Strategic choice model

Amemiya [1974] and Heckman [1978] suggest resolving identification problems in simultaneous probit models by making the model recursive. Bresnahan and Reiss [1990] show that this approach rules out interesting interactions in strategic choice models. Alternatively, they propose modifying the error structure to identify unique equilibria in strategic, multi-person choice models.

Statistical analysis of strategic choice extends random utility analysis by adding game structure and Nash equilibrium strategies (Bresnahan and Reiss [1990, 1991] and Berry [1992]). McKelvey and Palfrey [1995] proposed quantal response equilibrium analysis by assigning extreme value (logistic) distributed random errors to players' strategies. Strategic error by the players makes the model amenable to statistical analysis as the likelihood function does not degenerate. Signorino [2003] extends the idea to political science by replacing extreme value errors with assignment of normally distributed errors associated with analyst uncertainty and/or private information regarding the players' utility for outcomes. Since analyst error due to unobservable components is ubiquitous in business and economic data and private information problems are typical in settings where accounting plays an important role, we focus on the game setting with analyst error and private information.

A simple two player, sequential game with analyst error and private information (combined as π) is depicted in figure 1. Player A moves first

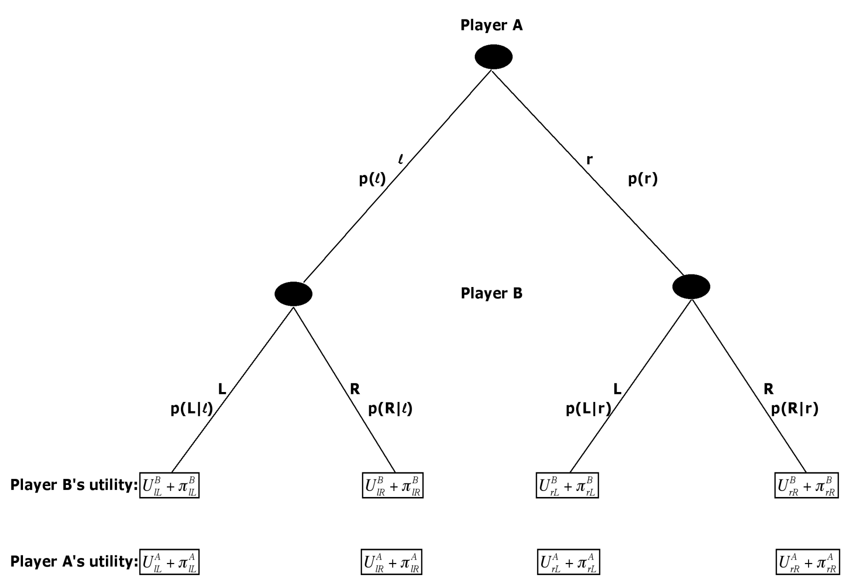

Figure 1: Strategic choice game tree

by playing either left (l) or right (r). Player B moves next but player

A's choice depends on the anticipated response of player B to player A's move. The unobservable (to the analyst) portion of each player's utility is assigned a normal distribution $\pi_i \sim N(0, \sigma^2 I)$ where

$$\pi_i^T = \left[\pi_{lLi}^A \, \pi_{lLi}^B \, \pi_{lRi}^A \, \pi_{lRi}^B \, \pi_{rLi}^A \, \pi_{rLi}^B \, \pi_{rRi}^A \, \pi_{rRi}^B\right]$$

Since choice is scale-free (see chapter 5) maximum likelihood estimation proceeds with σ^2 normalized to 1.

The log-likelihood is

$$\sum_{i=1}^{n} Y_{lLi} \log(P_{lLi}) + Y_{lRi} \log(P_{lRi}) + Y_{rLi} \log(P_{rLi}) + Y_{rRi} \log(P_{rRi})$$

where $Y_{jki} = 1$ if strategy j is played by A and k is played by B for sample i, and P_{jki} is the probability that strategy j is played by A and k is played by B for sample i. The latter requires some elaboration. Sequential play yields joint probabilities associated with the strategy pair $P_{jk} = P_{(k|j)} P_j$. Now, only the conditional and marginal probabilities remain to be identified. Player B's strategy depends on player A's observed move. If player A opts for l, then player B chooses L when $U_{lL}^B + \pi_{lL}^B - \left(U_{lR}^B + \pi_{lR}^B\right) > 0$ and R otherwise. Also, if player A opts for r, then player B chooses L when $U_{rL}^B + \pi_{rL}^B - \left(U_{rR}^B + \pi_{rR}^B\right) > 0$ and R otherwise. In the symmetric information with analyst error only game, π^B is unobserved by the analyst. On the other hand, in the private information game with analyst error, π^B is unobserved by both player A and the analyst. We continue with the private information and analyst error game (later we'll return to the symmetric information game).

1 Private information game with analyst error

First, we describe the data generating process (DGP) for the private information game. Then, we relate the DGP to the analyst's likelihood function.

1.1 DGP — players' strategies

Since player A moves first, player B's strategy is to select the maximum utility response to player A's revealed strategy. On the other hand, player A formulates a strategy based on the expected response of player B to A's strategy. Player A's strategy suffers an information disadvantage as player A does not observe π^B in the private information game. The private information game DGP is summarized below.

Let Y_{jki} be an indicator denoting the equilibrium strategy $j \in (l, r)$ for player A and the equilibrium response $k \in (L, R)$ for player B for pairing i.

$$Y_{lLi} = \begin{array}{ll} 1 \ \ if & \begin{array}{c} \left\{P_{(L|l)i}\left(U^A_{lLi}+\pi^A_{lLi}\right)+P_{(R|l)i}\left(U^A_{lRi}+\pi^A_{lRi}\right)\right\} \\ > \left\{P_{(L|r)i}\left(U^A_{rLi}+\pi^A_{rLi}\right)+P_{(R|r)i}\left(U^A_{rRi}+\pi^A_{rRi}\right)\right\} \\ \text{and } U^B_{lLi}+\pi^B_{lLi} > \left(U^B_{lRi}+\pi^B_{lRi}\right) \end{array} \\ 0 & \quad otherwise \end{array}$$

$$Y_{lRi} = \begin{array}{ll} 1 \ \ if & \begin{array}{c} \left\{P_{(L|l)i}\left(U^A_{lLi}+\pi^A_{lLi}\right)+P_{(R|l)i}\left(U^A_{lRi}+\pi^A_{lRi}\right)\right\} \\ > \left\{P_{(L|r)i}\left(U^A_{rLi}+\pi^A_{rLi}\right)+P_{(R|r)i}\left(U^A_{rRi}+\pi^A_{rRi}\right)\right\} \\ \text{and } U^B_{lLi}+\pi^B_{lLi} < \left(U^B_{lRi}+\pi^B_{lRi}\right) \end{array} \\ 0 & \quad otherwise \end{array}$$

$$Y_{rLi} = \begin{array}{ll} 1 \ \ if & \begin{array}{c} \left\{P_{(L|l)i}\left(U^A_{lLi}+\pi^A_{lLi}\right)+P_{(R|l)i}\left(U^A_{lRi}+\pi^A_{lRi}\right)\right\} \\ < \left\{P_{(L|r)i}\left(U^A_{rLi}+\pi^A_{rLi}\right)+P_{(R|r)i}\left(U^A_{rRi}+\pi^A_{rRi}\right)\right\} \\ \text{and } U^B_{rLi}+\pi^B_{rLi} > \left(U^B_{rRi}+\pi^B_{rRi}\right) \end{array} \\ 0 & \quad otherwise \end{array}$$

and

$$Y_{rRi} = \begin{array}{ll} 1 \ \ if & \begin{array}{c} \left\{P_{(L|l)i}\left(U^A_{lLi}+\pi^A_{lLi}\right)+P_{(R|l)i}\left(U^A_{lRi}+\pi^A_{lRi}\right)\right\} \\ < \left\{P_{(L|r)i}\left(U^A_{rLi}+\pi^A_{rLi}\right)+P_{(R|r)i}\left(U^A_{rRi}+\pi^A_{rRi}\right)\right\} \\ \text{and } U^B_{rLi}+\pi^B_{rLi} < \left(U^B_{rRi}+\pi^B_{rRi}\right) \end{array} \\ 0 & \quad otherwise \end{array}$$

where conditional probabilities $P_{(j|k)i}$ are defined below.

1.2 Likelihood function — analyst's strategy

In the private information game with analyst error, both player A and the analyst perceive the likelihood of player B's response to player A's move (due to unobserved π^B). When player A chooses l, then player B responds with L if $U^B_{lL}+\pi^B_{lL} > U^B_{lR}+\pi^B_{lR}$, or $U^B_{lL}-U^B_{lR} > \pi^B_{lR}-\pi^B_{lL} = \pi^B_l$ where $\pi^B_l \sim N\left(0, 2\sigma^2\right)$, and player B responds R otherwise. Similarly, when player A chooses r, then player B responds with L if $U^B_{rL}+\pi^B_{rL} > U^B_{rR}+\pi^B_{rR}$, or $U^B_{rL}-U^B_{rR} > \pi^B_{rR}-\pi^B_{rL} = \pi^B_r$ where $\pi^B_r \sim N\left(0, 2\sigma^2\right)$, and player B responds R otherwise. Hence, player A and the analyst assign the following conditional likelihoods (due to unobservable π^B)

$$\begin{aligned} P_{(L|l)} &= \Phi\left(\frac{U^B_{lL}-U^B_{lR}}{\sqrt{2\sigma^2}}\right) \\ P_{(R|l)} &= 1-P_{(L|l)} \\ P_{(R|r)} &= 1-P_{(L|r)} \\ P_{(L|r)} &= \Phi\left(\frac{U^B_{rL}-U^B_{rR}}{\sqrt{2\sigma^2}}\right) \end{aligned}$$

where $\Phi(\cdot)$ is the standard normal (cumulative) distribution function.

Player A's strategy however depends on B's response to A's move. Player A opts for strategy l if expected utility is greater than for strategy r.

$$\begin{aligned} & P_{(L|l)}\left(U_{lL}^A+\pi_{lL}^A\right)+P_{(R|l)}\left(U_{lR}^A+\pi_{lR}^A\right) \\ & >P_{(L|r)}\left(U_{rL}^A+\pi_{rL}^A\right)+P_{(R|r)}\left(U_{rR}^A+\pi_{rR}^A\right) \end{aligned}$$

Since the analyst doesn't observe π^A (or π^B), from the analyst's viewpoint we have

$$\begin{aligned} & P_{(L|l)}U_{lL}^A+P_{(R|l)}U_{lR}^A-\left(P_{(L|r)}U_{rL}^A+P_{(R|r)}U_{rR}^A\right) \\ & >-\left(P_{(L|l)}\pi_{lL}^A+P_{(R|l)}\pi_{lR}^A\right)+\left(P_{(L|r)}\pi_{rL}^A+P_{(R|r)}\pi_{rR}^A\right) \end{aligned}$$

where

$$\begin{aligned} & -\left(P_{(L|l)}\pi_{lL}^A+P_{(R|l)}\pi_{lR}^A\right)+\left(P_{(L|r)}\pi_{rL}^A+P_{(R|r)}\pi_{rR}^A\right) \\ & \sim N\left(0,\left(P_{(L|l)}^2+P_{(R|l)}^2+P_{(L|r)}^2+P_{(R|r)}^2\right)\sigma^2\right) \end{aligned}$$

Hence, the analyst assigns marginal probabilities

$$P_l=\Phi\left(\frac{P_{(L|l)}U_{lL}^A+P_{(R|l)}U_{lR}^A-\left(P_{(L|r)}U_{rL}^A+P_{(R|r)}U_{rR}^A\right)}{\sqrt{\left(P_{(L|l)}^2+P_{(R|l)}^2+P_{(L|r)}^2+P_{(R|r)}^2\right)\sigma^2}}\right)$$

and

$$P_r=1-P_l$$

Usually, the observable portion of expected utility is modeled as an index function; for Player B we have[1]

$$U_{jk}^B-U_{jk'}^B=\left(X_{jk}^B-X_{jk'}^B\right)\beta_j^B=X_j^B\beta_j^B$$

Since Player B moves following Player A, stochastic analysis of Player B's utility is analogous to the single-person binary discrete choice problem. That is,

$$\begin{aligned} P_{(L|l)} & =\Phi\left(\frac{U_{lL}^B-U_{lR}^B}{\sqrt{2\sigma^2}}\right) \\ & =\Phi\left(\frac{X_l^B\beta_l^B}{\sqrt{2}}\right) \end{aligned}$$

[1] Throughout this discussion, the intercept is implicit. In our numerical examples, various DGPs involve zero intercepts but the estimation includes an intercept.

and

$$P_{(L|r)} = \Phi\left(\frac{X_r^B \beta_r^B}{\sqrt{2}}\right)$$

However, analysis of Player A's utility is a little more subtle. Player A's expected utility depends on Player B's response to Player A's move. Since player A doesn't observe π^B in the private information game, Player A's expected utilities are weighted by the conditional probabilities associated with Player B's strategies as described above. Consequently, Player A's contribution to the likelihood function is a bit more complex than that representing Player B's utilities.[2] From the above, we have marginal probabilities describing the analyst's perception of player A's strategy.

$$\begin{aligned} P_l &= \Phi\left(\frac{P_{(L|l)}U_{lL}^A + P_{(R|l)}U_{lR}^A - \left(P_{(L|r)}U_{rL}^A + P_{(R|r)}U_{rR}^A\right)}{\sqrt{\left(P_{(L|l)}^2 + P_{(R|l)}^2 + P_{(L|r)}^2 + P_{(R|r)}^2\right)\sigma^2}}\right) \\ &= \Phi\left(\frac{\begin{array}{c} P_{(L|l)}X_{lL}^A\beta_{lL}^A + P_{(R|l)}X_{lR}^A\beta_{lR}^A \\ -\left(P_{(L|r)}X_{rL}^A\beta_{rL}^A + P_{(R|r)}X_{rR}^A\beta_{rR}^A\right)\end{array}}{\sqrt{\left(P_{(L|l)}^2 + P_{(R|l)}^2 + P_{(L|r)}^2 + P_{(R|r)}^2\right)}}\right) \end{aligned}$$

This completes the description of the analyst's log-likelihood function.

$$\begin{aligned} &\sum_{i=1}^{n} Y_{lLi}\log\left(P_{lLi}\right) + Y_{lRi}\log\left(P_{lRi}\right) + Y_{rLi}\log\left(P_{rLi}\right) + Y_{rRi}\log\left(P_{rRi}\right) \\ &= \sum_{i=1}^{n} Y_{lLi}\log\left(P_{li}P_{(L|l)i}\right) + Y_{lRi}\log\left(P_{li}P_{(R|l)i}\right) \\ &\quad + Y_{rLi}\log\left(P_{ri}P_{(L|r)i}\right) + Y_{rRi}\log\left(P_{ri}P_{(R|r)i}\right) \end{aligned}$$

1.3 Marginal probability effects

As in the case of conditionally-heteroskedastic probit (see chapter 5), marginal probability effects of regressors are likely to be nonmonotonic due to inter-agent probability interactions. Indeed, comparison of marginal effects for strategic probit with those of standard binary probit helps illustrate the contrast between statistical analysis of multi-person

[2] Recall the analysis is stochastic because the analyst doesn't observe part of the agents' utilities. Likewise, private information produces agent uncertainty regarding the other player's utility. Hence, private information also induces a stochastic component into the analysis. This probabilistic nature ensures that the likelihood doesn't degenerate even in a game of pure strategies.

strategic and single-person decisions. For the sequential strategic game above, the marginal probabilities for player A's regressors include

$$\frac{\partial P_{lLj}}{\partial X^A_{lkj}} = Den^{-1} P_{(L|l)j} \phi_{lj} P_{(k|l)j} \beta^A_{lk}$$

$$\frac{\partial P_{lLj}}{\partial X^A_{rkj}} = -Den^{-1} P_{(L|l)j} \phi_{lj} P_{(k|r)j} \beta^A_{rk}$$

$$\frac{\partial P_{lRj}}{\partial X^A_{lkj}} = Den^{-1} P_{(R|l)j} \phi_{lj} P_{(k|l)j} \beta^A_{lk}$$

$$\frac{\partial P_{lRj}}{\partial X^A_{rkj}} = -Den^{-1} P_{(R|r)j} \phi_{lj} P_{(k|r)j} \beta^A_{rk}$$

$$\frac{\partial P_{rLj}}{\partial X^A_{lkj}} = Den^{-1} P_{(L|r)j} \phi_{rj} P_{(k|l)j} \beta^A_{lk}$$

$$\frac{\partial P_{rLj}}{\partial X^A_{rkj}} = -Den^{-1} P_{(L|r)j} \phi_{rj} P_{(k|r)j} \beta^A_{rk}$$

$$\frac{\partial P_{rRj}}{\partial X^A_{lkj}} = Den^{-1} P_{(R|r)j} \phi_{rj} P_{(k|l)j} \beta^A_{lk}$$

$$\frac{\partial P_{rRj}}{\partial X^A_{rkj}} = -Den^{-1} P_{(R|r)j} \phi_{rj} P_{(k|r)j} \beta^A_{rk}$$

where P_{mnj}, ϕ_{ij} and $\phi_{(k|i)j}$ is the standard normal density function evaluated at the same arguments as P_{ij} and $P_{(k|i)j}$,

$$Den = \sqrt{P^2_{(L|l)j} + P^2_{(R|l)j} + P^2_{(L|r)j} + P^2_{(R|r)j}}$$

and

$$Num = \begin{array}{c} P_{(L|l)j} X^A_{lLj} \beta^A_{lL} + P_{(R|l)j} X^A_{lRj} \beta^A_{lR} \\ -\left(P_{(L|r)j} X^A_{rLj} \beta^A_{rL} + P_{(R|r)j} X^A_{rRj} \beta^A_{rR}\right) \end{array}$$

Similarly, the marginal probabilities with respect to player B's regressors include

$$\begin{aligned} \frac{\partial P_{lLj}}{\partial X^B_{lj}} = & \phi_{(L|l)j} \frac{\beta^B_l}{\sqrt{2}} P_{lj} \\ & + \phi_{(L|l)j} \frac{\beta^B_l}{\sqrt{2}} P_{(L|l)j} \phi_{lj} \left\{ \begin{array}{c} Den^{-1} \left(X^A_{lLj} \beta^A_{lL} - X^A_{lRj} \beta^A_{lR}\right) \\ -Den^{-3} Num \left(P_{(L|l)j} - P_{(R|l)j}\right) \end{array} \right\} \end{aligned}$$

$$\begin{aligned} \frac{\partial P_{lLj}}{\partial X^B_{rj}} = & P_{(L|l)j} \phi_{lj} \phi_{(L|r)j} \frac{\beta^B_r}{\sqrt{2}} \\ & \times \left\{ \begin{array}{c} -Den^{-1} \left(X^A_{rLj} \beta^A_{rL} - X^A_{rRj} \beta^A_{rR}\right) \\ -Den^{-3} Num \left(P_{(L|r)j} - P_{(R|r)j}\right) \end{array} \right\} \end{aligned}$$

$$\frac{\partial P_{lRj}}{\partial X^B_{lj}} = -\phi_{(R|l)j}\frac{\beta^B_l}{\sqrt{2}}P_{lj}$$

$$+\phi_{(R|l)j}\frac{\beta^B_l}{\sqrt{2}}P_{(R|l)j}\phi_{lj}\left\{\begin{array}{c} Den^{-1}\left(X^A_{lLj}\beta^A_{lL} - X^A_{lRj}\beta^A_{lR}\right) \\ -Den^{-3}Num\left(P_{(L|l)j} - P_{(R|l)j}\right)\end{array}\right\}$$

$$\frac{\partial P_{lRj}}{\partial X^B_{rj}} = -P_{(R|l)j}\phi_{lj}\phi_{(R|r)j}\frac{\beta^B_r}{\sqrt{2}}$$

$$\times\left\{\begin{array}{c} Den^{-1}\left(X^A_{rLj}\beta^A_{rL} - X^A_{rRj}\beta^A_{rR}\right) \\ +Den^{-3}Num\left(P_{(L|r)j} - P_{(R|r)j}\right)\end{array}\right\}$$

$$\frac{\partial P_{rLj}}{\partial X^B_{lj}} = P_{(L|r)j}\phi_{rj}\phi_{(L|l)j}\frac{\beta^B_l}{\sqrt{2}}$$

$$\times\left\{\begin{array}{c} -Den^{-1}\left(X^A_{lLj}\beta^A_{lL} - X^A_{lRj}\beta^A_{lR}\right) \\ +Den^{-3}Num\left(P_{(L|l)j} - P_{(R|l)j}\right)\end{array}\right\}$$

$$\frac{\partial P_{rLj}}{\partial X^B_{rj}} = \phi_{(L|r)j}\frac{\beta^B_r}{\sqrt{2}}P_{rj}$$

$$+\phi_{(L|r)j}\frac{\beta^B_r}{\sqrt{2}}P_{(L|r)j}\phi_{rj}\left\{\begin{array}{c} Den^{-1}\left(X^A_{rLj}\beta^A_{rL} - X^A_{rRj}\beta^A_{rR}\right) \\ +Den^{-3}Num\left(P_{(L|r)j} - P_{(R|r)j}\right)\end{array}\right\}$$

$$\frac{\partial P_{rRj}}{\partial X^B_{lj}} = -P_{(R|r)j}\phi_{rj}\phi_{(R|l)j}\frac{\beta^B_l}{\sqrt{2}}$$

$$\times\left\{\begin{array}{c} Den^{-1}\left(X^A_{lLj}\beta^A_{lL} - X^A_{lRj}\beta^A_{lR}\right) \\ -Den^{-3}Num\left(P_{(L|l)j} - P_{(R|l)j}\right)\end{array}\right\}$$

$$\frac{\partial P_{rRj}}{\partial X^B_{rj}} = -\phi_{(R|r)j}\frac{\beta^B_r}{\sqrt{2}}P_{rj}$$

$$+\phi_{(R|r)j}\frac{\beta^B_r}{\sqrt{2}}P_{(R|r)j}\phi_{rj}\left\{\begin{array}{c} Den^{-1}\left(X^A_{rLj}\beta^A_{rL} - X^A_{rRj}\beta^A_{rR}\right) \\ +Den^{-3}Num\left(P_{(L|r)j} - P_{(R|r)j}\right)\end{array}\right\}$$

Clearly, analyzing responses to anticipated moves by other agents who themselves are anticipating strategic responses changes the game. In other words, endogeneity is fundamental to the analysis of strategic play.

Example 1 (private information game with analyst error) *Consider a simple experiment comparing a sequential strategic two-person choice model with a single-person binary choice models for each player.*

We generated 200 *simulated samples of size* $n = 2,000$ *with uniformly distributed regressors and standard normal errors,* $\pi^k \sim N(0,1)$ *for k=A, B. In particular,*

$$X_l^B \sim U(-2,2)$$
$$X_r^B \sim U(-5,5)$$

$$X_{lL}^A, X_{lR}^A, X_{rL}^A, X_{rR}^A \sim U(-3,3)$$

and

$$\begin{array}{rl} & \beta_l^B = 1, \quad \beta_r^B = -1 \\ slopes: & \beta_{lL}^A = 1, \quad \beta_{lR}^A = 1 \\ & \beta_{rL}^A = -1, \quad \beta_{rR}^A = -1 \\ intercepts: & \beta_{l0}^B = 0, \beta_{r0}^B = 0, \beta_0^A = 0 \end{array}$$

where $U_j^B = X_j^B \beta_j^B$ *and* $U_{jk}^A = X_{jk}^A \beta_{jk}^A$*.Results (means, standard deviations, and the* 0.01 *and* 0.99 *quantiles) are reported in tables 1 and 2.*

Table 1: Strategic choice analysis for player B in the private information game

	β_{l0}^B	β_l^B	β_{r0}^B	β_r^B
parameter	0	1	0	-1
SC mean	-0.003	0.997	0.001	-1.007
DC mean	-0.003	0.707	0.003	-0.707
SC std dev	0.061	0.053	0.075	0.050
DC std dev	0.032	0.031	0.042	0.026
$SC \begin{pmatrix} 0.01, \\ 0.99 \end{pmatrix}$ quantiles	$\begin{pmatrix} -0.14, \\ 0.14 \end{pmatrix}$	$\begin{pmatrix} 0.89, \\ 1.13 \end{pmatrix}$	$\begin{pmatrix} -0.16, \\ 0.16 \end{pmatrix}$	$\begin{pmatrix} -1.15, \\ -0.91 \end{pmatrix}$
$DC \begin{pmatrix} 0.01, \\ 0.99 \end{pmatrix}$ quantiles	$\begin{pmatrix} -0.07, \\ -0.07 \end{pmatrix}$	$\begin{pmatrix} 0.65, \\ 0.77 \end{pmatrix}$	$\begin{pmatrix} -0.09, \\ 0.08 \end{pmatrix}$	$\begin{pmatrix} -0.76, \\ -0.65 \end{pmatrix}$

The single-person discrete choice (DC) estimates are more biased. Not surprisingly, this is particularly the case for player A. Tables 3 and 4 compare estimated marginal probability effects with marginal probability effects from the DGP for the strategic choice model (SC) and the single-person discrete choice models (DC). To simplify the discussion we only provide results for X_{ij}^A *and* X_i^B *where the* ij *strategy pair is played. Further, marginal probability effects are evaluated at the median for the regressor of focus for the* ij *subsample. Comparisons are reported as percent differences relative to the absolute value of the DGP parameter,* $\frac{est\ effect - DGP\ effect}{|DGP\ effect|}$*. The results clearly indicate not only are the*

Table 2: Strategic choice analysis for player A in the private information game

	β_0^A	β_{lL}^A	β_{lR}^A	β_{rL}^A	β_{rR}^A
parameter	0	1	1	-1	-1
SC mean	0.003	1.003	1.008	-1.005	-1.004
DC mean	0.078	0.277	0.281	0.435	0.434
SC std dev	0.048	0.063	0.064	0.060	0.055
DC std dev	0.036	0.034	0.034	0.047	0.041
$SC\begin{pmatrix}0.01,\\0.99\end{pmatrix}$ quantiles	$\begin{pmatrix}-0.10,\\0.12\end{pmatrix}$	$\begin{pmatrix}0.87,\\1.17\end{pmatrix}$	$\begin{pmatrix}0.88,\\1.15\end{pmatrix}$	$\begin{pmatrix}-1.15,\\-0.89\end{pmatrix}$	$\begin{pmatrix}-1.13,\\-0.89\end{pmatrix}$
$DC\begin{pmatrix}0.01,\\0.99\end{pmatrix}$ quantiles	$\begin{pmatrix}-0.01,\\0.18\end{pmatrix}$	$\begin{pmatrix}0.20,\\0.35\end{pmatrix}$	$\begin{pmatrix}0.20,\\0.36\end{pmatrix}$	$\begin{pmatrix}0.33,\\0.53\end{pmatrix}$	$\begin{pmatrix}0.34,\\0.52\end{pmatrix}$

single-person discrete choice parameter estimates more biased but their marginal probability effects are seriously misleading. On the other hand, estimated marginal probability effects for the strategic choice model are largely consistent with those for the DGP.

2 Symmetric information game with analyst error

First, we describe the data generating process (DGP) for the symmetric information game. Then, we relate the DGP to the analyst's likelihood function.

2.1 DGP — players' strategies

Since player A moves first, player B's strategy is to select the maximum utility response to player A's revealed strategy. On the other hand, player A formulates a strategy based on the anticipated response of player B to A's strategy. Unlike the private information game, player A knows player B's utilities including π^B in the symmetric information game. The symmetric information game DGP is summarized below.

Let Y_{jki} be an indicator denoting the equilibrium strategy $j \in (l, r)$ for player A and the equilibrium response $k \in (L, R)$ for player B for pairing i.

$$Y_{lLi} = \begin{array}{ll} 1 \ if & \begin{array}{c} \left\{\Im_{(L|l)i}\left(U_{lLi}^A + \pi_{lLi}^A\right) + \Im_{(R|l)i}\left(U_{lRi}^A + \pi_{lRi}^A\right)\right\} \\ > \left\{\Im_{(L|r)i}\left(U_{rLi}^A + \pi_{rLi}^A\right) + \Im_{(R|r)i}\left(U_{rRi}^A + \pi_{rRi}^A\right)\right\} \\ \text{and } U_{lLi}^B + \pi_{lLi}^B > \left(U_{lRi}^B + \pi_{lRi}^B\right) \end{array} \\ 0 & otherwise \end{array}$$

Table 3: Strategic marginal probability effects for player B in the private information game

marginal probability effect	$\frac{\partial P_{lL}}{\partial X_l^B}$	$\frac{\partial P_{lR}}{\partial X_l^B}$	$\frac{\partial P_{rL}}{\partial X_r^B}$	$\frac{\partial P_{rR}}{\partial X_r^B}$
DGP parameter mean	0.219	−0.219	−0.211	0.221
SC mean % difference	−0.005	−0.001	−0.015	0.003
DC mean % difference	0.272	−0.072	−0.388	0.647
SC std dev % difference	0.137	0.220	0.178	0.614
DC std dev % difference	6.418	4.139	3.616	7.292
SC $\begin{pmatrix} 0.01, \\ 0.99 \end{pmatrix}$ quantiles % difference	$\begin{pmatrix} -0.31, \\ -0.52 \end{pmatrix}$	$\begin{pmatrix} -0.55, \\ 0.26 \end{pmatrix}$	$\begin{pmatrix} -0.37, \\ 0.36 \end{pmatrix}$	$\begin{pmatrix} -0.24, \\ 1.00 \end{pmatrix}$
DC $\begin{pmatrix} 0.01, \\ 0.99 \end{pmatrix}$ quantiles % difference	$\begin{pmatrix} -0.95, \\ 13.6 \end{pmatrix}$	$\begin{pmatrix} -10.61, \\ 0.97 \end{pmatrix}$	$\begin{pmatrix} -18.1, \\ 0.94 \end{pmatrix}$	$\begin{pmatrix} -0.99, \\ 27.8 \end{pmatrix}$

$$Y_{lRi} = \begin{array}{ll} 1 \ \ if & \begin{array}{c} \left\{ \Im_{(L|l)i} \left(U^A_{lLi} + \pi^A_{lLi} \right) + \Im_{(R|l)i} \left(U^A_{lRi} + \pi^A_{lRi} \right) \right\} \\ > \left\{ \Im_{(L|r)i} \left(U^A_{rLi} + \pi^A_{rLi} \right) + \Im_{(R|r)i} \left(U^A_{rRi} + \pi^A_{rRi} \right) \right\} \\ \text{and } U^B_{lLi} + \pi^B_{lLi} < \left(U^B_{lRi} + \pi^B_{lRi} \right) \end{array} \\ 0 & \qquad\qquad otherwise \end{array}$$

$$Y_{rLi} = \begin{array}{ll} 1 \ \ if & \begin{array}{c} \left\{ \Im_{(L|l)i} \left(U^A_{lLi} + \pi^A_{lLi} \right) + \Im_{(R|l)i} \left(U^A_{lRi} + \pi^A_{lRi} \right) \right\} \\ < \left\{ \Im_{(L|r)i} \left(U^A_{rLi} + \pi^A_{rLi} \right) + \Im_{(R|r)i} \left(U^A_{rRi} + \pi^A_{rRi} \right) \right\} \\ \text{and } U^B_{rLi} + \pi^B_{rLi} > \left(U^B_{rRi} + \pi^B_{rRi} \right) \end{array} \\ 0 & \qquad\qquad otherwise \end{array}$$

and

$$Y_{rRi} = \begin{array}{ll} 1 \ \ if & \begin{array}{c} \left\{ \Im_{(L|l)i} \left(U^A_{lLi} + \pi^A_{lLi} \right) + \Im_{(R|l)i} \left(U^A_{lRi} + \pi^A_{lRi} \right) \right\} \\ < \left\{ \Im_{(L|r)i} \left(U^A_{rLi} + \pi^A_{rLi} \right) + \Im_{(R|r)i} \left(U^A_{rRi} + \pi^A_{rRi} \right) \right\} \\ \text{and } U^B_{rLi} + \pi^B_{rLi} < \left(U^B_{rRi} + \pi^B_{rRi} \right) \end{array} \\ 0 & \qquad\qquad otherwise \end{array}$$

where $\Im_{(k|j)i}$ is an indicator function equal to one if player B prefers k to k' given player A chooses stategy j and zero otherwise,[3] and conditional probabilities $P_{(j|k)i}$ are as defined above and again below.

2.2 Likelihood function — analyst's strategy

In the symmetric information game with analyst error, the analyst perceives the likelihood of player B's response to player A's move (due to

[3]As with conditional probabilities, $\Im_{(k'|j)i} = 1 - \Im_{(k|j)i}$.

Table 4: Strategic marginal probability effects for player A in the private information game

marginal probability effect	$\frac{\partial P_{lL}}{\partial X^A_{lL}}$	$\frac{\partial P_{lR}}{\partial X^A_{lR}}$	$\frac{\partial P_{rL}}{\partial X^A_{rL}}$	$\frac{\partial P_{rR}}{\partial X^A_{rR}}$
DGP parameter	0.073	0.080	-0.119	-0.115
SC mean % difference	0.003	-0.000	-0.019	0.008
DC mean % difference	8.130	3.878	17.15	11.55
SC std dev % difference	0.222	0.180	0.227	0.157
DC std dev % difference	74.48	18.74	152.9	65.20
SC $\begin{pmatrix} 0.01, \\ 0.99 \end{pmatrix}$ quantiles % difference	$\begin{pmatrix} -0.53, \\ 0.58 \end{pmatrix}$	$\begin{pmatrix} -0.39, \\ 0.87 \end{pmatrix}$	$\begin{pmatrix} -0.74, \\ 0.42 \end{pmatrix}$	$\begin{pmatrix} -0.38, \\ 0.46 \end{pmatrix}$
DC $\begin{pmatrix} 0.01, \\ 0.99 \end{pmatrix}$ quantiles % difference	$\begin{pmatrix} -0.84, \\ 110.5 \end{pmatrix}$	$\begin{pmatrix} -0.82, \\ 66.0 \end{pmatrix}$	$\begin{pmatrix} 1.32, \\ 78.1 \end{pmatrix}$	$\begin{pmatrix} 1.24, \\ 112.7 \end{pmatrix}$

unobserved π^B). When player A chooses l, then player B responds with L if $U^B_{lL} + \pi^B_{lL} > U^B_{lR} + \pi^B_{lR}$, or $U^B_{lL} - U^B_{lR} > \pi^B_{lR} - \pi^B_{lL} = \pi^B_l$ where $\pi^B_l \sim N(0, 2\sigma^2)$, and player B responds R otherwise. Similarly, when player A chooses r, then player B responds with L if $U^B_{rL} + \pi^B_{rL} > U^B_{rR} + \pi^B_{rR}$, or $U^B_{rL} - U^B_{rR} > \pi^B_{rR} - \pi^B_{rL} = \pi^B_r$ where $\pi^B_r \sim N(0, 2\sigma^2)$, and player B responds R otherwise. Hence, the analyst assigns the following conditional likelihoods (due to unobservable π^B)

$$\begin{aligned} P_{(L|l)} &= \Phi\left(\frac{U^B_{lL} - U^B_{lR}}{\sqrt{2\sigma^2}}\right) \\ P_{(R|l)} &= 1 - P_{(L|l)} \\ P_{(R|r)} &= 1 - P_{(L|r)} \\ P_{(L|r)} &= \Phi\left(\frac{U^B_{rL} - U^B_{rR}}{\sqrt{2\sigma^2}}\right) \end{aligned}$$

where $\Phi(\cdot)$ is the standard normal (cumulative) distribution function.

Player A's strategy however depends on B's response to A's move. Player A opts for strategy l if utility is greater than for strategy r. Suppose player A knows player B's equilibrium response to l is L and to r is L, then player A chooses strategy l if $\left(U^A_{lL} + \pi^A_{lL}\right) > \left(U^A_{rL} + \pi^A_{rL}\right)$ or $\left(U^A_{lL} - U^A_{rL}\right) > \left(\pi^A_{rL} - \pi^A_{lL}\right) = \pi^A_{LL}$ where $\pi^A_{LL} \sim N(0, 2\sigma^2)$ and strategy r otherwise. Likewise, if player A knows player B's equilibrium response to l is L and to r is R, then player A chooses strategy l if $\left(U^A_{lL} + \pi^A_{lL}\right) > \left(U^A_{rR} + \pi^A_{rR}\right)$ or $\left(U^A_{lL} - U^A_{rR}\right) > \left(\pi^A_{rR} - \pi^A_{lL}\right) = \pi^A_{LR}$ where

$\pi^A_{LR} \sim N(0, 2\sigma^2)$ and strategy r otherwise. If player A knows player B's equilibrium response to l is R and to r is L, then player A chooses strategy l if $\left(U^A_{lR} + \pi^A_{lR}\right) > \left(U^A_{rL} + \pi^A_{rL}\right)$ or $\left(U^A_{lR} - U^A_{rL}\right) > \left(\pi^A_{rL} - \pi^A_{lR}\right) = \pi^A_{RL}$ where $\pi^A_{RL} \sim N(0, 2\sigma^2)$ and strategy r otherwise. Finally, if player A knows player B's equilibrium response to l is R and to r is R, then player A chooses strategy l if $\left(U^A_{lR} + \pi^A_{lR}\right) > \left(U^A_{rR} + \pi^A_{rR}\right)$ or $\left(U^A_{lR} - U^A_{rR}\right) > \left(\pi^A_{rR} - \pi^A_{lR}\right) = \pi^A_{RR}$ where $\pi^A_{RR} \sim N(0, 2\sigma^2)$ and strategy r otherwise.

Since the analyst doesn't observe π^A or π^B, the analyst's analysis of the game is stochastic. Unobservability of π^B leads to a likelihood-weighted sum of the above pairwise comparisons[4]

$$\begin{aligned} &2\left\{P_{(L|l)}\left(U^A_{lL} + \pi^A_{lL}\right) + P_{(R|l)}\left(U^A_{lR} + \pi^A_{lR}\right)\right\} \\ &> 2\left\{P_{(L|r)}\left(U^A_{rL} + \pi^A_{rL}\right) + P_{(R|r)}\left(U^A_{rR} + \pi^A_{rR}\right)\right\} \end{aligned}$$

Hence, from the analyst's perspective, player A chooses strategy l if

$$\begin{aligned} &P_{(L|l)}\left(U^A_{lL} + \pi^A_{lL}\right) + P_{(R|l)}\left(U^A_{lR} + \pi^A_{lR}\right) \\ &> P_{(L|r)}\left(U^A_{rL} + \pi^A_{rL}\right) + P_{(R|r)}\left(U^A_{rR} + \pi^A_{rR}\right) \end{aligned}$$

and r otherwise. Unobservability of π^A leads to a complete, stochastic statement of the analyst's perception of player A's strategy

$$\begin{aligned} &P_{(L|l)}U^A_{lL} + P_{(R|l)}U^A_{lR} - \left(P_{(L|r)}U^A_{rL} + P_{(R|r)}U^A_{rR}\right) \\ &> -\left(P_{(L|l)}\pi^A_{lL} + P_{(R|l)}\pi^A_{lR}\right) + \left(P_{(L|r)}\pi^A_{rL} + P_{(R|r)}\pi^A_{rR}\right) \end{aligned}$$

where

$$\begin{aligned} &-\left(P_{(L|l)}\pi^A_{lL} + P_{(R|l)}\pi^A_{lR}\right) + \left(P_{(L|r)}\pi^A_{rL} + P_{(R|r)}\pi^A_{rR}\right) \\ &\sim N\left(0, \left(P^2_{(L|l)} + P^2_{(R|l)} + P^2_{(L|r)} + P^2_{(R|r)}\right)\sigma^2\right) \end{aligned}$$

Hence, the analyst assigns marginal probabilities

$$P_l = \Phi\left(\frac{P_{(L|l)}U^A_{lL} + P_{(R|l)}U^A_{lR} - \left(P_{(L|r)}U^A_{rL} + P_{(R|r)}U^A_{rR}\right)}{\sqrt{\left(P^2_{(L|l)} + P^2_{(R|l)} + P^2_{(L|r)} + P^2_{(R|r)}\right)\sigma^2}}\right)$$

and

$$P_r = 1 - P_l$$

Thus, the analyst's joint probabilities for the symmetric information setting are the same as for the private information setting, only the *DGP* differs between the two settings.[5]

[4]This parallels player A's (as well as the analyst's) analysis in the private information game.

[5]A complementary explanation of the likelihood functions for the two settings is that the analyst doesn't know whether it's a private or symmetric information game.

As in the private information setting the analyst's assessment of the conditional probabilities (for player B's strategy) is

$$\begin{aligned} P_{(L|l)} &= \Phi\left(\frac{U_{lL}^{B} - U_{lR}^{B}}{\sqrt{2\sigma^{2}}}\right) \\ &= \Phi\left(\frac{X_{l}^{B}\beta_{l}^{B}}{\sqrt{2}}\right) \end{aligned}$$

and

$$P_{(L|r)} = \Phi\left(\frac{X_{r}^{B}\beta_{r}^{B}}{\sqrt{2}}\right)$$

Also, the analyst's perception of player A's strategy is

$$\begin{aligned} P_{l} &= \Phi\left(\frac{P_{(L|l)}U_{lL}^{A} + P_{(R|l)}U_{lR}^{A} - \left(P_{(L|r)}U_{rL}^{A} + P_{(R|r)}U_{rR}^{A}\right)}{\sqrt{\left(P_{(L|l)}^{2} + P_{(R|l)}^{2} + P_{(L|r)}^{2} + P_{(R|r)}^{2}\right)\sigma^{2}}}\right) \\ &= \Phi\left(\frac{\begin{array}{c}\left(P_{(L|l)}X_{lL}^{A}\beta_{lL}^{A} + P_{(R|l)}X_{lR}^{A}\beta_{lR}^{A}\right) \\ - \left(P_{(L|r)}X_{rL}^{A}\beta_{rL}^{A} + P_{(R|r)}X_{rR}^{A}\beta_{rR}^{A}\right)\end{array}}{\sqrt{\left(P_{(L|l)}^{2} + P_{(R|l)}^{2} + P_{(L|r)}^{2} + P_{(R|r)}^{2}\right)}}\right) \end{aligned}$$

This completes the description of the analyst's log-likelihood function for the symmetric information setting.

$$\begin{aligned} &\sum_{i=1}^{n} Y_{lLi}\log\left(P_{lLi}\right) + Y_{lRi}\log\left(P_{lRi}\right) + Y_{rLi}\log\left(P_{rLi}\right) + Y_{rRi}\log\left(P_{rRi}\right) \\ =&\sum_{i=1}^{n} Y_{lLi}\log\left(P_{li}P_{(L|l)i}\right) + Y_{lRi}\log\left(P_{li}P_{(R|l)i}\right) \\ &+Y_{rLi}\log\left(P_{ri}P_{(L|r)i}\right) + Y_{rRi}\log\left(P_{ri}P_{(R|r)i}\right) \end{aligned}$$

2.3 Marginal probability effects

Since the private and symmetric information settings joint probabilities are the same, they have the same marginal probability effects associated with the regressors (observables).

Example 2 (symmetric information game with analyst error) *Repeat the simple experiment comparing a sequential strategic two-person choice model with a single-person binary choice models for each player* *for the symmetric information setting. We generated* 200 *simulated samples of size* $n = 2,000$ *with uniformly distributed regressors and standard normal errors,* $\pi^k \sim N(0,1)$ *for k=A, B. In particular,*

$$X_l^B \sim U(-2,2)$$
$$X_r^B \sim U(-5,5)$$

$$X_{lL}^A, X_{lR}^A, X_{rL}^A, X_{rR}^A \sim U(-3,3)$$

and

$$\begin{array}{rl} & \beta_l^B = 1, \quad \beta_r^B = -1 \\ slopes: & \beta_{lL}^A = 1, \quad \beta_{lR}^A = 1 \\ & \beta_{rL}^A = -1, \quad \beta_{rR}^A = -1 \\ intercepts: & \beta_{l0}^B = 0, \beta_{r0}^B = 0, \beta_0^A = 0 \end{array}$$

where $U_j^B = X_j^B \beta_j^B$ *and* $U_{jk}^A = X_{jk}^A \beta_{jk}^A$*.Results (means, standard deviations, and the* 0.01 *and* 0.99 *quantiles) are reported in tables 5 and 6. The single-person discrete choice (DC) estimates are more biased.*

Table 5: Strategic choice analysis for player B in the symmetric information game

	β_{l0}^B	β_l^B	β_{r0}^B	β_r^B
parameter	0	1	0	-1
SC mean	0.001	0.991	-0.000	-1.004
DC mean	-0.002	0.709	0.003	-0.711
SC std dev	0.065	0.059	0.097	0.060
DC std dev	0.031	0.031	0.048	0.030
SC (0.01, 0.99) quantiles	$(-0.12, 0.15)$	$(0.87, 1.13)$	$(-0.22, 0.20)$	$(-1.16, -0.87)$
DC (0.01, 0.99) quantiles	$(-0.06, 0.08)$	$(0.64, 0.78)$	$(-0.12, 0.10)$	$(-0.79, -0.64)$

Not surprisingly, this is particularly the case for player A where β_r *are the wrong sign. Tables 7 and 8 compare estimated marginal probability effects with marginal probability effects from the DGP for the strategic choice model (SC) and the single-person discrete choice models (DC). To simplify the discussion we only provide results for* X_{ij}^A *and* X_i^B *where the* ij *strategy pair is played. Further, marginal probability effects are evaluated at the median for the regressor of focus for the* ij *subsample. Comparisons are reported as percent differences relative to the absolute value of the DGP parameter,* $\frac{est\ effect - DGP\ effect}{|DGP\ effect|}$*. The results clearly in-*

Table 6: Strategic choice analysis for player A in the symmetric information game

	β_0^A	β_{lL}^A	β_{lR}^A	β_{rL}^A	β_{rR}^A
parameter	0	1	1	-1	-1
SC mean	0.002	0.648	0.653	-0.642	-0.637
DC mean	0.001	0.430	0.428	0.430	0.428
SC std dev	0.042	0.043	0.047	0.043	0.046
DC std dev	0.039	0.041	0.042	0.045	0.041
$SC \begin{pmatrix} 0.01, \\ 0.99 \end{pmatrix}$ quantiles	$\begin{pmatrix} -0.09, \\ 0.10 \end{pmatrix}$	$\begin{pmatrix} 0.55, \\ 0.74 \end{pmatrix}$	$\begin{pmatrix} 0.56, \\ 0.77 \end{pmatrix}$	$\begin{pmatrix} -0.74, \\ -0.55 \end{pmatrix}$	$\begin{pmatrix} -0.74, \\ -0.54 \end{pmatrix}$
$DC \begin{pmatrix} 0.01, \\ 0.99 \end{pmatrix}$ quantiles	$\begin{pmatrix} -0.08, \\ 0.09 \end{pmatrix}$	$\begin{pmatrix} 0.34, \\ 0.54 \end{pmatrix}$	$\begin{pmatrix} 0.35, \\ 0.53 \end{pmatrix}$	$\begin{pmatrix} 0.34, \\ 0.55 \end{pmatrix}$	$\begin{pmatrix} 0.33, \\ 0.52 \end{pmatrix}$

dicate not only are the single-person discrete choice parameter estimates more biased but their marginal probability effects are seriously misleading in the symmetric information setting as they are in the private information setting. The results for the strategic choice model in the symmetric information setting are poorer than those for the private information setting. On the surface this may be surprising. However, the analyst suffers a more severe information deficit in the symmetric information setting as the players observe everything (including not only their own π but also the others π) while the analyst doesn't observe π^A or π^B.

3 Summary

Multi-person strategic choice models can be extended in a variety of ways including simultaneous move games, games with learning, games with private information, games with multiple equilibria, etc. (Bresnahan and Reiss [1990], Tamer [2003]). The key point is that strategic interaction is endogenous and standard, single-person discrete choice models (as well as simultaneous probit models) ignore this source of endogeneity.

Table 7: Strategic marginal probability effects for player B in the symmetric information game

marginal probability effect	$\frac{\partial P_{lL}}{\partial X_l^B}$	$\frac{\partial P_{lR}}{\partial X_l^B}$	$\frac{\partial P_{rL}}{\partial X_r^B}$	$\frac{\partial P_{rR}}{\partial X_r^B}$
DGP parameter mean	0.196	−0.207	−0.202	0.193
SC mean % difference	0.431	−0.114	−1.213	0.352
DC mean % difference	3.119	−2.913	−19.89	1.898
SC std dev % difference	2.910	1.952	7.918	2.007
DC std dev % difference	37.59	32.33	229.4	12.51
SC $\begin{pmatrix} 0.01, \\ 0.99 \end{pmatrix}$ quantiles % difference	$\begin{pmatrix} -0.29, \\ 4.76 \end{pmatrix}$	$\begin{pmatrix} -7.41, \\ 0.31 \end{pmatrix}$	$\begin{pmatrix} -20.3, \\ 0.30 \end{pmatrix}$	$\begin{pmatrix} -0.38, \\ 9.99 \end{pmatrix}$
DC $\begin{pmatrix} 0.01, \\ 0.99 \end{pmatrix}$ quantiles % difference	$\begin{pmatrix} -1.00, \\ 16.5 \end{pmatrix}$	$\begin{pmatrix} -37.6, \\ 0.98 \end{pmatrix}$	$\begin{pmatrix} -235., \\ 1.08 \end{pmatrix}$	$\begin{pmatrix} -0.96, \\ 74.2 \end{pmatrix}$

Table 8: Strategic marginal probability effects for player A in the symmetric information game

marginal probability effect	$\frac{\partial P_{lL}}{\partial X_{lL}^A}$	$\frac{\partial P_{lR}}{\partial X_{lR}^A}$	$\frac{\partial P_{rL}}{\partial X_{rL}^A}$	$\frac{\partial P_{rR}}{\partial X_{rR}^A}$
DGP parameter	0.070	0.075	−0.107	−0.107
SC mean % difference	1.969	1.491	−1.405	−1.565
DC mean % difference	13.46	11.82	12.85	24.24
SC std dev % difference	6.794	7.483	5.121	9.156
DC std dev % difference	75.69	96.54	58.44	259.3
SC $\begin{pmatrix} 0.01, \\ 0.99 \end{pmatrix}$ quantiles % difference	$\begin{pmatrix} -0.38, \\ 9.99 \end{pmatrix}$	$\begin{pmatrix} -0.41, \\ 38.3 \end{pmatrix}$	$\begin{pmatrix} -0.39, \\ 19.5 \end{pmatrix}$	$\begin{pmatrix} -10.7, \\ 0.43 \end{pmatrix}$
DC $\begin{pmatrix} 0.01, \\ 0.99 \end{pmatrix}$ quantiles % difference	$\begin{pmatrix} -0.96, \\ 74.2 \end{pmatrix}$	$\begin{pmatrix} -0.75, \\ 454. \end{pmatrix}$	$\begin{pmatrix} -0.86, \\ 234. \end{pmatrix}$	$\begin{pmatrix} 1.25, \\ 58.4 \end{pmatrix}$