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Appendix I
Quantum information

Quantum information follows from physical theory and experiments. Unlike classical information which is framed in the language and algebra of set theory, quantum information is framed in the language and algebra of vector spaces. To begin to appreciate the difference, consider a set versus a tuple (or vector). For example, the tuple $\begin{bmatrix} 1 & 1 \end{bmatrix}$ is different than the tuple $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ but the sets, $\{1, 1\}$ and $\{1, 1, 1\}$ are the same as set $\{1\}$.

Quantum information is axiomatic. Its richness and elegance is demonstrated in that only four axioms make it complete.[1]

I.1 Quantum information axioms

I.1.1 The superposition axiom

A quantum unit (qubit) is specified by a two element vector, say $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$, with $|\alpha|^2 + |\beta|^2 = 1$.

[1]Some argue that the measurement axiom is contained in the transformation axiom, hence requiring only three axioms. Even though we appreciate the merits of the argument, we'll proceed with four axioms. Feel free to count them as only three if you prefer.

Let $|\psi\rangle \equiv \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \alpha|0\rangle + \beta|1\rangle$,[2] $\langle\psi| = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}^\dagger$ where † is the adjoint (conjugate transpose) operation.

I.1.2 The transformation axiom

A transformation of a quantum unit is accomplished by unitary (length-preserving) matrix multiplication. The Pauli matrices provide a basis of unitary operators.

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

where $i = \sqrt{-1}$. The operations work as follows: $I\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$, $X\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \beta \\ \alpha \end{bmatrix}$, $Y\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = -\begin{bmatrix} \beta i \\ \alpha i \end{bmatrix}$, and $Z\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = -\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$. Other useful single qubit transformations are $H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $\Theta = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & 1 \end{bmatrix}$. Examples of these transformations in Dirac notation are[3]

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}};\ H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$$\Theta|0\rangle = e^{i\theta}|0\rangle;\ \Theta|1\rangle = |1\rangle$$

I.1.3 The measurement axiom

Measurement occurs via interaction with the quantum state as if a linear projection is applied to the quantum state.[4] The set of projection matrices

[2] Dirac notation is a useful descriptor, as $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

[3] A summary table for common quantum operators expressed in Dirac notation is provided in section I.2.

[4] This is a compact way of describing measurement. However, conservation of information (a principle of quantum information) demands that operations be reversible. In other words, all transformations (including interactions to measure) be unitary — projections are not unitary. However, there always exist unitary operators that produce the same post-measurement state as that indicated via projection. Hence, we treat projections as an expedient for describing measurement.

are complete as they add to the identity matrix.

$$\sum_m M_m^\dagger M_m = I$$

where $M_m^\dagger$ is the adjoint (conjugate transpose) of projection matrix M_m The probability of a particular measurement occurring is the squared absolute value of the projection. (An implication of the axiom not explicitly used here is that the post-measurement state is the projection appropriately normalized; this effectively rules out multiple measurement.)

For example, let the projection matrices be $M_0 = |0\rangle\langle 0| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $M_1 = |1\rangle\langle 1| = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Note that M_0 projects onto the $|0\rangle$ vector and M_1 projects onto the $|1\rangle$ vector. Also note that $M_0^\dagger M_0 + M_1^\dagger M_1 = M_0 + M_1 = I$. For $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, the projection of $|\psi\rangle$ onto $|0\rangle$ is $M_0|\psi\rangle$. The probability of $|0\rangle$ being the result of the measurement is $\langle\psi| M_0 |\psi\rangle = |\alpha|^2$.

I.1.4 The combination axiom

Qubits are combined by tensor multiplication. For example, two $|0\rangle$ qubits are combined as $|0\rangle \otimes |0\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ denoted $|00\rangle$. It is often useful to transform one qubit in a combination and leave another unchanged; this can also be accomplished by tensor multiplication. Let H_1 denote a Hadamard transformation on the first qubit. Then applied to a two qubit system, $H_1 = H \otimes I = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$ and $H_1|00\rangle = \frac{|00\rangle + |10\rangle}{\sqrt{2}}$.

Another important two qubit transformation is the controlled not operator,

$$Cnot = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The controlled not operator flips the target, second qubit, if the control, first qubit, equals $|1\rangle$ and otherwise leaves the target unchanged: $Cnot\,|00\rangle = |00\rangle$, $Cnot\,|01\rangle = |01\rangle$, $Cnot\,|10\rangle = |11\rangle$, and $Cnot\,|11\rangle = |10\rangle$,

Entangled two qubit states or Bell states are defined as follows,

$$|\beta_{00}\rangle = Cnot\ H_1\,|00\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

and more generally,

$$\left|\beta_{ij}\right\rangle = Cnot\ H_1\,|ij\rangle \text{ for } i, j = 0, 1$$

The four two qubit Bell states form an orthonormal basis.

I.2 Summary of quantum "rules"

Below we tabulate two qubit quantum operator rules. The column heading indicates the initial state while the row value corresponding the operator identifies the transformed state. Of course, the same rules apply to one qubit (except *Cnot*) or many qubits, we simply have to exercise care to identify which qubit is transformed by the operator (we continue to denote the target qubit via the subscript on the operator).

operator	$\|00\rangle$	$\|01\rangle$	$\|10\rangle$	$\|11\rangle$
I_1 or I_2	$\|00\rangle$	$\|01\rangle$	$\|10\rangle$	$\|11\rangle$
X_1	$\|10\rangle$	$\|11\rangle$	$\|00\rangle$	$\|01\rangle$
X_2	$\|01\rangle$	$\|00\rangle$	$\|11\rangle$	$\|10\rangle$
Z_1	$\|00\rangle$	$\|01\rangle$	$-\|10\rangle$	$-\|11\rangle$
Z_2	$\|00\rangle$	$-\|01\rangle$	$\|10\rangle$	$-\|11\rangle$
Y_1	$i\|10\rangle$	$i\|11\rangle$	$-i\|00\rangle$	$-i\|01\rangle$
Y_2	$i\|01\rangle$	$-i\|00\rangle$	$i\|11\rangle$	$-i\|10\rangle$
H_1	$\frac{\|00\rangle+\|10\rangle}{\sqrt{2}}$	$\frac{\|01\rangle+\|11\rangle}{\sqrt{2}}$	$\frac{\|00\rangle-\|10\rangle}{\sqrt{2}}$	$\frac{\|01\rangle-\|11\rangle}{\sqrt{2}}$
H_2	$\frac{\|00\rangle+\|01\rangle}{\sqrt{2}}$	$\frac{\|00\rangle-\|01\rangle}{\sqrt{2}}$	$\frac{\|10\rangle+\|11\rangle}{\sqrt{2}}$	$\frac{\|10\rangle-\|11\rangle}{\sqrt{2}}$
Θ_1	$e^{i\theta}\|00\rangle$	$e^{i\theta}\|01\rangle$	$\|10\rangle$	$\|11\rangle$
Θ_2	$e^{i\theta}\|00\rangle$	$\|01\rangle$	$e^{i\theta}\|10\rangle$	$\|11\rangle$
$Cnot$	$\|00\rangle$	$\|01\rangle$	$\|11\rangle$	$\|10\rangle$

common quantum operator rules

I.3 Observables and expected payoffs

Measurements involve real values drawn from observables. To ensure measurements lead to real values, observables are also represented by Hermitian matrices. A Hermitian matrix is one in which the complex conjugate of the matrix equals the original matrix, $M^\dagger = M$. Hermitian matrices have real eigenvalues and eigenvalues are the values realized via measurement. Suppose we are working with observable M in state $|\psi\rangle$ where

$$\begin{aligned} M &= \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} \\ &= \lambda_1 |00\rangle \langle 00| + \lambda_2 |01\rangle \langle 01| + \lambda_3 |10\rangle \langle 10| + \lambda_4 |11\rangle \langle 11| \end{aligned}$$

The expected payoff is

$$\begin{aligned} \langle M \rangle &= \langle \psi | M | \psi \rangle \\ &= \lambda_1 \langle \psi | 00 \rangle \langle 00 | \psi \rangle + \lambda_2 \langle \psi | 01 \rangle \langle 01 | \psi \rangle \\ &\quad + \lambda_3 \langle \psi | 10 \rangle \langle 10 | \psi \rangle + \lambda_4 \langle \psi | 11 \rangle \langle 11 | \psi \rangle \end{aligned}$$

In other words, λ_1 is observed with probability $\langle \psi | 00 \rangle \langle 00 | \psi \rangle$, λ_2 is observed with probability $\langle \psi | 01 \rangle \langle 01 | \psi \rangle$, λ_3 is observed with probability $\langle \psi | 10 \rangle \langle 10 | \psi \rangle$, and λ_4 is observed with probability $\langle \psi | 11 \rangle \langle 11 | \psi \rangle$.

I.4 Density operators and quantum entropy

To this point we've focused on pure states. How do we proceed if our state of knowledge indicates a mixture of states (that is, a probability weighted average of states)? Density operators supply the frame for mixed as well as pure states. A density operator is defined as

$$\rho = \sum_{i=1}^{n} p_i \left|\psi_i\right\rangle \left\langle\psi_i\right|$$

where the trace (sum of the diagonal elements) equals one, $tr\left(\rho\right) = 1$. For density operator ρ the expected value associated with observable $M = \sum_{k=1}^{n} \lambda_k \left|k\right\rangle \left\langle k\right|$ (via spectral decomposition where $\left|k\right\rangle$ is an orthonormal basis) is $\left\langle M\right\rangle = tr\left(\rho M\right) = tr\left(M\rho\right)$.

This follows as the probability of observing λ_k equals

$$\Pr\left(\lambda_k\right) = \left\langle k\right| \rho \left|k\right\rangle = \sum_{i=1}^{n} p_i \left\langle k\right| \psi_i\rangle \left\langle\psi_i\right| k\rangle$$

and useful properties of the trace.

$$tr\left(BA\right) = tr\left(AB\right)$$

To see this, recognize $\left(AB\right)_{ik} = \sum_j a_{ij}b_{jk}$ then $tr\left(AB\right) = \sum_{i,j} a_{ij}b_{ji}$ and $tr\left(BA\right) = \sum_{i,j} b_{ji}a_{ij}$ which is the same as $\sum_{i,j} a_{ij}b_{ji}$. Now, let $B \equiv \left|\psi\right\rangle \left\langle\psi\right|$

$$\begin{aligned} tr\left(AB\right) &= tr\left(A\left|\psi\right\rangle \left\langle\psi\right|\right) \\ &= \sum_i \left\langle i\right| A \left|\psi\right\rangle \left\langle\psi\right| i\rangle \\ &= \left\langle\psi\right| A \left|\psi\right\rangle \end{aligned}$$

where $\left|i\right\rangle$ is an orthonormal basis for $\left|\psi\right\rangle$ with first element $\left|\psi\right\rangle$.[5] The second line implements $tr\left(AB\right) = \sum_{i,j} a_{ij}b_{ji}$ while the third line follows from orthogonality of the basis for $\left|\psi\right\rangle$, that is, $\left\langle\psi\right| i\rangle$ is either 0 or 1.

[5] Notice for a pure state $\left|\psi\right\rangle$, $\left\langle\psi\right| A \left|\psi\right\rangle = \left\langle A\right\rangle_{\left|\psi\right\rangle}$, the expected value of the observable when the system is in state $\left|\psi\right\rangle$ equals $tr\left(A\left|\psi\right\rangle \left\langle\psi\right|\right)$.

Then, applying the first result followed by the third result in reverse and repeating based on the second result we have

$$\begin{aligned}
\langle M\rangle &= \sum_{k=1}^{n} \Pr(\lambda_k)\,\lambda_k \\
&= \sum_{k=1}^{n}\sum_{i=1}^{n} p_i \langle k|\,\psi_i\rangle \langle \psi_i|\,k\rangle \lambda_k \\
&= \sum_{k=1}^{n} \langle k|\,\rho\,|k\rangle\,\lambda_k \\
&= tr\,(M\rho) \\
&= \sum_{k=1}^{n}\sum_{i=1}^{n} p_i \langle \psi_i|\,k\rangle \langle k|\,\psi_i\rangle \lambda_k \\
&= \sum_{i=1}^{n} p_i \langle \psi_i|\,M\,|\psi_i\rangle \\
&= tr\,(\rho M)
\end{aligned}$$

Consider an example. Suppose

$$\rho = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.6 \end{bmatrix} = 0.4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} + 0.6 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix}$$

and

$$M = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} = 3 \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} - 1 \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Then,

$$\begin{aligned}
\Pr(\lambda = 3) &= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0.4 & 0 \\ 0 & 0.6 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \\
&= 0.4 \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \\
&\quad +0.6 \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \\
&= \frac{1}{2}
\end{aligned}$$

and

$$\begin{aligned}
\Pr(\lambda = -1) &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0.4 & 0 \\ 0 & 0.6 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \\
&= 0.4 \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \\
&\quad +0.6 \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \\
&= \frac{1}{2}
\end{aligned}$$

Hence, the expected value for the observable M is

$$\begin{aligned}
\langle M \rangle &= \sum_{k=1}^{2} \Pr(\lambda_k)\,\lambda_k \\
&= \frac{1}{2}3 + \frac{1}{2}(-1) = 1 \\
&= tr(\rho M) = tr(M\rho) \\
&= \begin{bmatrix} 0.4 & -0.8 \\ -1.2 & 0.6 \end{bmatrix} = \begin{bmatrix} 0.4 & -1.2 \\ -0.8 & 0.6 \end{bmatrix} = 1
\end{aligned}$$

or

$$\begin{aligned}
\langle M \rangle &= \sum_{i,j} p_j \langle i | M | \psi \rangle \langle \psi | i \rangle \\
&= 0.4 \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&\quad +0.4 \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&\quad 0.6 \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&\quad +0.6 \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \sum_{j} p_j \langle \psi | M | \psi \rangle \\
&= 0.4 \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&\quad +0.6 \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= 0.4 + 0.6 = 1
\end{aligned}$$

I.4.1 Quantum entropy

von Neumann defines quantum entropy as

$$S = -tr\left(\rho \log \rho\right)$$

For pure states, ρ is a Hermitian matrix whose logarithm (discussed earlier) involves its spectral decomposition $\rho = Q\Lambda Q^T$ where Λ has one eigenvalue equal to unity and the remainder equal to zero. Hence, $\log \rho = Q \log \Lambda Q^T = 0$ (by convention, $0 \log 0 = 0$) so that von Neumann entropy is zero for pure states.

On the other hand, for mixed states spectral decomposition of ρ is $\sum_{j=1}^{n} \lambda_j \left|j\right\rangle \left\langle j\right|$. This involves nonzero eigenvalues so that

$$\begin{aligned} S &= -tr\left(\rho \log \rho\right) \\ &= -tr\left(\sum_{j=1}^{n} \lambda_j \left|j\right\rangle \left\langle j\right| \sum_{j=1}^{n} \log\left(\lambda_j\right) \left|j\right\rangle \left\langle j\right|\right) \\ &= -tr\left(\sum_{j=1}^{n} \lambda_j \log\left(\lambda_j\right) \left|j\right\rangle \left\langle j\right| \left|j\right\rangle \left\langle j\right|\right) \\ &= -tr\left(\sum_{j=1}^{n} \lambda_j \log\left(\lambda_j\right) \left|j\right\rangle \left\langle j\right|\right) \\ &= -\sum_{i=1}^{n} \lambda_i \log \lambda_i \end{aligned}$$

The latter result follows from equality of the sum of the eigenvalues and the trace of a matrix.

I.5 Some trigonometric identities

Most trigonometric identities follow directly from Euler's equation:

$$e^{\pm i\theta} = \cos\theta \pm i \sin\theta$$

Remark 1 $\cos^2\theta + \sin^2\theta = 1$

Proof.

$$\begin{aligned} e^{i\theta}e^{-i\theta} &= e^0 = 1 \\ \left(\cos\theta + i\sin\theta\right)\left(\cos\theta - i\sin\theta\right) &= \cos^2\theta + i\cos\theta\sin\theta - i\cos\theta\sin\theta + \sin^2\theta \\ &= \cos^2\theta + \sin^2\theta \end{aligned}$$

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Remark 2 $\frac{1-\cos\theta}{2} = \sin^2\frac{\theta}{2}$

Proof.

$$\begin{aligned}
\frac{2-\left(e^{i\theta}+e^{-i\theta}\right)}{4} &= \frac{2-2\cos\theta}{4} = \frac{1-\cos\theta}{2} \\
\frac{2-\left[\left(e^{\frac{i\theta}{2}}\right)^2+\left(e^{-\frac{i\theta}{2}}\right)^2\right]}{4} &= \frac{2-\left[2-4\sin^2\frac{\theta}{2}\right]}{4} \\
&= \sin^2\frac{\theta}{2}
\end{aligned}$$

Details related to the second line are below.

$$\begin{aligned}
\left(e^{\frac{i\theta}{2}}\right)^2 &= \left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)^2 \\
&= \cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}+2i\cos\frac{\theta}{2}\sin\frac{\theta}{2}
\end{aligned}$$

$$\begin{aligned}
\left(e^{-\frac{i\theta}{2}}\right)^2 &= \left(\cos\frac{\theta}{2}-i\sin\frac{\theta}{2}\right)^2 \\
&= \cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}-2i\cos\frac{\theta}{2}\sin\frac{\theta}{2}
\end{aligned}$$

$$\begin{aligned}
\left(e^{\frac{i\theta}{2}}\right)^2+\left(e^{-\frac{i\theta}{2}}\right)^2 &= 2\cos^2\frac{\theta}{2}-2\sin^2\frac{\theta}{2} \\
&= 2-4\sin^2\frac{\theta}{2}
\end{aligned}$$

Remark 3 $\frac{1-\cos\theta_1\cos\theta_2}{2} = \cos^2\frac{\theta_1}{2}\sin^2\frac{\theta_2}{2}+\sin^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2}$

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Proof. From the two preceding identities

$$\frac{1-\cos\theta}{2} = \sin^2\frac{\theta}{2} = 1-\cos^2\frac{\theta}{2}$$

and

$$\begin{aligned}
\cos\theta &= 1-2\sin^2\frac{\theta}{2} \\
&= 2\cos^2\frac{\theta}{2}-1
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{1-\cos\theta_1\cos\theta_2}{2} &= \frac{1-\left(1-2\sin^2\frac{\theta_1}{2}\right)\left(2\cos^2\frac{\theta_2}{2}-1\right)}{2}\\
&= \frac{1-\left(2\cos^2\frac{\theta_2}{2}-2\sin^2\frac{\theta_1}{2}2\cos^2\frac{\theta_2}{2}-1+2\sin^2\frac{\theta_1}{2}\right)}{2}\\
&= -\cos^2\frac{\theta_2}{2}+2\sin^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2}+1-\sin^2\frac{\theta_1}{2}\\
&= \cos^2\frac{\theta_2}{2}\left(2\sin^2\frac{\theta_1}{2}-1\right)+\cos^2\frac{\theta_1}{2}\\
&= \cos^2\frac{\theta_2}{2}\left(2\sin^2\frac{\theta_1}{2}-\cos^2\frac{\theta_1}{2}-\sin^2\frac{\theta_1}{2}\right)+\cos^2\frac{\theta_1}{2}\\
&= \cos^2\frac{\theta_2}{2}\left(\sin^2\frac{\theta_1}{2}-\cos^2\frac{\theta_1}{2}\right)+\cos^2\frac{\theta_1}{2}\\
&= \cos^2\frac{\theta_2}{2}\sin^2\frac{\theta_1}{2}-\cos^2\frac{\theta_2}{2}\cos^2\frac{\theta_1}{2}+\cos^2\frac{\theta_1}{2}\\
&= \sin^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2}+\cos^2\frac{\theta_1}{2}\left(1-\cos^2\frac{\theta_2}{2}\right)\\
&= \cos^2\frac{\theta_1}{2}\sin^2\frac{\theta_2}{2}+\sin^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2}
\end{aligned}$$

■

Remark 4 $\sin 2\theta = 2\cos\theta\sin\theta$

Proof.

$$\begin{aligned}
\frac{e^{i2\theta}-e^{-i2\theta}}{2i} &= \frac{\cos 2\theta + i\sin 2\theta - (\cos 2\theta - i\sin 2\theta)}{2i}\\
&= \frac{2i\sin 2\theta}{2i}\\
&= \sin 2\theta\\
\frac{\left(e^{i\theta}\right)^2-\left(e^{-i\theta}\right)^2}{2i} &= \frac{(\cos\theta+i\sin\theta)^2-(\cos\theta-i\sin\theta)^2}{2i}\\
&= \frac{\cos^2\theta+2i\cos\theta\sin\theta-\sin^2\theta-\left(\cos^2\theta-2i\cos\sin\theta-\sin^2\theta\right)}{2i}\\
&= \frac{4i\cos\theta\sin\theta}{2i}\\
&= 2\cos\theta\sin\theta
\end{aligned}$$

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