# Structural models

Structural models depict causal relations amongst a set of variables. For example, suppose we have a structural model of equilibrium (supply equals demand)

$$\begin{aligned} q &= \alpha_1 p + \alpha_2 x_1 + \varepsilon_1 \\ p &= \beta_1 q + \beta_2 x_2 + \varepsilon_2 \end{aligned}$$

where $q$, quantity, and $p$, price, are endogenous variables and $x_1$ and $x_2$ are exogenous influences on supply and demand, with $\varepsilon_1$ and $\varepsilon_2$ unobservable quantities unrelated to the exogenous variables. As written, the model likely suffers an omitted, correlated variable (endogeneity) problem. However, reduced form parameters can be identified and structural parameters recovered if rank and order conditions are satisfied.

## 1 Reduced form and rank condition

First, rewrite the model in reduced form by moving the endogenous components to the left-hand side.

$$\begin{bmatrix} 1 & -\alpha_1 \\ -\beta_1 & 1 \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} = \begin{bmatrix} \alpha_2 & 0 \\ 0 & \beta_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}$$

where $Var \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} = \begin{bmatrix} v_{11} & v_{12} \\ v_{12} & v_{22} \end{bmatrix}$. Provided the equations on the left are linearly independent ($\alpha_1 \beta_1 \neq 1$; the rank condition is satisfied), the reduced form is identified (typically estimated via 2SLS-GLS to address the error structure implied by simultaneity; see $Var[\delta]$ below).

$$\begin{aligned} \begin{bmatrix} q \\ p \end{bmatrix} &= \begin{bmatrix} 1 & -\alpha_1 \\ -\beta_1 & 1 \end{bmatrix}^{-1} \left\{ \begin{bmatrix} \alpha_2 & 0 \\ 0 & \beta_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} \right\} \\ \begin{bmatrix} q \\ p \end{bmatrix} &= \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} \end{aligned}$$

where

$$\begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix} = \frac{1}{1 - \alpha_1 \beta_1} \begin{bmatrix} \alpha_2 & \alpha_1 \beta_2 \\ \alpha_2 \beta_1 & \beta_2 \end{bmatrix}$$

and

$$\begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} = \frac{1}{1 - \alpha_1 \beta_1} \begin{bmatrix} \varepsilon_1 + \alpha_1 \varepsilon_2 \\ \beta_1 \varepsilon_1 + \varepsilon_2 \end{bmatrix}$$

Since there are four reduced form parameters from which to recover the four structural parameters, we can proceed as follows.

$$\begin{aligned} \frac{\pi_{21}}{\pi_{11}} &= \beta_1 \\ \frac{\pi_{12}}{\pi_{22}} &= \alpha_1 \end{aligned}$$

This identifies the denominator $(1-\alpha_1\beta_1)$ so that

$$\begin{aligned}\alpha_2 &= (1-\alpha_1\beta_1)\,\pi_{11} \\ &= \left(1-\frac{\pi_{12}}{\pi_{22}}\frac{\pi_{21}}{\pi_{11}}\right)\pi_{11}\end{aligned}$$

and

$$\begin{aligned}\beta_2 &= (1-\alpha_1\beta_1)\,\pi_{22} \\ &= \left(1-\frac{\pi_{12}}{\pi_{22}}\frac{\pi_{21}}{\pi_{11}}\right)\pi_{22}\end{aligned}$$

Also, the variance of the unobservables for the structural model can similarly be recovered from the variance of the reduced form model.

$$\begin{aligned}
& Var\begin{bmatrix}\delta_1 \\ \delta_2\end{bmatrix} \\
= & \begin{bmatrix} c_{11} & c_{12} \\ c_{12} & c_{22}\end{bmatrix} \\
= & Var\left[\frac{1}{1-\alpha_1\beta_1}\begin{bmatrix}\varepsilon_1+\alpha_1\varepsilon_2 \\ \beta_1\varepsilon_1+\varepsilon_2\end{bmatrix}\right] \\
= & \left(\frac{1}{1-\alpha_1\beta_1}\right)^2\begin{bmatrix}1 & \alpha_1 \\ \beta_1 & 1\end{bmatrix}\begin{bmatrix}v_{11} & v_{12} \\ v_{12} & v_{22}\end{bmatrix}\begin{bmatrix}1 & \alpha_1 \\ \beta_1 & 1\end{bmatrix}^T \\
= & \left(\frac{1}{1-\frac{\pi_{12}}{\pi_{22}}\frac{\pi_{21}}{\pi_{11}}}\right)^2\begin{bmatrix}1 & \frac{\pi_{12}}{\pi_{22}} \\ \frac{\pi_{21}}{\pi_{11}} & 1\end{bmatrix}\begin{bmatrix}v_{11} & v_{12} \\ v_{12} & v_{22}\end{bmatrix}\begin{bmatrix}1 & \frac{\pi_{12}}{\pi_{22}} \\ \frac{\pi_{21}}{\pi_{11}} & 1\end{bmatrix}^T \\
= & \left(\frac{1}{1-\frac{\pi_{12}}{\pi_{22}}\frac{\pi_{21}}{\pi_{11}}}\right)^2\times \\
& \begin{bmatrix} v_{11}+\left(\frac{\pi_{12}}{\pi_{22}}\right)^2 v_{22}+2\frac{\pi_{12}}{\pi_{22}}v_{12} & \frac{\pi_{21}}{\pi_{11}}v_{11}+\frac{\pi_{12}}{\pi_{22}}v_{22}+\left(1+\frac{\pi_{12}}{\pi_{22}}\frac{\pi_{21}}{\pi_{11}}\right)v_{12} \\ \frac{\pi_{21}}{\pi_{11}}v_{11}+\alpha_1 v_{22}+\left(1+\frac{\pi_{12}}{\pi_{22}}\frac{\pi_{21}}{\pi_{11}}\right)v_{12} & \left(\frac{\pi_{21}}{\pi_{11}}\right)^2 v_{11}+v_{22}+2\frac{\pi_{21}}{\pi_{11}}v_{12}\end{bmatrix}
\end{aligned}$$

The variance of the structural model can be recovered from the following linear system of equations.

$$\begin{bmatrix}c_{11} \\ c_{12} \\ c_{22}\end{bmatrix} = \left(\frac{1}{1-\frac{\pi_{12}}{\pi_{22}}\frac{\pi_{21}}{\pi_{11}}}\right)^2\begin{bmatrix}1 & 2\frac{\pi_{12}}{\pi_{22}} & \left(\frac{\pi_{12}}{\pi_{22}}\right)^2 \\ \frac{\pi_{21}}{\pi_{11}} & \left(1+\frac{\pi_{12}}{\pi_{22}}\frac{\pi_{21}}{\pi_{11}}\right) & \frac{\pi_{12}}{\pi_{22}} \\ \left(\frac{\pi_{21}}{\pi_{11}}\right)^2 & 2\frac{\pi_{21}}{\pi_{11}} & 1\end{bmatrix}\begin{bmatrix}v_{11} \\ v_{12} \\ v_{22}\end{bmatrix}$$

Solving yields

$$\begin{bmatrix}v_{11} \\ v_{12} \\ v_{22}\end{bmatrix} = \begin{bmatrix}\frac{c_{22}\pi_{12}^2-2c_{12}\pi_{12}\pi_{22}+c_{11}\pi_{22}^2}{\pi_{22}^2} \\ \frac{c_{12}(\pi_{11}\pi_{22}+\pi_{12}\pi_{21})-c_{11}\pi_{21}\pi_{22}-c_{22}\pi_{11}\pi_{12}}{\pi_{11}\pi_{22}} \\ \frac{c_{22}\pi_{11}^2-2c_{12}\pi_{11}\pi_{21}+c_{11}\pi_{21}^2}{\pi_{11}^2}\end{bmatrix}$$

## 2 Order condition

Alternatively, suppose $x_1 = x_2 = x$ (the order condition fails), we have only one exogenous variable (the same variable for each equation). Then, we have only two reduced form parameters and recovery of the four structural parameters is out of reach.

$$\begin{aligned} \begin{bmatrix} q \\ p \end{bmatrix} &= \begin{bmatrix} 1 & -\alpha_1 \\ -\beta_1 & 1 \end{bmatrix}^{-1} \left\{ \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix} x + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} \right\} \\ &= \frac{1}{1-\alpha_1\beta_1} \begin{bmatrix} \alpha_2 + \alpha_1\beta_2 \\ \alpha_2\beta_1 + \beta_2 \end{bmatrix} x + \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} \end{aligned}$$

The order condition is more subtle than it may appear. The zeroes in the matrix of structural parameters, $\begin{bmatrix} \alpha_2 & 0 \\ 0 & \beta_2 \end{bmatrix}$, are key as the order condition says we must exclude as many exogenous variables from each equation as there are endogenous variables (in the equation). For instance, suppose there are two exogenous variables but both are included in the first equation.

$$\begin{aligned} \begin{bmatrix} q \\ p \end{bmatrix} &= \begin{bmatrix} 1 & -\alpha_1 \\ -\beta_1 & 1 \end{bmatrix}^{-1} \left\{ \begin{bmatrix} \alpha_2 & \alpha_3 \\ 0 & \beta_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} \right\} \\ \begin{bmatrix} q \\ p \end{bmatrix} &= \frac{1}{1-\alpha_1\beta_1} \begin{bmatrix} \alpha_2 & \alpha_1\beta_2 + \alpha_3 \\ \alpha_2\beta_1 & \beta_2 + \alpha_3\beta_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} \end{aligned}$$

The problem is we are attempting to recover five structural parameters from four reduced form parameters. We don't have sufficient information as in the case above with only one exogenous variable.[1] The order condition is simply a counting rule.

[1] As before, we can recover $\beta_1$ from the first column of reduced form parameters but we're unable to untangle $\alpha_1$ and the remaining structural parameters.