Class Notes -- Best Depreciation Schedules:

A Solution to an Allocation and a Statistical Problem

1. Introduction

> Depreciation is probably the most discussed and disputatious topic in all accounting. Sydney Davidson (1957, p. 191).

Accounting for long-lived or depreciable assets is difficult to explain as Davidson's quote suggests. Sometimes it is argued that depreciation should be the decline in value of the asset. However, secondary markets for these assets sometimes don't exist and even when they do market value is ambiguous (buying and selling prices can be quite different). Our objective in this note is to offer an explanation for why depreciation practice persists -- or more broadly, why accruals persist, why accruals are useful? It is frequently asserted that financial statements should provide information that is useful in assessing the expected (future) cash flows of a firm. In this note, depreciation is both an allocation of cost (of services of the asset) and can be a basis for efficiently assessing expected cash outflows related to investment.

The note is organized as follows. First, the idea of steady state accounting is illustrated via a classic taxicab example that involves certain periodic investments. Second, steady state is revisited in an uncertain world. Third, the idea of best depreciation is explored in the context of a stationary (constant mean) investment population. Best depreciation is defined to be the minimum variance unbiased estimate of the unknown periodic investment population mean; straight-line depreciation is best in this stationary setting. Fourth, best depreciation is re-examined when the population means change (stochastically; that is, the means are subject to random shocks) through time. Best depreciation is accelerated in the nonstationary setting. Fifth, a simple investment problem illustrates why investors prefer minimum variance statistics -- they minimize the expected opportunity costs associated with making potentially incorrect decisions.

2. The Question

The certainty case

In some circumstances annual depreciation equals in value the annual investment. The simplest illustration of this is provided in Hatfield (1971, pp. 140-141). A cab driver purchases a cab at the beginning of every year for $1000. Each cab lasts four years. At the end of four years, the first cab will, by hypothesis, be worn out, and new cabs continue to be purchased. The same circumstances will exist at the end of the fifth year and at the end of each following year. This is referred to as steady state: starting year four, one cab is fully depreciated and one new cab is purchased every year and the plant consists of four cabs. In steady state the annual investment is $1000 and the annual depreciation using the straight-line method is also $1000 ($250 for each of the four cabs). In fact, annual investment equals annual depreciation under any depreciation method as long as each cab is fully depreciated over four years.

time	periodic investment	periodic depreciation charge	book value of depreciable asset
1	$1,000	$ 250	$ 750
2	$1,000	$ 500	$1,250
3	$1,000	$ 750	$1,500
4*	$1,000	$1,000	$1,500
5*	$1,000	$1,000	$1,500
⋮*	$1,000	$1,000	$1,500

* denotes steady state

Denote by I_t and Dep_t the investment and depreciation amounts in period t. Each investment is assumed to have a useful life of T periods. Steady state is both implied by and implies the following condition.

Steady state condition: $Dep_t = I_t$ for all $t \geq T$.

Uncertainty case

If cash outflows are uncertain or stochastic (subject to random shocks), then an analogous definition of steady state can be written by applying an expectations operator to the above condition. Say, the investment in period t, I_t, is drawn from a population of investment opportunities that is randomly distributed with an *unknown* mean $\bar{I}_t$. I_t can be expressed as: $I_t = \bar{I}_t + e_t$, where e_t is randomly distributed with mean 0 and variance σ_e^2. The steady state condition can now be written as follows.

Stochastic steady state condition: $E(Dep_t) = \bar{I}_t$ for all $t \geq T$.

In the nonstochastic case the book value of the asset stays the same period to period. In the stochastic case, the same is true of the expected book value. This is because the expected depreciation in each period is the same as expected investment. In other words, the steady state condition is equivalent to depreciation being an unbiased estimate of the investment mean. Suppose the example above involves uncertain investment amounts but whose mean is, say, $1,000. The stochastic steady condition is illustrated below.

time	mean of periodic investment	expected value of periodic depreciation charge	expected book value of depreciable asset
1	$1,000	$ 250	$ 750
2	$1,000	$ 500	$1,250
3	$1,000	$ 750	$1,500
4*	$1,000	$1,000	$1,500
5*	$1,000	$1,000	$1,500
⋮*	$1,000	$1,000	$1,500

* denotes steady state

Of course, the actual depreciation amount may differ considerably from $\bar{I}_t$. A measure of this dispersion is $E[(Dep_t - \bar{I}_t)^2]$. This number is the variance associated with any unbiased estimate of $\bar{I}_t$. An additional, and seemingly natural, condition to impose on depreciation is that this variance be made as small as possible.[1]

Assume each investment has 0 salvage value. Alternatively, I_t can be viewed as being net of salvage value. The firm chooses a depreciation (rate) schedule $(d_1, d_2,...,d_T)$, $d_i \geq 0$, i = 1,2…,T. Each investment, say investment I_t, is depreciated an amount $d_1 I_t$ in its first year of existence, $d_2 I_t$ in its second year and so on till year T. The investment is fully depreciated after T periods: $\sum_{t=1}^{T} d_t = 1$. From the Tth period on, the firm is in (stochastic) steady state. The depreciation amount in period t (t ≥ T) is denoted by Dep_t.

$$Dep_t = d_1 I_t + d_2 I_{t-1} + ... + d_T I_{t-T+1}.$$

Depreciation amounts are linear transformations of investments. We term as "best" a depreciation schedule such that the resulting depreciation charge in each period is (1) an unbiased estimate of $\bar{I}_t$ and (2) among all unbiased estimates (that employ the most recent T realizations), it has minimum variance. Deriving best depreciation schedules is an exercise in Gaussian estimation -- depreciation charge in each period is required to be the BLU (best linear unbiased) estimate of the period's unknown population mean of investment.

Returning to the taxicab example, assume cash outflows are stochastic, i.e., it is not certain how much a cab will cost every period. The questions we confront in this note are as follows. Given a particular stochastic cash outflow process, what is the best depreciation schedule? Does the best depreciation schedule resemble what we observe?

3. The Best Depreciation Schedule

The objective is to choose the best depreciation schedule. Recall, a best depreciation schedule results in Dep_t being an unbiased, minimum variance estimate of the unknown underlying population mean $\bar{I}_t$. We present the result via a numerical example. Assume T = 4.

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[1] The utility of this assumption is illustrated in section 4.

The stationary case

The investment population mean, $\bar{I}_t$, is not known but the cash outflow process is

$$I_t = \bar{I} + e_t.$$

Since the population mean is constant, there is only one parameter to be estimated. Denote this unknown population mean by $\bar{I}$. The BLU estimate for this parameter computed at time t using the last T (= 4) investment observations is the OLS (ordinary least squares) solution to the following problem and $\sigma_e^2 = 1$.

$$I = H_s \bar{I} + \eta, \text{ where } I = \begin{bmatrix} I_{t-3} \\ I_{t-2} \\ I_{t-1} \\ I_t \end{bmatrix}, H_S = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \text{ and } \eta = \begin{bmatrix} e_{t-3} \\ e_{t-2} \\ e_{t-1} \\ e_t \end{bmatrix}.$$

The error vector η has mean 0 and its variance-covariance matrix is the identity matrix. The OLS estimate for $\bar{I}$ is $\hat{\bar{I}} = (H_S^T H_S)^{-1} H_S^T I = (\frac{1}{4})I_t + (\frac{1}{4})I_{t-1} + (\frac{1}{4})I_{t-2} + (\frac{1}{4})I_{t-3}$. Our goal is to choose a depreciation schedule, d_1 through d_4, such that $Dep_t = \hat{\bar{I}}$. This can be accomplished only by setting $d_1 = d_2 = d_3 = d_4 = 1/4$. The best depreciation schedule is straight-line. The variance of the estimate is $(H_S^T H_S)^{-1}$. For our example this is 1/4, which is equal to the depreciation rate times σ_e^2 (1/4 x 1 = 1/4).

The nonstationary case

Again, in the nonstationary case the investment population mean, $\bar{I}_t$, is not known. The cash outflow process is similar to that above but with the addition of (potential) stochastic shifts in the mean from period-to-period.

$$I_t = \bar{I}_t + e_t$$

$$\bar{I}_{t+1} = \bar{I}_t + \varepsilon_{t+1},$$

where ε_{t+1} is randomly distributed with mean 0 and variance σ_ε^2. The error terms, e_t and $\varepsilon_{t'}$, are assumed to be independent for all t and t'.

The nonstationarity in the population means increases the potential number of parameters that can be estimated; however, for our purposes we are usually only concerned with the most recent mean. Hence, we'll pay particular attention to estimating the most recent mean (and carry the others along as well). These population means can be viewed as signals which now have to be filtered (separated) from the noise ε. The BLU estimate can be found using a Kalman filter. To see why this technique yields a BLU estimate, we recast the system in terms of a problem where OLS can be applied. The only difference is that instead of estimating only one population mean we now estimate the vector of population means $\bar{I}$.

The system of linear equations to be solved in our example (assume $\sigma_e^2 = \sigma_\varepsilon^2 = 1$) is:

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\begin{aligned}
&I_{t-3} = \bar{I}_{t-3} + e_{t-3} \\
&0 = \bar{I}_{t-2} - \bar{I}_{t-3} - \varepsilon_{t-2} \\
&I_{t-2} = \bar{I}_{t-2} + e_{t-2} \\
&0 = \bar{I}_{t-1} - \bar{I}_{t-2} - \varepsilon_{t-1} \\
&I_{t-1} = \bar{I}_{t-1} + e_{t-1} \\
&0 = \bar{I}_{t} - \bar{I}_{t-1} - \varepsilon_{t} \\
&I_{t} = \bar{I}_{t} + e_{t}
\end{aligned}
$$

In matrix form, $I = H_N \bar{I} + \eta$, where

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I = \begin{bmatrix} I_{t-3} \\ 0 \\ I_{t-2} \\ 0 \\ I_{t-1} \\ 0 \\ I_t \end{bmatrix}, H_N = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \eta = \begin{bmatrix} e_{t-3} \\ -\varepsilon_{t-2} \\ e_{t-2} \\ -\varepsilon_{t-1} \\ e_{t-1} \\ -\varepsilon_t \\ e_t \end{bmatrix}, \text{ and } \bar{I} = \begin{bmatrix} \bar{I}_{t-3} \\ \bar{I}_{t-2} \\ \bar{I}_{t-1} \\ \bar{I}_t \end{bmatrix}.
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The error vector η has mean 0 and its variance-covariance matrix is the identity matrix. The OLS estimate for $\bar{I}$ is $\hat{\bar{I}} = (H_N^T H_N)^{-1} H_N^T I$. The last element in the vector $\hat{\bar{I}}_t$ is the BLU estimate for $\bar{I}_t$. For our example, $\hat{\bar{I}}_t = (\frac{13}{21})I_t + (\frac{5}{21})I_{t-1} + (\frac{2}{21})I_{t-2} + (\frac{1}{21})I_{t-3}$. Setting $Dep_t = \hat{\bar{I}}_t$, implies $d_1 = 13/21$, $d_2 = 5/21$, $d_3 = 2/21$, $d_4 = 1/21$. The best depreciation schedule is accelerated -- depreciation rates are higher in early years. Note that the OLS weights are positive and sum to one: $\frac{13}{21} + \frac{5}{21} + \frac{2}{21} + \frac{1}{21} = 1$. The weights are valid depreciation rates.

Fibonacci depreciation rates

The accelerated depreciation schedule that arises in the example deserves comment. The numerators and denominators in the fractions that specify the depreciation schedule are Fibonacci numbers. And the same holds for any choice of T (as long as $\sigma_e^2 = \sigma_\varepsilon^2$). In fact, for a given T, the best depreciation schedule $(d_1, d_2, \ldots, d_T) = (\frac{F_{2T-1}}{F_{2T}}, \frac{F_{2T-3}}{F_{2T}}, \ldots, \frac{F_1}{F_{2T}})$, where $(F_0, F_1, F_2, F_3, F_4, \ldots) = (0,1,1,2,3,\ldots)$ is the Fibonacci sequence ($F_t = F_{t-1} + F_{t-2}$). For T = 4, $(d_1, d_2, d_3, d_4) = (\frac{F_7}{F_8}, \frac{F_5}{F_8}, \frac{F_3}{F_8}, \frac{F_1}{F_8}) = (\frac{13}{21}, \frac{5}{21}, \frac{2}{21}, \frac{1}{21})$, which is what we derived using the Kalman filter. The variance of $\hat{I}_t$ is the lower diagonal element in $(H_N^T H_N)^{-1}$. For our example the variance of the estimate is 13/21 -- again, it is d_1 times σ_e^2 (13/21 x 1 = 13/21).[2]

When the depreciation schedule is Fibonacci, another elegant number, the golden mean $g = (1+\sqrt{5})/2 \approx 1.618$, naturally arises.[3] As n becomes large, the ratio of successive Fibonacci numbers F_{n+1}/F_n approaches g. In our example, for large T, the depreciation rate in year one approaches 1/g, the depreciation rate in year two is $1/g^3$ and so on.[4]

Group depreciation

The same Dep_t charge can then be obtained by following a declining balance (group) depreciation schedule. The sum of beginning book value and the investment in the period is depreciated using a constant group depreciation rate $1/g \approx .618$. Note that declining balance depreciation results in I_t being depreciated at the rate of 1/g in its first

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2 If straight-line depreciation is used in this nonstationary example the variance of depreciation would be 9/8 which is greater than the variance associated with the best depreciation schedule. In general, the variance of straight-line depreciation in the nonstationary setting is $\frac{\sigma_e^2}{T} + \frac{1}{T^2}\sum_{k=1}^{T-1} k^2 \sigma_\varepsilon^2$.

3 The most elegant rectangles have their sides in the ratio g to 1. "It is said that some of the measurements of Greek vases, also the proportion of temples, exemplify the golden section; and one prominent psychologist even claimed to have proved that the pleasure experienced on viewing a masterpiece alleged to be constructed according to the golden section is a necessary consequence of the solid geometry of the rods and cones in the eye." (Bell, 1992, p. 115).

4 $\frac{F_{2T-3}}{F_{2T}} = \frac{F_{2T-3}}{F_{2T-2}} \frac{F_{2T-2}}{F_{2T-1}} \frac{F_{2T-1}}{F_{2T}}$ which approaches $\frac{1}{g^3}$.

period, at the rate of $(1-1/g)(1/g) = 1/g^3$ in its second period and so on.[5] Notice that reference to the assets' economic lives is not needed for group depreciation here.

In the nonstationary case, if $\sigma_e^2 \neq \sigma_\varepsilon^2$, the depreciation schedule is no longer Fibonacci; however, it continues to be accelerated. Generally stated, the best declining balance (group) rate is $\dfrac{2}{1+\sqrt{1+4\dfrac{\sigma_e^2}{\sigma_\varepsilon^2}}}$ and the variance of depreciation is equal to the rate times σ_e^2.

4. Why minimum variance?

One might question why we're interested in constructing depreciation, or more generally accruals, to yield the minimum variance estimate amongst all unbiased estimators of the mean of cash flows. Consider the valuation role of accounting.[6] The usefulness of the minimum variance estimator is demonstrated if investment decisions based exclusively on accounting information yield the minimum expected opportunity loss. That is, expected opportunity loss is minimized when the minimum variance estimate of expected free cash flows is employed to value the firm.

This goes as follows. Opportunity losses occur when either one invests (a_1) and the current price exceeds the (per share) value implied by the underlying (unobservable) mean of free cash flows. Also, opportunity losses occur when one doesn't invest (a_2) and

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5 The equality follows, since for the golden mean $\frac{1}{g} = g - 1$. The succeeding period is calculated as follows: $\left(1-\frac{1}{g}-\frac{1}{g^3}\right)\frac{1}{g} = \left(\frac{1}{g^2}-\frac{1}{g^3}\right)\frac{1}{g} = \left(\frac{g-1}{g^3}\right)\frac{1}{g} = \frac{1}{g^5}$.

6 We're going outside of the depreciation context to illustrate that these ideas may be important for understanding accruals more broadly (e.g., in this context one can think of accrual 'income' as a statistic or estimate for expected cash flows). Also, a similar argument can be made regarding the stewardship role of accounting. For instance, if the owners believe that the world is populated by more and less talented managers. Then, accounting information can help to discriminate whether a particular manager (or management team) belongs to the more talented or less talented poplutation and reward them accordingly. Frequently, accounting information that provides a low variance estimate is desirable for such performance evaluation activities.

the (per share) value implied by the underlying (unobservable) mean of free cash flows exceeds the current price.[7] Denote these opportunity losses as follows.

$$L(a_1) = P - b'$$

$$L(a_2) = b' - P$$

where P = share price for the firm's stock and $b' = \frac{b}{r}$ (b = unobserved mean of free cash flows -- assumed to be a perpetuity for simplicity, and r = constant discount rate). For simplicity, assume that b' conditioned on the estimator $\frac{\hat{b}}{r}$ is normally distributed with mean $\frac{\hat{b}}{r}$ and variance $\frac{1}{r^2}\mathrm{Var}[\hat{b}]$.

The expected value of the opportunity loss indicates the magnitude of opporunity losses from investing or not investing, on average, over a number of such investment decisions. This is the metric of interest to a risk neutral investor who wishes to increase his/her wealth. The expected opportunity loss associated with investing (a_1) is the expected loss P - b' given that value b' (unobserved) is less than observed price P.

$$EL(a_1) = E[\, P - b' \mid b' < P]$$

To solve for $EL(a_1)$ let $z = \frac{P - \hat{b}'}{\left[\mathrm{Var}\left(\hat{b}'\right)\right]^{1/2}}$ (i.e., let z be a standard normal random variable).

Following Schlaifer (1959), $EL(a_1) = P - \left\{\hat{b}' - \left[\mathrm{Var}\,(\hat{b}')\right]^{1/2} \frac{f(z)}{F(z)}\right\}$

$$= \left(P - \hat{b}'\right) + \left[\mathrm{Var}(\hat{b}')\right]^{1/2} \frac{f(z)}{F(z)}$$

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7 Standard valuation of the firm's stock is based on the present value of future dividends, or equivalently, the present value of future 'free' cash flows. If expected free cash flows are a perpetuity (a constant indefinitely into the future) and discount rates are constant, then value is equal to the perpetual amount of expected free cash flows divided by the discount rate.

where $F(\bullet)$ and $f(\bullet)$ are the cumulative distribution function and probability density function, respectively, for the standard normal distribution.

Similarly, the expected opportunity loss associated with not investing (a_2) is the expected opportunity loss from not investing (b' - P) given that value b' (unobserved) exceeds the observed price P.

$$EL(a_2) = \left(\hat{b}' - P\right) + \left[Var(\hat{b}')\right]^{1/2} \frac{f(z)}{\left[1 - F(z)\right]}$$

Since the minimum expected opportunity loss is found by selecting the smaller of $EL(a_1)$ or $EL(a_2)$ and each of these are increasing in $Var[\hat{b}']$, even a risk neutral investor prefers the minimum variance estimator. In other words, (everything else equal) investors prefer that accounting provides minimum variance statistics for expected cash flows. And, we've discovered that carefully constructed accruals can be minimum variance estimates of cash flow means.

Example

Suppose one estimates expected periodic free cash flows (the amount of the perpetuity) two different ways and each way the estimate is 1 (of course, different estimation methods will usually produce different assessments). However the variance of the first method (accelerated) is 13/21 and the variance of the second method (straight-line) is 9/8. Further, suppose the observed price is 9 and the discount rate is a constant 10% per period. Therefore, the imputed value of the stock b' based on either of these estimates is 10 and the stock is acquired (b' > P).

The **expected value** of the opportunity loss associated with the first method (accelerated) is $EL(a_1|acc, invest) = \left(P - \hat{b}'_{acc}\right) + \left[Var(\hat{b}'_{acc})\right]^{1/2} \frac{f(z_{acc})}{F(z_{acc})} \approx$ $(-1) + \sqrt{\frac{13}{21}}\left(\frac{.1779}{.1019}\right) = .3739$. Likewise, the **expected value** of the opportunity loss associated with the second method (straight-line) is $EL(a_1|SL, invest) =$ $\left(P - \hat{b}'_{SL}\right) + \left[Var(\hat{b}'_{SL})\right]^{1/2} \frac{f(z_{SL})}{F(z_{SL})} \approx (-1) + \sqrt{\frac{9}{8}}\left(\frac{.2558}{.1729}\right) = .5693$. Clearly, if one relies on one of the estimates to guide decisions then the opportunity loss associated with reliance on the estimate with smaller variance is smaller, on average.

5. Conclusion

This note views depreciation as a joint problem of allocating historical cost and of constructing a statistic that conveys information of an unknown population cost parameter. Both objectives are met: investments are depreciated fully over their useful life and the depreciation amount in each period is an unbiased, minimum variance estimate of the population mean. The depreciation schedules we observe in practice, straight-line and accelerated, are shown to be best in particular settings.

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