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Appendix C
Multivariate normal theory

The Gaussian or normal probability distribution is ubiquitous when the data have continuous support as seen in the Central Limit theorems and the resilience to transformation of Gaussian random variables. When confronted with a vector of variables (multivariate), it often is sensible to think of a joint normal probability assignment to describe their stochastic properties. Let $W = \begin{bmatrix} X \\ Z \end{bmatrix}$ be an m-element vector (X has m_1 elements and Z has m_2 variables such that $m_1 + m_2 = m$) with joint normal probability, then the density function is

$$f_W(w) = \frac{1}{(2\pi)^{m/2} |\Sigma|^{1/2}} \exp\left[-\frac{1}{2} (w - \mu)^T \Sigma^{-1} (w - \mu) \right]$$

where

$$\mu = \begin{bmatrix} \mu_X \\ \mu_Z \end{bmatrix}$$

is the m-element vector of means for W and

$$\Sigma = \begin{bmatrix} \Sigma_{XX} & \Sigma_{XZ} \\ \Sigma_{ZX} & \Sigma_{ZZ} \end{bmatrix}$$

is the $m \times m$ variance-covariance matrix for W with m linearly independent rows and columns.

Of course, the density integrates to unity, $\int_X \int_Z f_W(w)\, dz dx = 1$. And, the marginal densities are found by integrating out the other variables, for example, $f_X(x) = \int_Z f_W(w)\, dz$ and $f_Z(z) = \int_X f_W(w)\, dx$.

Importantly, as it unifies linear regression, the conditional distributions are also Gaussian.

$$\begin{aligned} f_X(x \mid Z = z) &= \frac{f_W(w)}{f_Z(z)} \\ &\sim N(E[X \mid Z = z], Var[X \mid Z]) \end{aligned}$$

where

$$E[X \mid Z = z] = \mu_X + \Sigma_{XZ}\Sigma_{ZZ}^{-1}(z - \mu_Z)$$

and

$$Var[X \mid Z] = \Sigma_{XX} - \Sigma_{XZ}\Sigma_{ZZ}^{-1}\Sigma_{ZX}$$

Also,

$$f_Z(z \mid X = x) \sim N(E[Z \mid X = x], Var[Z \mid X])$$

where

$$E[Z \mid X = x] = \mu_Z + \Sigma_{ZX}\Sigma_{XX}^{-1}(x - \mu_X)$$

and

$$Var[Z \mid X] = \Sigma_{ZZ} - \Sigma_{ZX}\Sigma_{XX}^{-1}\Sigma_{XZ}$$

From the conditional expectation (or regression) function we see when the data are Gaussian, linearity imposes no restriction.

$$E[Z \mid X = x] = \mu_Z + \Sigma_{ZX}\Sigma_{XX}^{-1}(x - \mu_X)$$

is often written

$$\begin{aligned} E[Z \mid X = x] &= \left\{\mu_Z - \Sigma_{ZX}\Sigma_{XX}^{-1}\mu_X\right\} + \left\{\Sigma_{ZX}\Sigma_{XX}^{-1}x\right\} \\ &= \alpha + \beta^T x \end{aligned}$$

where α corresponds to an intercept (or vector of intercepts) and $\beta^T x$ corresponds to weighted regressors. Applied linear regression estimates the sample analogs to the above parameters, α and β.

C.1 Conditional distribution

Next, we develop more carefully the result for $f_X(x \mid Z = z)$; the result for $f_Z(z \mid X = x)$ follows in analogous fashion.

$$\begin{aligned}
f_X(x \mid Z) &= \frac{f_W(w)}{f_Z(z)} \\
&= \frac{\frac{1}{(2\pi)^{m/2}|\Sigma|^{1/2}} \exp\left[-\frac{1}{2}(w-\mu)^T \Sigma^{-1}(w-\mu)\right]}{\frac{1}{(2\pi)^{m_2/2}|\Sigma_{ZZ}|^{1/2}} \exp\left[-\frac{1}{2}(z-\mu_Z)^T \Sigma_{ZZ}^{-1}(z-\mu_z)\right]} \\
&= \frac{1}{(2\pi)^{m_1/2}|Var[X \mid Z]|^{1/2}} \\
&\quad \times \exp\left[-\frac{1}{2}(x - E[X \mid Z])^T Var[X \mid Z]^{-1}(x - E[X \mid Z])\right]
\end{aligned}$$

The normalizing constants are identified almost immediately since

$$\frac{(2\pi)^{m/2}}{(2\pi)^{m_2/2}} = \frac{(2\pi)^{(m_1+m_2)/2}}{(2\pi)^{m_2/2}} = (2\pi)^{m_1/2}$$

for the leading term and by theorem 1 in section A.6.2 we have

$$|\Sigma| = |\Sigma_{ZZ}| \left|\Sigma_{XX} - \Sigma_{XZ}\Sigma_{ZZ}^{-1}\Sigma_{ZX}\right|$$

since Σ_{XX}, Σ_{ZZ}, and Σ are positive definite, their determinants are positive and their square roots are real. Hence,

$$\begin{aligned}
\frac{|\Sigma|^{\frac{1}{2}}}{|\Sigma_{ZZ}|^{\frac{1}{2}}} &= \frac{|\Sigma_{ZZ}|^{\frac{1}{2}} \left|\Sigma_{XX} - \Sigma_{XZ}\Sigma_{ZZ}^{-1}\Sigma_{ZX}\right|^{\frac{1}{2}}}{|\Sigma_{ZZ}|^{\frac{1}{2}}} \\
&= \left|\Sigma_{XX} - \Sigma_{XZ}\Sigma_{ZZ}^{-1}\Sigma_{ZX}\right|^{\frac{1}{2}} \\
&= |Var[X \mid Z]|^{\frac{1}{2}}
\end{aligned}$$

This leaves the exponential terms

$$\begin{aligned}
&\frac{\exp\left[-\frac{1}{2}(w-\mu)^T \Sigma^{-1}(w-\mu)\right]}{\exp\left[-\frac{1}{2}(z-\mu_Z)^T \Sigma_{ZZ}^{-1}(z-\mu_z)\right]} \\
= \; &\exp\left[-\frac{1}{2}(w-\mu)^T \Sigma^{-1}(w-\mu) + \frac{1}{2}(z-\mu_Z)^T \Sigma_{ZZ}^{-1}(z-\mu_z)\right]
\end{aligned}$$

which require a bit more foundation. We begin with a lemma for the inverse of a partitioned matrix.

Lemma 1 *Let a symmetric, positive definite matrix H be partitioned as* $\begin{bmatrix} A & B^T \\ B & C \end{bmatrix}$ *where A and C are square $n_1 \times n_1$ and $n_2 \times n_2$ positive definite matrices (their inverses exist). Then,*

$$\begin{aligned} H^{-1} &= \begin{bmatrix} \left(A - B^T C^{-1} B\right)^{-1} & -A^{-1}B^T\left(C - BA^{-1}B^T\right)^{-1} \\ -C^{-1}B\left(A - B^T C^{-1} B\right)^{-1} & \left(C - BA^{-1}B^T\right)^{-1} \end{bmatrix} \\ &= \begin{bmatrix} \left(A - B^T C^{-1} B\right)^{-1} & -\left(A - B^T C^{-1} B\right)^{-1} B^T C^{-1} \\ -C^{-1}B\left(A - B^T C^{-1} B\right)^{-1} & \left(C - BA^{-1}B^T\right)^{-1} \end{bmatrix} \\ &= \begin{bmatrix} \left(A - B^T C^{-1} B\right)^{-1} & -A^{-1}B^T\left(C - BA^{-1}B^T\right)^{-1} \\ -\left(C - BA^{-1}B^T\right)^{-1} BA^{-1} & \left(C - BA^{-1}B^T\right)^{-1} \end{bmatrix} \end{aligned}$$

Proof. H is symmetric and the inverse of a symmetric matrix is also symmetric. Hence, the second and third lines follow from symmetry and the first line. Since H is symmetric, positive definite,

$$\begin{aligned} H &= LDL^T \\ &= \begin{bmatrix} I & 0 \\ BA^{-1} & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & C - BA^{-1}B^T \end{bmatrix} \begin{bmatrix} I & A^{-1}B^T \\ 0 & I \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} H^{-1} &= \left(L^T\right)^{-1} D^{-1} L^{-1} \\ &= \begin{bmatrix} I & -A^{-1}B^T \\ 0 & I \end{bmatrix} \begin{bmatrix} A^{-1} & 0 \\ 0 & \left(C - BA^{-1}B^T\right)^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -BA^{-1} & I \end{bmatrix} \end{aligned}$$

Expanding gives

$$H^{-1} = \begin{bmatrix} X & -A^{-1}B^T\left(C - BA^{-1}B^T\right)^{-1} \\ -\left(C - BA^{-1}B^T\right)^{-1} BA^{-1} & \left(C - BA^{-1}B^T\right)^{-1} \end{bmatrix}$$

where

$$X = A^{-1} + A^{-1}B^T\left(C - BA^{-1}B^T\right)^{-1} BA^{-1} = \left(A - B^T C^{-1} B\right)^{-1}$$

The latter equality follows from some linear algebra. Suppose it's true

$$\left(A - B^T C^{-1} B\right)^{-1} = A^{-1} + A^{-1}B^T\left(C - BA^{-1}B^T\right)^{-1} BA^{-1}$$

pre- and post-multiply both sides by A

$$A\left(A - B^T C^{-1} B\right)^{-1} A = A + B^T\left(C - BA^{-1}B^T\right)^{-1} B$$

post multiply both sides by $A^{-1}\left(A - B^T C^{-1} B\right)$

$$\begin{aligned} A &= \left(A - B^T C^{-1} B\right) \\ &\quad + B^T\left(C - BA^{-1}B^T\right)^{-1} BA^{-1}\left(A - B^T C^{-1} B\right) \\ 0 &= -B^T C^{-1} B + B^T\left(C - BA^{-1}B^T\right)^{-1} BA^{-1}\left(A - B^T C^{-1} B\right) \end{aligned}$$

Expanding the right hand side gives

$$\begin{aligned} 0 &= -B^T C^{-1} B + B^T \left(C - BA^{-1}B^T\right)^{-1} B \\ &\quad -B^T \left(C - BA^{-1}B^T\right)^{-1} BA^{-1}B^T C^{-1} B \end{aligned}$$

Collecting terms gives

$$0 = -B^T C^{-1} B + B^T \left(C - BA^{-1}B^T\right)^{-1} \left(I - BA^{-1}B^T C^{-1}\right) B$$

Rewrite I as CC^{-1} and substitute

$$0 = -B^T C^{-1} B + B^T \left(C - BA^{-1}B^T\right)^{-1} \left(CC^{-1} - BA^{-1}B^T C^{-1}\right) B$$

Factor

$$\begin{aligned} 0 &= -B^T C^{-1} B + B^T \left(C - BA^{-1}B^T\right)^{-1} \left(C - BA^{-1}B^T\right) C^{-1} B \\ 0 &= -B^T C^{-1} B + B^T C^{-1} B = 0 \end{aligned}$$

This completes the lemma. ■

Now, we write out the exponential terms and utilize the lemma to simplify.

$$\begin{aligned}
& \exp\left[-\frac{1}{2}(w-\mu)^T \Sigma^{-1}(w-\mu) + \frac{1}{2}(z-\mu_Z)^T \Sigma_{ZZ}^{-1}(z-\mu_z)\right] \\
= \; & \exp\left\{\begin{array}{c} -\frac{1}{2}\left[\begin{array}{cc} (x-\mu_X)^T & (z-\mu_Z)^T \end{array}\right] \\ \times \left[\begin{array}{cc} \Sigma_{XX\cdot Z}^{-1} & -\Sigma_{XX\cdot Z}^{-1}\Sigma_{XZ}\Sigma_{ZZ}^{-1} \\ -\Sigma_{ZZ}^{-1}\Sigma_{ZX}\Sigma_{XX\cdot Z}^{-1} & \Sigma_{ZZ\cdot X}^{-1} \end{array}\right]\left[\begin{array}{c} x-\mu_X \\ z-\mu_Z \end{array}\right] \\ +\frac{1}{2}(z-\mu_Z)^T \Sigma_{ZZ}^{-1}(z-\mu_z) \end{array}\right\} \\
= \; & \exp\left\{\begin{array}{c} -\frac{1}{2}\left[\begin{array}{c} (x-\mu_X)^T \Sigma_{XX\cdot Z}^{-1}(x-\mu_X) \\ -(x-\mu_X)^T \Sigma_{XX\cdot Z}^{-1}\Sigma_{XZ}\Sigma_{ZZ}^{-1}(z-\mu_Z) \\ -(z-\mu_Z)^T \Sigma_{ZZ}^{-1}\Sigma_{ZX}\Sigma_{XX\cdot Z}^{-1}(x-\mu_X) \\ +(z-\mu_Z)^T \Sigma_{ZZ\cdot X}^{-1}(z-\mu_Z) \end{array}\right] \\ +\frac{1}{2}(z-\mu_Z)^T \Sigma_{ZZ}^{-1}(z-\mu_z) \end{array}\right\} \\
= \; & \exp\left\{-\frac{1}{2}\left[\begin{array}{c} (x-\mu_X)^T \Sigma_{XX\cdot Z}^{-1}(x-\mu_X) \\ -(x-\mu_X)^T \Sigma_{XX\cdot Z}^{-1}\Sigma_{XZ}\Sigma_{ZZ}^{-1}(z-\mu_Z) \\ -(z-\mu_Z)^T \Sigma_{ZZ}^{-1}\Sigma_{ZX}\Sigma_{XX\cdot Z}^{-1}(x-\mu_X) \\ +(z-\mu_Z)^T\left(\Sigma_{ZZ\cdot X}^{-1} - \Sigma_{ZZ}^{-1}\right)(z-\mu_Z) \end{array}\right]\right\}
\end{aligned}$$

where $\Sigma_{XX\cdot Z} = \Sigma_{XX} - \Sigma_{XZ}\Sigma_{ZZ}^{-1}\Sigma_{ZX}$ and $\Sigma_{ZZ\cdot X} = \Sigma_{ZZ} - \Sigma_{ZX}\Sigma_{XX}^{-1}\Sigma_{XZ}$. From the last term, write

$$\Sigma_{ZZ\cdot X}^{-1} - \Sigma_{ZZ}^{-1} = \Sigma_{ZZ}^{-1}\Sigma_{ZX}\Sigma_{XX\cdot Z}^{-1}\Sigma_{XZ}\Sigma_{ZZ}^{-1}$$

To see this utilize the lemma for the inverse of a partitioned matrix. By symmetry

$$\Sigma_{ZZ}^{-1}\Sigma_{ZX}\Sigma_{XX\cdot Z}^{-1} = \Sigma_{ZZ\cdot X}^{-1}\Sigma_{ZX}\Sigma_{XX}^{-1}$$

Post multiply both sides by $\Sigma_{XZ}\Sigma_{ZZ}^{-1}$ and simplify

$$\begin{aligned}
\Sigma_{ZZ}^{-1}\Sigma_{ZX}\Sigma_{XX\cdot Z}^{-1}\Sigma_{XZ}\Sigma_{ZZ}^{-1} &= \Sigma_{ZZ\cdot X}^{-1}\Sigma_{ZX}\Sigma_{XX}^{-1}\Sigma_{XZ}\Sigma_{ZZ}^{-1} \\
&= \Sigma_{ZZ\cdot X}^{-1}\left(\Sigma_{ZZ} - \Sigma_{ZZ\cdot X}\right)\Sigma_{ZZ}^{-1} \\
&= \Sigma_{ZZ\cdot X}^{-1} - \Sigma_{ZZ}^{-1}
\end{aligned}$$

Now, substitute this into the exponential component

$$\exp\left\{-\frac{1}{2}\left[\begin{array}{c}
(x-\mu_X)^T\Sigma_{XX\cdot Z}^{-1}(x-\mu_X) \\
-(x-\mu_X)^T\Sigma_{XX\cdot Z}^{-1}\Sigma_{XZ}\Sigma_{ZZ}^{-1}(z-\mu_Z) \\
-(z-\mu_Z)^T\Sigma_{ZZ}^{-1}\Sigma_{ZX}\Sigma_{XX\cdot Z}^{-1}(x-\mu_X) \\
+(z-\mu_Z)^T\Sigma_{ZZ}^{-1}\Sigma_{ZX}\Sigma_{XX\cdot Z}^{-1}\Sigma_{XZ}\Sigma_{ZZ}^{-1}(z-\mu_Z)
\end{array}\right]\right\}$$

Combining the first and second terms and combine the third and fourth terms gives

$$\exp\left\{-\frac{1}{2}\left[\begin{array}{c}
(x-\mu_X)^T\Sigma_{XX\cdot Z}^{-1}\left(x-\mu_X-\Sigma_{XZ}\Sigma_{ZZ}^{-1}(z-\mu_Z)\right) \\
-(z-\mu_Z)^T\Sigma_{ZZ}^{-1}\Sigma_{ZX}\Sigma_{XX\cdot Z}^{-1}\left(x-\mu_X-\Sigma_{XZ}\Sigma_{ZZ}^{-1}(z-\mu_Z)\right)
\end{array}\right]\right\}$$

Then, since

$$(x-\mu_X)^T - (z-\mu_Z)^T\Sigma_{ZZ}^{-1}\Sigma_{ZX} = \left(x-\mu_X-\Sigma_{XZ}\Sigma_{ZZ}^{-1}(z-\mu_Z)\right)^T$$

combining these two terms simplifies as

$$\exp\left[-\frac{1}{2}\left(x-E\left[x\mid Z=z\right]\right)^T\Sigma_{XX\cdot Z}^{-1}\left(x-E\left[x\mid Z=z\right]\right)\right]$$

where $E\left[x \mid Z=z\right] = \mu_X - \Sigma_{XZ}\Sigma_{ZZ}^{-1}(z-\mu_Z)$. Therefore, the result matches the claim for $f_X(x \mid Z=z)$, the conditional distribution of X given $Z=z$ is normally distributed with mean $E\left[x \mid Z=z\right]$ and variance $Var\left[x \mid Z\right]$.

C.2 Special case of precision

Now, we consider a special case of Bayesian normal updating expressed in terms of precision of variables (inverse variance) along with variance representation above. Suppose a variable of interest x is observed with error

$$Y = x + \varepsilon$$

where

$$x \sim N\left(\mu_x, \sigma_x^2 = \frac{1}{\tau_x}\right)$$

and

$$\varepsilon \sim N\left(0, \sigma_\varepsilon^2 = \frac{1}{\tau_\varepsilon}\right)$$

ε independent of x, σ_j^2 refers to variance, and τ_j refers to precision of variable j. This implies

$$\begin{aligned} Var\begin{bmatrix} Y \\ x \end{bmatrix} &= \begin{bmatrix} E\left[(Y-\mu_x)^2\right] & E\left[(Y-\mu_x)(x-\mu_x)\right] \\ E\left[(x-\mu_x)(Y-\mu_x)\right] & E\left[(x-\mu_x)^2\right] \end{bmatrix} \\ &= \begin{bmatrix} \sigma_x^2+\sigma_\varepsilon^2 & \sigma_x^2 \\ \sigma_x^2 & \sigma_x^2 \end{bmatrix}. \end{aligned}$$

Then, the posterior or updated distribution for x given $Y = y$ is normal.

$$(x \mid Y = y) \sim N\left(E\left[x \mid Y = y\right], Var\left[x \mid Y\right]\right)$$

where

$$\begin{aligned} E\left[x \mid Y = y\right] &= \mu_x + \frac{\sigma_x^2}{\sigma_x^2+\sigma_\varepsilon^2}\left(y-\mu_x\right) \\ &= \frac{\sigma_\varepsilon^2\mu_x + \sigma_x^2 y}{\sigma_x^2+\sigma_\varepsilon^2} \\ &= \frac{\tau_x\mu_x + \tau_\varepsilon y}{\tau_x+\tau_\varepsilon} \end{aligned}$$

and

$$\begin{aligned} Var\left[x \mid Y\right] &= \sigma_x^2 - \frac{\left(\sigma_x^2\right)^2}{\sigma_x^2+\sigma_\varepsilon^2} \\ &= \frac{\sigma_x^2\left(\sigma_x^2+\sigma_\varepsilon^2\right) - \left(\sigma_x^2\right)^2}{\sigma_x^2+\sigma_\varepsilon^2} \\ &= \frac{\sigma_x^2\sigma_\varepsilon^2}{\sigma_x^2+\sigma_\varepsilon^2} \\ &= \frac{1}{\tau_x+\tau_\varepsilon} \end{aligned}$$

For both the conditional expectation and variance, the penultimate line expresses the quantity in terms of variance and the last line expresses the same quantity in terms of precision. The precision of x given Y is $\tau_{x|Y} = \tau_x + \tau_\varepsilon$.

C.3 Truncated normal distribution

Suppose we have a continuum of states that map one-to-one into an unbounded random variable, x, with mean μ and variance σ^2. Our natural (maximum entropy) probability assignment for x is a normal distribution with mean μ and variance σ^2. The density function for x is

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right], \quad -\infty < x < \infty$$

Suppose we have information that partitions the states, and therefore x, into two regions around t creating two truncated distributions for x. The density functions are

$$f(x \mid x < t) = \frac{1}{\sqrt{2\pi}\sigma F\left(\frac{t-\mu}{\sigma}\right)} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right], \quad -\infty < x < t$$

and

$$f(x \mid t < x) = \frac{1}{\sqrt{2\pi}\sigma\left[1-F\left(\frac{t-\mu}{\sigma}\right)\right]} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right], \quad t < x < \infty$$

where $F(\cdot)$ is the cumulative standard normal distribution. Of course, the rescaling by $F(\cdot)$ normalizes each distribution such that it integrates to one over the region of support.

Often we're interested in the expected value and, possibly, variance of the truncated outcome random variable x. First, we state the result then provide brief derivations followed by a numerical example.

Let

$$\ell(t) = -\frac{\phi\left(\frac{t-\mu}{\sigma}\right)}{F\left(\frac{t-\mu}{\sigma}\right)}, \quad -\infty < x < t$$

and

$$u(t) = \frac{\phi\left(\frac{t-\mu}{\sigma}\right)}{1-F\left(\frac{t-\mu}{\sigma}\right)}, \quad t < x < \infty$$

where $\phi(\cdot)$ is the standard normal density function with mean zero and variance one. Then,

$$\begin{aligned} E[x \mid x < t] &= \mu + \sigma\ell(t) \\ &= \mu - \sigma\frac{\phi\left(\frac{t-\mu}{\sigma}\right)}{F\left(\frac{t-\mu}{\sigma}\right)} \end{aligned}$$

and

$$\begin{aligned} E[x \mid x > t] &= \mu + \sigma u(t) \\ &= \mu + \sigma\frac{\phi\left(\frac{t-\mu}{\sigma}\right)}{1 - F\left(\frac{t-\mu}{\sigma}\right)} \end{aligned}$$

Notice, iterated expectations produces the mean of the untruncated random variable.

$$\begin{aligned} E_t\left[E\left[x \mid t\right]\right] &= F\left(\frac{t-\mu}{\sigma}\right) E\left[x \mid x<t\right] + \left[1-F\left(\frac{t-\mu}{\sigma}\right)\right] E\left[x \mid x>t\right] \\ &= F\left(\frac{t-\mu}{\sigma}\right)\left[\mu - \sigma\frac{\phi\left(\frac{t-\mu}{\sigma}\right)}{F\left(\frac{t-\mu}{\sigma}\right)}\right] \\ &\quad + \left[1-F\left(\frac{t-\mu}{\sigma}\right)\right]\left[\mu + \sigma\frac{\phi\left(\frac{t-\mu}{\sigma}\right)}{1-F\left(\frac{t-\mu}{\sigma}\right)}\right] \\ &= \mu \end{aligned}$$

Variances for the truncated distributions are

$$\begin{aligned} Var\left[x \mid x<t\right] &= \sigma^2\left[1-\ell\left(t\right)\left(\ell\left(t\right)-\frac{t-\mu}{\sigma}\right)\right] \\ &= \sigma^2\left[1+\frac{\phi\left(\frac{t-\mu}{\sigma}\right)}{F\left(\frac{t-\mu}{\sigma}\right)}\left(-\frac{\phi\left(\frac{t-\mu}{\sigma}\right)}{F\left(\frac{t-\mu}{\sigma}\right)}-\frac{t-\mu}{\sigma}\right)\right] \end{aligned}$$

and

$$\begin{aligned} Var\left[x \mid x>t\right] &= \sigma^2\left[1-u\left(t\right)\left(u\left(t\right)-\frac{t-\mu}{\sigma}\right)\right] \\ &= \sigma^2\left[1-\frac{\phi\left(\frac{t-\mu}{\sigma}\right)}{1-F\left(\frac{t-\mu}{\sigma}\right)}\left(\frac{\phi\left(\frac{t-\mu}{\sigma}\right)}{1-F\left(\frac{t-\mu}{\sigma}\right)}-\frac{t-\mu}{\sigma}\right)\right] \end{aligned}$$

To derive these results it's convenient to transform variables. Let $z = \frac{x-\mu}{\sigma}$, or $x = \sigma z + \mu$ so that $dx = \sigma dz$ and $f\left(x\right) dx = \sigma f\left(z\right) dz \equiv \phi\left(z\right) dz = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{z^2}{2}\right] dz$.

$$E\left[x \mid x<t\right] = \frac{1}{F\left(\frac{t-\mu}{\sigma}\right)}\int_{-\infty}^{t} x f\left(x\right) dx$$

Now, transform from x to z and utilize $\int z \exp\left[-\frac{z^2}{2}\right] dz = -\exp\left[-\frac{z^2}{2}\right]$.

$$\begin{aligned} E\left[x \mid x<t\right] &= \frac{1}{F\left(\frac{t-\mu}{\sigma}\right)}\int_{-\infty}^{\frac{t-\mu}{\sigma}}\left(\sigma z+\mu\right)\phi\left(z\right) dz \\ &= \frac{1}{F\left(\frac{t-\mu}{\sigma}\right)}\left\{\mu\int_{-\infty}^{\frac{t-\mu}{\sigma}}\phi\left(z\right) dz + \sigma\int_{-\infty}^{\frac{t-\mu}{\sigma}} z\phi\left(z\right) dz\right\} \\ &= \frac{F\left(\frac{t-\mu}{\sigma}\right)}{F\left(\frac{t-\mu}{\sigma}\right)}\mu - \frac{\sigma\phi\left(z\right)\mid_{-\infty}^{\frac{t-\mu}{\sigma}}}{F\left(\frac{t-\mu}{\sigma}\right)} \\ &= \mu - \sigma\frac{\phi\left(\frac{t-\mu}{\sigma}\right)}{F\left(\frac{t-\mu}{\sigma}\right)} \\ &= \mu - \sigma\ell\left(t\right) \end{aligned}$$

Similarly, the upper support expectation is

$$\begin{aligned}
E\left[x \mid x > t\right] &= \frac{1}{1-F\left(\frac{t-\mu}{\sigma}\right)} \int_{\frac{t-\mu}{\sigma}}^{\infty} \left(\sigma z + \mu\right) \phi\left(z\right) dz \\
&= \frac{1}{1-F\left(\frac{t-\mu}{\sigma}\right)} \left\{ \mu \int_{\frac{t-\mu}{\sigma}}^{\infty} \phi\left(z\right) dz + \sigma \int_{\frac{t-\mu}{\sigma}}^{\infty} z\phi\left(z\right) dz \right\} \\
&= \frac{1-F\left(\frac{t-\mu}{\sigma}\right)}{1-F\left(\frac{t-\mu}{\sigma}\right)} \mu - \frac{\sigma\phi\left(z\right) \mid_{\frac{t-\mu}{\sigma}}^{\infty}}{1-F\left(\frac{t-\mu}{\sigma}\right)} \\
&= \mu + \sigma \frac{\phi\left(\frac{t-\mu}{\sigma}\right)}{1-F\left(\frac{t-\mu}{\sigma}\right)} \\
&= \mu + \sigma u\left(t\right)
\end{aligned}$$

Variances of the truncated distributions involve

$$Var\left[x \mid x < t\right] = \int_{-\infty}^{t} x^2 f\left(x\right) dx - E\left[x \mid x < t\right]^2$$

and

$$Var\left[x \mid x > t\right] = \int_{t}^{\infty} x^2 f\left(x\right) dx - E\left[x \mid x > t\right]^2$$

As we have expressions for the truncated means, we focus on the second moments and then combine the results.

$$\begin{aligned}
E\left[x^2 \mid x < t\right] &= \frac{1}{F\left(\frac{t-\mu}{\sigma}\right)} \int_{-\infty}^{t} x^2 f\left(x\right) dx \\
&= \frac{1}{F\left(\frac{t-\mu}{\sigma}\right)} \int_{-\infty}^{\frac{t-\mu}{\sigma}} \left(\sigma z + \mu\right)^2 \phi\left(z\right) dz \\
&= \frac{1}{F\left(\frac{t-\mu}{\sigma}\right)} \int_{-\infty}^{\frac{t-\mu}{\sigma}} \left[\sigma^2 z^2 + 2\sigma\mu z + \mu^2\right] \phi\left(z\right) dz \\
&= \mu^2 - 2\sigma\mu \frac{\phi\left(\frac{t-\mu}{\sigma}\right)}{F\left(\frac{t-\mu}{\sigma}\right)} + \frac{\sigma^2}{F\left(\frac{t-\mu}{\sigma}\right)} \int_{-\infty}^{\frac{t-\mu}{\sigma}} z^2 \phi\left(z\right) dz
\end{aligned}$$

Focusing on the last term, integration by parts produces

$$\begin{aligned}
\int_{-\infty}^{\frac{t-\mu}{\sigma}} z^2 \phi\left(z\right) dz &= \int_{-\infty}^{\frac{t-\mu}{\sigma}} z\left[z\phi\left(z\right)\right] dz \\
&= -z\phi\left(z\right) \mid_{-\infty}^{\frac{t-\mu}{\sigma}} - \int_{-\infty}^{\frac{t-\mu}{\sigma}} -\phi\left(z\right) dz \\
&= -\frac{t-\mu}{\sigma} \phi\left(\frac{t-\mu}{\sigma}\right) + F\left(\frac{t-\mu}{\sigma}\right)
\end{aligned}$$

Hence,

$$\begin{aligned}
E\left[x^2 \mid x < t\right] &= \mu^2 - 2\sigma\mu \frac{\phi\left(\frac{t-\mu}{\sigma}\right)}{F\left(\frac{t-\mu}{\sigma}\right)} \\
&\quad + \frac{\sigma^2}{F\left(\frac{t-\mu}{\sigma}\right)} \left[F\left(\frac{t-\mu}{\sigma}\right) - \frac{t-\mu}{\sigma}\phi\left(\frac{t-\mu}{\sigma}\right)\right] \\
&= \mu^2 + \sigma^2 - \sigma^2 \frac{\phi\left(\frac{t-\mu}{\sigma}\right)}{F\left(\frac{t-\mu}{\sigma}\right)} \frac{t+\mu}{\sigma} \\
&= \mu^2 + \sigma^2 + \sigma^2 \ell(t) \frac{t+\mu}{\sigma}
\end{aligned}$$

and

$$\begin{aligned}
Var\left[x \mid x < t\right] &= \mu^2 + \sigma^2 + \sigma^2 \ell(t) \frac{t+\mu}{\sigma} - \left(\mu + \sigma\ell(t)\right)^2 \\
&= \mu^2 + \sigma^2 + \sigma^2 \ell(t) \frac{t+\mu}{\sigma} \\
&\quad - \left(\mu^2 + 2\mu\sigma\ell(t) + \sigma^2\ell(t)^2\right) \\
&= \sigma^2 + \sigma^2\ell(t)\frac{t+\mu}{\sigma} - \left(2\mu\sigma\ell(t) + \sigma^2\ell(t)^2\right) \\
&= \sigma^2 \left\{1 - \ell(t)\left[\ell(t) - \frac{t-\mu}{\sigma}\right]\right\}
\end{aligned}$$

Variance for upper support is analogous.

$$\begin{aligned}
E\left[x^2 \mid x > t\right] &= \frac{1}{1 - F\left(\frac{t-\mu}{\sigma}\right)} \int_t^\infty x^2 f(x)\, dx \\
&= \frac{1}{1 - F\left(\frac{t-\mu}{\sigma}\right)} \int_{\frac{t-\mu}{\sigma}}^\infty (\sigma z + \mu)^2 \phi(z)\, dz \\
&= \frac{1}{1 - F\left(\frac{t-\mu}{\sigma}\right)} \int_{\frac{t-\mu}{\sigma}}^\infty \left[\sigma^2 z^2 + 2\sigma\mu z + \mu^2\right] \phi(z)\, dz \\
&= \mu^2 + 2\sigma\mu \frac{\phi\left(\frac{t-\mu}{\sigma}\right)}{1 - F\left(\frac{t-\mu}{\sigma}\right)} + \frac{\sigma^2}{1 - F\left(\frac{t-\mu}{\sigma}\right)} \int_{\frac{t-\mu}{\sigma}}^\infty z^2 \phi(z)\, dz
\end{aligned}$$

Focusing on the last term, integration by parts produces

$$\begin{aligned}
\int_{\frac{t-\mu}{\sigma}}^\infty z^2 \phi(z)\, dz &= \int_{\frac{t-\mu}{\sigma}}^\infty z\left[z\phi(z)\right] dz \\
&= -z\phi(z) \mid_{\frac{t-\mu}{\sigma}}^\infty - \int_{\frac{t-\mu}{\sigma}}^\infty -\phi(z)\, dz \\
&= \frac{t-\mu}{\sigma}\phi\left(\frac{t-\mu}{\sigma}\right) + \left[1 - F\left(\frac{t-\mu}{\sigma}\right)\right]
\end{aligned}$$

Hence,

$$
\begin{aligned}
E\left[x^2 \mid x > t\right] &= \mu^2 + 2\sigma\mu \frac{\phi\left(\frac{t-\mu}{\sigma}\right)}{1 - F\left(\frac{t-\mu}{\sigma}\right)} \\
&\quad + \frac{\sigma^2}{1 - F\left(\frac{t-\mu}{\sigma}\right)} \left\{ \frac{t-\mu}{\sigma} \phi\left(\frac{t-\mu}{\sigma}\right) + \left[1 - F\left(\frac{t-\mu}{\sigma}\right)\right] \right\} \\
&= \mu^2 + \sigma^2 + \sigma^2 \frac{\phi\left(\frac{t-\mu}{\sigma}\right)}{1 - F\left(\frac{t-\mu}{\sigma}\right)} \frac{t+\mu}{\sigma} \\
&= \mu^2 + \sigma^2 + \sigma^2 u(t) \frac{t+\mu}{\sigma}
\end{aligned}
$$

and

$$
\begin{aligned}
Var\left[x \mid x > t\right] &= \mu^2 + \sigma^2 + \sigma^2 u(t) \frac{t+\mu}{\sigma} - \left(\mu + \sigma u(t)\right)^2 \\
&= \mu^2 + \sigma^2 + \sigma^2 u(t) \frac{t+\mu}{\sigma} \\
&\quad - \left(\mu^2 + 2\mu\sigma u(t) + \sigma^2 u(t)^2\right) \\
&= \sigma^2 + \sigma^2 u(t) \frac{t+\mu}{\sigma} - \left(2\mu\sigma u(t) + \sigma^2 u(t)^2\right) \\
&= \sigma^2 \left\{ 1 - u(t) \left[u(t) - \frac{t-\mu}{\sigma} \right] \right\}
\end{aligned}
$$

The various components are connected via variance decomposition.

$$
Var[x] = E_t\left[Var[x \mid t]\right] + Var_t\left[E[x \mid t]\right]
$$

where

$$
\begin{aligned}
E_t\left[Var[x \mid t]\right] &= F\left(\frac{t-\mu}{\sigma}\right) \sigma^2 \left\{ 1 - \ell(t) \left[\ell(t) - \frac{t-\mu}{\sigma} \right] \right\} \\
&\quad + \left(1 - F\left(\frac{t-\mu}{\sigma}\right)\right) \sigma^2 \left\{ 1 - u(t) \left[u(t) - \frac{t-\mu}{\sigma} \right] \right\} \\
&= \sigma^2 + \sigma^2 \left\{ \begin{array}{c} \phi\left(\frac{t-\mu}{\sigma}\right) \left[\ell(t) - \frac{t-\mu}{\sigma}\right] \\ -\phi\left(\frac{t-\mu}{\sigma}\right) \left[u(t) - \frac{t-\mu}{\sigma}\right] \end{array} \right\} \\
&= \sigma^2 + \sigma^2 \phi\left(\frac{t-\mu}{\sigma}\right) \left(\ell(t) - u(t)\right)
\end{aligned}
$$

and

$$\begin{aligned}
Var_t\left[E\left[x \mid t\right]\right] &= F\left(\frac{t-\mu}{\sigma}\right)\left[\mu + \sigma\ell\left(t\right) - \mu\right]^2 \\
&\quad + \left(1 - F\left(\frac{t-\mu}{\sigma}\right)\right)\left[\mu + \sigma u\left(t\right) - \mu\right]^2 \\
&= F\left(\frac{t-\mu}{\sigma}\right)\left[\sigma\ell\left(t\right)\right]^2 + \left(1 - F\left(\frac{t-\mu}{\sigma}\right)\right)\left[\sigma u\left(t\right)\right]^2 \\
&= \sigma^2\left\{F\left(\frac{t-\mu}{\sigma}\right)\ell\left(t\right)^2 + \left(1 - F\left(\frac{t-\mu}{\sigma}\right)\right)u\left(t\right)^2\right\} \\
&= \sigma^2\phi\left(\frac{t-\mu}{\sigma}\right)\left\{-\ell\left(t\right) + u\left(t\right)\right\}
\end{aligned}$$

Then,

$$\begin{aligned}
Var\left[x\right] &= E_t\left[Var\left[x \mid t\right]\right] + Var_t\left[E\left[x \mid t\right]\right] \\
&= \sigma^2 + \sigma^2\phi\left(\frac{t-\mu}{\sigma}\right)\left(\ell\left(t\right) - u\left(t\right)\right) \\
&\quad + \sigma^2\phi\left(\frac{t-\mu}{\sigma}\right)\left\{-\ell\left(t\right) + u\left(t\right)\right\} \\
&= \sigma^2
\end{aligned}$$

Example 13 *Suppose $x \sim N\left(\mu = 10, \sigma = 2\right)$ and the distribution is truncated at $t = 5$. The density function at lower support is*

$$f\left(x \mid x < 5\right) = \tfrac{1}{2\sqrt{2\pi}(0.00621)}\exp\left[-\tfrac{(x-10)^2}{8}\right], \quad -\infty < x < 5$$

and at upper support is

$$f\left(x \mid x > 5\right) = \tfrac{1}{2\sqrt{2\pi}(0.99379)}\exp\left[-\tfrac{(x-10)^2}{8}\right], \quad 5 < x < \infty$$

Means of the truncated random variable are

$$\begin{aligned}
E\left[x \mid x < 5\right] &= \mu + \sigma\ell\left(t\right) \\
&= 10 - 2\frac{\phi\left(\frac{5-10}{2}\right)}{F\left(\frac{5-10}{2}\right)} \\
&= 4.35451
\end{aligned}$$

and

$$\begin{aligned}
E\left[x \mid x > 5\right] &= \mu + \sigma u\left(t\right) \\
&= 10 + 2\frac{\phi\left(\frac{5-10}{2}\right)}{1 - F\left(\frac{5-10}{2}\right)} \\
&= 10.03528
\end{aligned}$$

Iterated expectations provides a consistency check.

$$\begin{aligned} E_t\left[E\left[x \mid t\right]\right] &= F\left(\frac{5-10}{2}\right) E\left[x \mid x<5\right] + \left[1 - F\left(\frac{5-10}{2}\right)\right] E\left[x \mid x>5\right] \\ &= (0.00621)\, 4.35451 + (0.99379)\, 10.03528 \\ &= 10 \end{aligned}$$

Variances of the truncated random variable are

$$\begin{aligned} Var\left[x \mid x<5\right] &= \sigma^2 \left\{1 - \ell(t)\left[\ell(t) - \frac{t-\mu}{\sigma}\right]\right\} \\ &= 4\left\{1 + \frac{\phi\left(\frac{5-10}{2}\right)}{F\left(\frac{5-10}{2}\right)}\left[-\frac{\phi\left(\frac{5-10}{2}\right)}{F\left(\frac{5-10}{2}\right)} - \frac{5-10}{2}\right]\right\} \\ &= 0.3558952 \end{aligned}$$

and

$$\begin{aligned} Var\left[x \mid x>5\right] &= \sigma^2 \left\{1 - u(t)\left[u(t) - \frac{t-\mu}{\sigma}\right]\right\} \\ &= 4\left\{1 - \frac{\phi\left(\frac{5-10}{2}\right)}{1-F\left(\frac{5-10}{2}\right)}\left[\frac{\phi\left(\frac{5-10}{2}\right)}{1-F\left(\frac{5-10}{2}\right)} - \frac{5-10}{2}\right]\right\} \\ &= 3.822377 \end{aligned}$$

Variance decomposition provides a consistency check.

$$Var\left[x\right] = E_t\left[Var\left[x \mid t\right]\right] + Var_t\left[E\left[x \mid t\right]\right]$$

$$\begin{aligned} E_t\left[Var\left[x \mid t\right]\right] &= F\left(\frac{t-\mu}{\sigma}\right)\sigma^2\left\{1 - \ell(t)\left[\ell(t) - \frac{t-\mu}{\sigma}\right]\right\} \\ &\quad + \left(1 - F\left(\frac{t-\mu}{\sigma}\right)\right)\sigma^2\left\{1 - u(t)\left[u(t) - \frac{t-\mu}{\sigma}\right]\right\} \\ &= F\left(\frac{5-10}{2}\right) 4\left\{1 + \frac{\phi\left(\frac{5-10}{2}\right)}{F\left(\frac{5-10}{2}\right)}\left[-\frac{\phi\left(\frac{5-10}{2}\right)}{F\left(\frac{5-10}{2}\right)} - \frac{5-10}{2}\right]\right\} \\ &\quad + \left(1 - F\left(\frac{5-10}{2}\right)\right) 4 \\ &\quad \times \left\{1 - \frac{\phi\left(\frac{5-10}{2}\right)}{1-F\left(\frac{5-10}{2}\right)}\left[\frac{\phi\left(\frac{5-10}{2}\right)}{1-F\left(\frac{5-10}{2}\right)} - \frac{5-10}{2}\right]\right\} \\ &= 3.800852 \end{aligned}$$

and

$$
\begin{aligned}
Var_t\left[E\left[x \mid t\right]\right] &= F\left(\frac{t-\mu}{\sigma}\right)\left[\sigma\ell\left(t\right)\right]^2 + \left(1-F\left(\frac{t-\mu}{\sigma}\right)\right)\left[\sigma u\left(t\right)\right]^2 \\
&= F\left(\frac{5-10}{2}\right)\left[2\left(-\frac{\phi\left(\frac{5-10}{2}\right)}{F\left(\frac{5-10}{2}\right)}\right)\right]^2 \\
&\quad + \left(1-F\left(\frac{5-10}{2}\right)\right)\left[2\frac{\phi\left(\frac{5-10}{2}\right)}{1-F\left(\frac{5-10}{2}\right)}\right]^2 \\
&= 0.199148
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
Var\left[x\right] &= E_t\left[Var\left[x \mid t\right]\right] + Var_t\left[E\left[x \mid t\right]\right] \\
&= 3.800852 + 0.199148 \\
&= 4
\end{aligned}
$$