methods to evaluate information value, etc. In spite of this apparent setback, we're still able to identify the optimal project portfolio for maximizing long-run wealth and avoid bankruptcy. Recall, maximization of the geometric mean steers clear of nonpositive returns in any state (a state involving a zero return produces an absorbing state while a negative return results in an imaginary objective function value).

Example 6 (spanning by perturbation) *Continue with example 5. The manager may expand the project set to span the state space by aggregating existing projects and writing/revising contracts so as to perturb the payoffs. Suppose project 4 is equal parts projects 1 and 2 plus the following perturbation, ε, of the resultant project's payoffs/returns.*

$$\varepsilon = \begin{bmatrix} 0 & 0 & 0.1 & -0.1 \end{bmatrix}$$

Hence, the payoff matrix is

$$A = \begin{bmatrix} & s_1 & s_2 & s_3 & s_4 \\ project_1 & 1.1 & \frac{1}{1.1} & \frac{1}{1.1} & \frac{1}{1.1} \\ project_2 & \frac{1}{1.1} & 1.1 & \frac{1}{1.1} & \frac{1}{1.1} \\ project_3 & \frac{1}{\sqrt{1.1}} & \frac{1}{\sqrt[4]{1.1}} & \sqrt{1.1} & \sqrt[4]{1.1} \\ project_4 & 0.55 + \frac{0.5}{1.1} & 0.55 + \frac{0.5}{1.1} & \frac{1}{1.1} + 0.1 & \frac{1}{1.1} - 0.1 \end{bmatrix}$$

with normalized cost

$$v = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

The Arrow-Debreu investment portfolios are formed as

$$A^{-1} = \begin{bmatrix} 3.8615 & -1.0427 & -2.7727 & 0.3426 \\ -1.3766 & 4.1954 & -2.7727 & 0.3426 \\ -3.5979 & -3.9669 & 3.0638 & 4.6217 \\ 1.4021 & 1.0331 & 3.0638 & -5.3783 \end{bmatrix}$$

and state prices are

$$y = A^{-1}v = \begin{bmatrix} 0.3885 \\ 0.3885 \\ 0.1207 \\ 0.1207 \end{bmatrix}$$

The returns on each Arrow-Debreu portfolio are

$$\begin{bmatrix} \frac{1}{0.3885} & 0 & 0 & 0 \\ 0 & \frac{1}{0.3885} & 0 & 0 \\ 0 & 0 & \frac{1}{0.1207} & 0 \\ 0 & 0 & 0 & \frac{1}{0.1207} \end{bmatrix} = \begin{bmatrix} 2.5742 & 0 & 0 & 0 \\ 0 & 2.5742 & 0 & 0 \\ 0 & 0 & 8.2824 & 0 \\ 0 & 0 & 0 & 8.2824 \end{bmatrix}$$

Since the Kelly criterion indicates the optimal weight k equals the state probability, $k = p = 0.25$ for each Arrow-Debreu portfolio, the expected returns are

$$\begin{aligned} E[r] &= 0.25\log(0.25\times 2.5742) + 0.25\log(0.25\times 2.5742) \\ &\quad +0.25\log(0.25\times 8.2824) + 0.25\log(0.25\times 8.2824) \\ &= 0.1435 \end{aligned}$$

and the geometric mean of returns is

$$\begin{aligned} G[r] &= (0.25\times 2.5742)^{0.25} \times (0.25\times 2.5742)^{0.25} \\ &\quad \times (0.25\times 8.2824)^{0.25} \times (0.25\times 8.2824)^{0.25} \\ &= \exp(E[r]) = 1.1543 \end{aligned}$$

The portfolio weights, w, that maximize the expected compound return on the projects are

$$w = \begin{bmatrix} -2.9474 \\ -4.0457 \\ 9.1190 \\ -1.1260 \end{bmatrix}$$

$$\begin{aligned} \prod_{i=1}^{4} \left(w^T r_i\right)^{p_i} &= (0.6435)^{0.25} \times (0.6435)^{0.25} \times (2.0706)^{0.25} \times (2.0706)^{0.25} \\ &= G[r] = 1.1543 \end{aligned}$$

and

$$\ln\left(\prod_{i=1}^{4} \left(w^T r_i\right)^{p_i}\right) = E[r] = 0.1435$$

Left/right signals are uninformative so mutual information is zero.

$$\begin{aligned} I(info; y) &= H(info) + H(y) - H(info, y) \\ &= \left(-\frac{1}{2}\ln\frac{1}{2} - \frac{1}{2}\ln\frac{1}{2}\right) \\ &\quad + \left(-\frac{1}{4}\ln\frac{1}{4} - \frac{1}{4}\ln\frac{1}{4} - \frac{1}{4}\ln\frac{1}{4} - \frac{1}{4}\ln\frac{1}{4}\right) \\ &\quad - \begin{pmatrix} -\frac{1}{8}\ln\frac{1}{8} - \frac{1}{8}\ln\frac{1}{8} - \frac{1}{8}\ln\frac{1}{8} - \frac{1}{8}\ln\frac{1}{8} \\ -\frac{1}{8}\ln\frac{1}{8} - \frac{1}{8}\ln\frac{1}{8} - \frac{1}{8}\ln\frac{1}{8} - \frac{1}{8}\ln\frac{1}{8} \end{pmatrix} \\ &= \ln 2 + \ln 4 - \ln 8 = 0 \end{aligned}$$

However, perturbation through contract design allows us to quickly assess the value of information.

While there is no information in the preceding example, perturbation through contract design allows us to quickly assess the value of information gathering (experimentation) which is, in turn, fulfilled by project rebalancing. Much of the remaining coverage addresses experimental design and interpretation of evidence but we'll try to keep the firm's objective (long-run wealth maximization) in view.