Smooth accruals

Now, we explore valuation and evaluation roles of smooth accruals in a simple, yet dynamic setting with informed priors regarding the initial mean of cash flows. Accruals smooth cash flows to summarize the information content regarding expected cash flows from the past cash flow history. This is similar in spirit to Arya et al [2002]. In addition, we show in a moral hazard setting that the foregoing accrual statistic can be combined with current cash flows and non-accounting contractible information to efficiently (subject to LEN model restrictions[1]) supply incentives to replacement agents via sequential spot contracts. Informed priors regarding the permanent component of cash flows facilitates performance evaluation. The LEN (linear exponential normal) model application is similar to Arya et al [2004]. It is not surprising that accruals can serve as statistics for valuation or evaluation, rather the striking contribution here is that the same accrual statistic can serve both purposes without loss of efficiency.

0.1 DGP

The data generating process (DGP) is as follows. Period t cash flows (excluding the agent's compensation s) includes a permanent component m_t that derives from productive capital, the agent's contribution a_t, and a stochastic error e_t.

$$cf_t = m_t + a_t + e_t$$

The permanent component (mean) is subject to stochastic shocks.

$$m_t = g\ m_{t-1} + \epsilon_t$$

where m_0 is common knowledge (strongly informed priors), g is a deterministic growth factor, and stochastic shock ϵ_t. In addition, there exists contractible, non-accounting information that is informative of the agent's action a_t with noise μ_t.

$$y_t = a_t + \mu_t$$

The errors, e, ϵ, and μ are jointly normally distributed with mean zero and variance-covariance matrix Σ. The DGP is common knowledge to management and the auditor. Hence, the auditor's role is simply manager's report compliance with the predetermined accounting system.[2]

The agent has reservation wage RW and is evaluated subject to moral haz-

[1] See Holmstrom and Milgrom [1987], for details on the strengths and limitations of the LEN (linear exponential normal) model.

[2] Importantly, this eliminates strategic reporting considerations typically associated with equilibrium earnings management.

ard. The agent's action is binary $a \in \{a_H, a_L\}$, $a_H > a_L$, with personal cost $c(a), c(a_H) > c(a_L)$, and the agent's preferences for payments s and actions are CARA $U(s,a) = -exp\{-r[s - c(a)]\}$. Payments are linear in performance measures w (with weights γ_t) plus flat wage $\delta_t, s_t = \delta_t + \gamma_t^T w$.

The valuation role of accruals is to summarize next period's unknown expected cash flow m_{t+1} based on the history of cash flows through time t (restricted recognition). The incentive-induced equilibrium agent action a_t^* is effectively known for valuation purposes. Hence, the observable cash flow history at time t is $\{cf_1 - a_1^*, cf_2 - a_2^*, \ldots, cf_t - a_t^*\}$.

0.2 Valuation results

For the case $\Sigma = D$ where D is a diagonal matrix comprised of $\sigma_e^2, \sigma_\epsilon^2$, and σ_μ^2 (appropriately aligned), the following OLS regression identifies the most efficient valuation usage of the past cash flow history. The current accrual equals the estimate of the current mean of cash flows scaled by g^{t-1} — the last element of the vector, $accruals_t = \frac{1}{g^{t-1}} \widehat{m}_t$ where

$$\widehat{m}_t = (H^T H)^{-1} H^T z,$$

$$H = \begin{bmatrix} -\nu & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \nu g & -\nu & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \nu g & -\nu \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}, z = \begin{bmatrix} -\nu g \; m_0 \\ cf_1 - a_1^* \\ 0 \\ cf_2 - a_2^* \\ \vdots \\ 0 \\ cf_t - a_t^* \end{bmatrix}, \text{ and } \nu = \frac{\sigma_e}{\sigma_\varepsilon}.$$

The valuation role of accruals is synthesized in proposition 1. Accruals can supply a sufficient summary of the cash flow history for the cash flow mean.[3] Next, we analyze accruals where the DGP involves unequal error variances and deterministic (as well as stochastic) growth in cash flows.

0.2.1 Growth with unequal variances

Consider a data generating process with $\Sigma = D$ involving unequal variances between e, ϵ, and μ, and a deterministic (as well as stochastic) growth element g. Let

$m_t = g \; m_{t-1} + \epsilon_t, \nu = \frac{\sigma_e}{\sigma_\epsilon}$, and $\phi = \frac{\sigma_e}{\sigma_\mu}$. Also, $B = \begin{bmatrix} 1+\nu^2 & \nu^2 \\ g^2 & g^2\nu^2 \end{bmatrix} =$

[3] As the agent's equilibrium contribution a^* is known, expected cash flow for the current period is estimated by $\widehat{m}_t + a_t^*$ and next period's expected cash flow is predicted by $g \; \widehat{m}_t + a_{t+1}^*$.

$S\Lambda S^{-1}$ where

$$\Lambda = \left[\begin{array}{cc} \frac{1+\nu^2+g^2\nu^2-\sqrt{(1+\nu^2+g^2\nu^2)^2-4g^2\nu^4}}{2} & 0 \\ 0 & \frac{1+\nu^2+g^2\nu^2+\sqrt{(1+\nu^2+g^2\nu^2)^2-4g^2\nu^4}}{2} \end{array} \right]$$

and

$$S = \left[\begin{array}{cc} \frac{1+\nu^2-g^2\nu^2-\sqrt{(1+\nu^2+g^2\nu^2)^2-4g^2\nu^4}}{2g^2} & \frac{1+\nu^2-g^2\nu^2+\sqrt{(1+\nu^2+g^2\nu^2)^2-4g^2\nu^4}}{2g^2} \\ 1 & 1 \end{array} \right].$$

Now, define the difference equations by

$$\left[\begin{array}{c} den_t \\ num_t \end{array} \right] = B^t \left[\begin{array}{c} den_0 \\ num_0 \end{array} \right] = S\Lambda^t S^{-1} \left[\begin{array}{c} 1 \\ 0 \end{array} \right].$$

Accruals as a valuation statistic

The primary result involving accruals as a valuation statistic is presented in proposition 1.

Proposition 1 *Let $m_t = g\ m_{t-1} + e_t$, $\Sigma = D$, and $\nu = \frac{\sigma_e}{\sigma_\epsilon}$. Then, $accruals_{t-1}$ and cf_t are, collectively, sufficient statistics for the mean of cash flows m_t based on the history of cash flows and $g^{t-1}accruals_t$ is an efficient statistic for m_t*

$$\begin{aligned} [\widehat{m}_t | cf_1, ..., cf_t] &= g^{t-1} accruals_t \\ &= \frac{1}{den_t} \left\{ \frac{num_t}{g^2}(cf_t - a_t^*) + g^{t-1}\nu^2 den_{t-1} accruals_{t-1} \right\} \end{aligned}$$

where $accruals_0 = m_0$, and $\left[\begin{array}{c} den_t \\ num_t \end{array} \right] = B^t \left[\begin{array}{c} den_0 \\ num_0 \end{array} \right] = S\Lambda^t S \left[\begin{array}{c} 1 \\ 0 \end{array} \right]$. The variance of accruals is equal to the variance of the estimate of the mean of cash flows multiplied by $g^{2(t-1)}$; the variance of the estimate of the mean of cash flows equals the coefficient on current cash flow multiplied by σ_e^2, $Var[\widehat{m}_t] = \frac{num_t}{den_t g^2}\sigma_e^2$.

To appreciate the tidiness property of accruals in this setting it is instructive to consider the weight placed on the most recent cash flow as the number of periods becomes large. This limiting result is expressed in corollary 1.

Corollary 1 *As t becomes large, the weight on current cash flows for the efficient estimator of the mean of cash flows approaches*

$$\frac{2}{1 + (1 - g^2)\nu^2 + \sqrt{(1 + (1 + g^2)\nu^2)^2 + 4g^2\nu^4}}$$

and the variance of the estimate approaches

$$\frac{2}{1+(1-g^2)\,\nu^2+\sqrt{\left(1+(1+g^2)\,\nu^2\right)^2+4g^2\nu^4}}\sigma_e^2.$$

Tidy accruals

Accruals, as identified above, are tidy in the sense that each period's cash flow is ultimately recognized in accounting income or remains as a "permanent" amount on the balance sheet.[4] This permanent balance is approximately

$$\sum_{t=1}^{k-1} cf_t \left[1-\frac{num_t}{num_k}-num_t\sum_{n=t}^{k-1}\frac{g^{n-t-2}\nu^{2(n-1)}}{g^{n-1}den_n}\right]$$

where k is the first period where $\frac{num_t}{g^2 den_t}$ is well approximated by the asymptotic rate identified in corollary 1 and the estimate of expected cash flow $\widehat{m}_t$ is identified from tidy accruals as $g^{t-1}accruals_t$.[5] In the benchmark case, this balance reduces to

$$\sum_{t=1}^{k-1} cf_t \left[1-\frac{F_{2t}}{F_{2k}}-F_{2t}\sum_{n=t}^{k-1}\frac{1}{F_{2n+1}}\right]$$

where the estimate of expected cash flow $\widehat{m}_t$ is equal to tidy $accruals_t$.

0.3 Performance evaluation

On the other hand, the evaluation role of accruals must regard a_t as unobservable while previous actions of this or other agents are at the incentive-induced equilibrium action a^*, and all observables are potentially (conditionally) informative: $\{cf_1-a_1^*, cf_2-a_2^*, \ldots, cf_t\}$, and $\{y_1-a_1^*, y_2-a_2^*, \ldots, y_t\}$.[6]

For the case $\Sigma = D$, the most efficient linear contract can be found by determining the incentive portion of compensation via OLS and then plugging a constant δ to satisfy individual rationality.[7] The (linear) incentive payments are equal to the OLS estimator, the final element of $\widehat{a}_t$, multiplied by $\Delta = \frac{c(a_H)-c(a_L)}{a_H-a_L}$, $\gamma_t = \Delta\ \widehat{a}_t$[8] where

$$\widehat{a}_t = (H_a^T H_a)^{-1} H_a^T w,$$

[4] The permanent balance is of course settled up on dissolution of the firm.

[5] Cash flows beginning with period k and after are fully accrued as the asymptotic rate effectively applies each period. Hence, a convergent geometric series is formed that sums to one. On the other hand, the permanent balance arises as a result of the influence of the common knowledge initial expected cash flow m_0.

[6] For the case $\Sigma = D$, past y's are uninformative of the current period's act.

[7] Individual rationality is satisfied if $\delta = RW - \{E[incentivepayments|a] - \frac{1}{2}rVar[s] - c(a)\}$.

[8] The nuisance parameters (the initial $2t$ elements of $\widehat{a}_t$) could be avoided if one employs GLS in place of OLS.

$$H_a = \begin{bmatrix} -\nu & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \nu g & -\nu & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \nu g & -\nu & 0 \\ 0 & 0 & \cdots & 0 & 1 & 1 \\ 0 & 0 & \cdots & 0 & 0 & \phi \end{bmatrix}, w = \begin{bmatrix} -\nu g \; m_0 \\ cf_1 - a_1^* \\ 0 \\ cf_2 - a_2^* \\ \vdots \\ 0 \\ cf_t \\ \phi y_t \end{bmatrix}, \text{ and } \phi = \frac{\sigma_e}{\sigma_\varepsilon}.$$

Further, the variance of the incentive payments equals the last row, column element of $\Delta^2 (H_a^T H_a)^{-1} \sigma_e^2$.

In a moral hazard setting, the incentive portion of the LEN contract based on cash flow and other monitoring information history is identified in proposition 2. Incentive payments depend only on two realizations: unexpected cash flow and other monitoring information for period t. Unexpected cash flow at time t is

$$\begin{aligned} cf_t - E[cf_t | cf_1, \ldots, cf_{t-1}] &= cf_t - g^{t-1} accruals_{t-1} \\ &= cf_t - \widehat{m}_{t-1} \\ &= cf_t - [\widehat{m}_t | cf_1, \ldots, cf_{t-1}]. \end{aligned}$$

As a result, sequential spot contracting with replacement agents has a particularly streamlined form. Accounting accruals supply a convenient and sufficient summary of the cash flow history for the cash flow mean. Hence, the combination of last period's accruals with current cash flow yields the pivotal unexpected cash flow variable.

Proposition 2 *Let $m_t = g \; m_{t-1} + e_t$, $\Sigma = D$, $\nu = \frac{\sigma_e}{\sigma_\epsilon}$, and $\phi = \frac{\sigma_e}{\sigma_\mu}$. Then, $accruals_{t-1}$, cf_t, and y_t, collectively, are sufficient statistics for evaluating the agent with incentive payments given by*

$$\gamma_t = \Delta \frac{1}{\nu^2 den_{t-1} + \phi^2 den_t} \left[\phi^2 den_t y_t + \nu^2 den_{t-1} \left(cf_t - g^{t-1} accruals_{t-1} \right) \right]$$

and variance of payments equal to

$$Var[\gamma_t^T w] = \Delta^2 \frac{den_t}{\nu^2 den_{t-1} + \phi^2 den_t} \sigma_e^2$$

where $accruals_{t-1}$ and den_t are as defined in proposition 1, and $\Delta = \frac{c(a_H) - c(a_L)}{aH - aL}$.

Benchmark case

Suppose $\Sigma = \sigma_e^2 I$ ($\nu = \phi = 1$) and $g = 1$. This benchmark case highlights the key informational structure in the data. Corollary 2 identifies the linear combination of current cash flows and last period's accruals employed to estimate the current cash flow mean conditional on cash flow history for this benchmark case.

Corollary 2 *For the benchmark case $\Sigma = \sigma_e^2 I$ $(\nu = \phi = 1)$ and $g = 1$, accruals at time t are an efficient summary of past cash flow history for the cash flow mean if*

$$\begin{aligned} [\widehat{m}_t | cf_1, ..., cf_t] &= accruals_t \\ &= \frac{F_{2t}}{F_{2t+1}} (cf_t - a_t^*) + \frac{F_{2t-1}}{F_{2t+1}} accruals_{t-1} \end{aligned}$$

where $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$ (the Fibonacci series), and the sequence is initialized with $accruals_0 = m_0$ (common knowledge mean beliefs). Then,variance of accruals equals $Var[\widehat{m}_t] = \frac{F_{2t}}{F_{2t+1}} \sigma_e^2$.

For the benchmark case, the evaluation role of accruals is synthesized in corollary 3.

Corollary 3 *For the benchmark case $\Sigma = \sigma_e^2 I$ $(\nu = \phi = 1)$ and $g = 1$, $accruals_{t-1}$, cf_t, and y_t are, collectively, sufficient statistics for evaluating the agent with incentive payments given by*

$$\gamma_t = \Delta \left\{ \frac{F_{2t+1}}{L_{2t}} y_t + \frac{F_{2t-1}}{L_{2t}} (cf_t - accruals_{t-1}) \right\}$$

and variance of payments equals $\Delta^2 \frac{F_{2t+1}}{L_{2t}} \sigma_e^2$ where $accruals_{t-1}$ is as defined in corollary 2, $L_n = L_{n-1} + L_{n-2}$, $L_0 = 2$, $L_1 = 1$ (the Lucas series), and $\Delta = \frac{c(a_H) - c(a_L)}{a_H - a_L}$.[9]

0.4 Summary

A positive view of accruals is outlined above. Accruals combined with current cash flow can serve as sufficient statistics of the cash flow history for the mean of cash flows. Further, in a moral hazard setting accruals can be combined with current cash flow and other monitoring information to efficiently evaluate replacement agents via sequential spot contracts. Informed priors regarding the contaminating permanent component facilitates this performance evaluation exercise. Notably, the same accrual statistic serves both valuation and evaluation purposes. Next, we relax common knowledge of the *DGP* by both management and the auditor to explore strategic reporting equilibria albeit with a simpler *DGP*.

[9] The Lucas and Fibonacci series are related by $L_n = F_{n-1} + F_{n+1}$, for $n = 1, 2, ...$.

Appendix

Proposition 1. *Let* $m_t = g\ m_{t-1} + e_t$, $\Sigma = D$, *and* $\nu = \frac{\sigma_e}{\sigma_\epsilon}$. *Then,* $accruals_{t-1}$ *and* cf_t *are, collectively, sufficient statistics for the mean of cash flows* m_t *based on the history of cash flows and* $g^{t-1}accruals_t$ *is an efficient statistic for* m_t

$$\begin{aligned}[\widehat{m}_t|cf_1,...,cf_t] &= g^{t-1}accruals_t \\ &= \frac{1}{den_t}\left\{\frac{num_t}{g^2}(cf_t - a_t^*) + g^{t-1}\nu^2 den_{t-1}accruals_{t-1}\right\}\end{aligned}$$

where $accruals_0 = m_0$, *and* $\begin{bmatrix} den_t \\ num_t \end{bmatrix} = B^t \begin{bmatrix} den_0 \\ num_0 \end{bmatrix} = S\Lambda^t S \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. *The variance of accruals is equal to the variance of the estimate of the mean of cash flows multiplied by* $g^{2(t-1)}$; *the variance of the estimate of the mean of cash flows equals the coefficient on current cash flow multiplied by* σ_e^2, $Var\left[\widehat{m}_t\right] = \frac{num_t}{den_t g^2}\sigma_e^2$.

Proof. Outline of the proof:

1. Since the data are multivariate normally distributed, BLU estimation is efficient (achieves the Cramer-Rao lower bound amongst consistent estimators; see Greene [1997], p. 300-302).

2. BLU estimation is written as a recursive least squares exercise (see Strang [1986], p. 146-148).

3. The proof is completed by induction. That is, the difference equation solution is shown, by induction, to be equivalent to the recursive least squares estimator. A key step is showing that the information matrix $\Im$ and its inverse can be derived in recursive fashion via LDL^T factorization (i.e., $D^{-1}L^{-1}\Im = L^T$).

Recursive least squares. Let $H_1 = \begin{bmatrix} -\nu \\ 1 \end{bmatrix}$ (a 2 by 1 matrix), $H_2 = \begin{bmatrix} g\nu & -\nu \\ 0 & 1 \end{bmatrix}$ (a 2 by 2 matrix), $H_t = \begin{bmatrix} 0 & \cdots & 0 & g\nu & -\nu \\ 0 & \cdots & 0 & 0 & 1 \end{bmatrix}$ (a 2 by t matrix with $t-2$ leading columns of zeroes), $z_1 = \begin{bmatrix} -g\nu m_0 \\ cf_1 - a_1^* \end{bmatrix}$, $z_2 = \begin{bmatrix} 0 \\ cf_2 - a_2^* \end{bmatrix}$, and $z_t = \begin{bmatrix} 0 \\ cf_t - a_t^* \end{bmatrix}$. The information matrix for a t-period cash flow history is

$$\begin{aligned}\Im_t &= \Im_{t-1}^a + H_t^T H_t \\ &= \begin{bmatrix} 1+\nu^2+g^2\nu^2 & -g\nu^2 & 0 & \cdots & 0 \\ -g\nu^2 & 1+\nu^2+g^2\nu^2 & -g\nu^2 & \ddots & \vdots \\ 0 & -g\nu^2 & \ddots & -g\nu^2 & 0 \\ \vdots & \ddots & -g\nu^2 & 1+\nu^2+g^2\nu^2 & -g\nu^2 \\ 0 & \cdots & 0 & -g\nu^2 & 1+\nu^2 \end{bmatrix},\end{aligned}$$

a symmetric tri-diagonal matrix, where $\Im^a_{t-1}$ is $\Im_{t-1}$ augmented with a row and column of zeroes to conform with $\Im_t$. For instance, $\Im_1 = \left[1+\nu^2\right]$ and $\Im^a_1 = \begin{bmatrix} 1+\nu^2 & 0 \\ 0 & 0 \end{bmatrix}$. The estimate of the mean of cash flows is derived recursively as

$$b_t = b^a_{t-1} + k_t \left(z_t - H_t b^a_{t-1}\right)$$

for $t > 1$ where $k_t = \Im_t^{-1} H_t^T$, the gain matrix, and b^a_{t-1} is b_{t-1} augmented with a zero to conform with b_t. The best linear unbiased estimate of the current mean is the last element in the vector b_t and its variance is the last row-column element of $\Im_t^{-1}$ multiplied by σ_e^2.

Difference equations. The difference equations are

$$\begin{bmatrix} den_t \\ num_t \end{bmatrix} = \begin{bmatrix} 1+\nu^2 & \nu^2 \\ g^2 & g^2\nu^2 \end{bmatrix} \begin{bmatrix} den_{t-1} \\ num_{t-1} \end{bmatrix}$$

with $\begin{bmatrix} den_0 \\ num_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The difference equations estimator for the current mean of cash flows and its variance are

$$\begin{aligned} \widehat{m}_t &= \frac{1}{den_t}\left(\frac{num_t}{g^2}\left(cf_t - a_t^*\right) + g\nu^2 den_{t-1}\widehat{m}_{t-1}\right) \\ &= g^{t-1}\, accruals_t = \frac{1}{den_t}\left(\frac{num_t}{g^2}\left(cf_t - a_t^*\right) + g^{t-1}\nu^2 den_{t-1} accruals_{t-1}\right) \end{aligned}$$

where $accruals_0 = m_0$, and

$$Var\left[\widehat{m}_t\right] = g^{2(t-1)} Var\left[accruals_t\right] = \sigma_e^2 \frac{num_t}{g^2 den_t}.$$

Induction steps. Assume

$$\begin{aligned} \widehat{m}_t &= \frac{1}{den_t}\left(\frac{num_t}{g^2}\left(cf_t - a_t^*\right) + g\nu^2 den_{t-1}\widehat{m}_{t-1}\right) \\ &= g^{t-1}\, accruals_t = \frac{1}{den_t}\left(\frac{num_t}{g^2}\left(cf_t - a_t^*\right) + g^{t-1}\nu^2 den_{t-1} accruals_{t-1}\right) \\ &= \left[b^a_{t-1} + k_t\left(z_t - H_t b^a_{t-1}\right)\right][t] \end{aligned}$$

and

$$Var\left[\widehat{m}_t\right] = g^{2(t-1)} Var\left[accruals_t\right] = Var\left[b_t\right][t,t]$$

where $[t]$ ($[t,t]$) refers to element t (t,t) in the vector (matrix). The above is clearly true for the base case, $t = 1$ and $t = 2$. Now, show

$$\begin{aligned} \widehat{m}_{t+1} &= \frac{1}{den_{t+1}}\left(\frac{num_{t+1}}{g^2}\left(cf_{t+1} - a_{t+1}^*\right) + g^t\nu^2 den_t accruals_t\right) \\ &= \left[b^a_t + k_{t+1}\left(z_{t+1} - H_{t+1} b^a_t\right)\right][t+1]. \end{aligned}$$

Recall $z_{t+1} = \begin{bmatrix} 0 \\ c_{t+1} - a^*_{t+1} \end{bmatrix}$ and $H_{t+1} = \begin{bmatrix} 0 & \cdots & 0 & g\nu & -\nu \\ 0 & \cdots & 0 & 0 & 1 \end{bmatrix}$. From LDL^T factorization of $\Im_{t+1}$ (recall $L^T = D^{-1}L^{-1}\Im$ where L^{-1} is simply products of matrices reflecting successive row eliminations - no row exchanges are involved due to the tri-diagonal structure and D^{-1} is the reciprocal of the diagonal elements remaining following eliminations) the last row of $\Im^{-1}_{t+1}$ is

$$\begin{bmatrix} \frac{g^{t-1}\nu^{2(t-1)}num_1}{g^2 den_{t+1}} & \cdots & \frac{g^2\nu^4 num_{t-1}}{g^2 den_{t+1}} & \frac{g\nu^2 num_t}{g^2 den_{t+1}} & \frac{num_{t+1}}{g^2 den_{t+1}} \end{bmatrix}.$$

This immediately identifies the variance associated with the estimator as the last term in $\Im^{-1}_{t+1}$ multiplied by the variance of cash flows, $\frac{num_{t+1}}{g^2 den_{t+1}}\sigma_e^2$. Hence, the difference equation and the recursive least squares variance estimators are equivalent.

Since $H^T_{t+1} z_{t+1} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ cf_{t+1} - a^*_{t+1} \end{bmatrix}$, the lead term on the RHS of the $[t+1]$ mean estimator is $\frac{num_{t+1}}{g^2 den_{t+1}}\left(cf_{t+1} - a^*_{t+1}\right)$ which is identical to the lead term on the left hand side (LHS). Similarly, the second term on the RHS (recall the focus is on element t, the last element of b^a_t is 0) is

$$\begin{aligned}
&\left[\left(I - k_{t+1}H_{t+1}\right) b^a_t\right][t+1] \\
&= \left[\left(I - \Im^{-1}_{t+1} \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & \vdots & \ddots & \vdots & \vdots \\ \vdots & 0 & \ddots & 0 & 0 \\ 0 & \cdots & 0 & g^2\nu^2 & -g\nu^2 \\ 0 & \cdots & 0 & -g\nu^2 & 1+\nu^2 \end{bmatrix} \right) b^a_t \right][t+1] \\
&= \left(\frac{-g^3\nu^4 num_t}{g^2 den_{t+1}} + \frac{g\nu^2 num_{t+1}}{g^2 den_{t+1}} \right) \widehat{m}_t \\
&= \left(\frac{-g^3\nu^4 num_t + g\nu^2 num_{t+1}}{g^2 den_{t+1}} \right) g^{t-1}\ accruals_t.
\end{aligned}$$

The last couple of steps involves substitution of $\widehat{m}_t$ for $b^a_t\ [t+1]$ and $g^{t-1}accruals_t$ for $\widehat{m}_t$ on the right hand side (RHS). The difference equation relation, $num_{t+1} = g^2 den_t + g^2\nu^2 num_t$, implies

$$\begin{aligned}
\frac{-g^3\nu^4 num_t + g\nu^2 num_{t+1}}{g^2 den_{t+1}}\widehat{m}_t &= \frac{1}{den_{t+1}} g\nu^2 den_t\ \widehat{m}_t = \\
&= \frac{1}{den_{t+1}} g^t\nu^2 den_t\ accruals_t
\end{aligned}$$

the second term on the LHS. This completes the induction steps. ■

Corollary 1. *As t becomes large, the weight on current cash flows for the* *efficient estimator of the mean of cash flows approaches*

$$\frac{2}{1+\left(1-g^2\right)\nu^2+\sqrt{\left(1+\left(1+g^2\right)\nu^2\right)^2+4g^2\nu^4}}$$

and the variance of the estimate approaches

$$\frac{2}{1+\left(1-g^2\right)\nu^2+\sqrt{\left(1+\left(1+g^2\right)\nu^2\right)^2+4g^2\nu^4}}\sigma_e^2.$$

Proof. The difference equations

$$\begin{aligned}\begin{bmatrix} den_t \\ num_t \end{bmatrix} &= S\Lambda^t S^{-1}\begin{bmatrix} den_0 \\ num_0 \end{bmatrix} \\ &= S\Lambda^t S^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = S\Lambda^t c\end{aligned}$$

imply

$$c = S^{-1}\begin{bmatrix} den_0 \\ num_0 \end{bmatrix} = \begin{bmatrix} \frac{-g^2}{1+(1+g^2)\nu^2+\sqrt{(1+(1+g^2)\nu^2)^2-4g^2\nu^4}} \\ \frac{g^2}{1+(1+g^2)\nu^2+\sqrt{(1+(1+g^2)\nu^2)^2-4g^2\nu^4}} \end{bmatrix}.$$

Thus,

$$\begin{bmatrix} den_t \\ num_t \end{bmatrix} = S\begin{bmatrix} \lambda_1^t & 0 \\ 0 & \lambda_2^t \end{bmatrix} c$$

$$= \begin{bmatrix} \frac{\lambda_2^t\left(1+\left(1-g^2\right)\nu^2+\sqrt{(1+(1+g^2)\nu^2)^2-4g^2\nu^4}\right)-\lambda_1^t\left(1+\left(1-g^2\right)\nu^2-\sqrt{(1+(1+g^2)\nu^2)^2-4g^2\nu^4}\right)}{2\sqrt{(1+(1+g^2)\nu^2)^2-4g^2\nu^4}} \\ \frac{g^2\left(\lambda_2^t-\lambda_1^t\right)}{\sqrt{(1+(1+g^2)\nu^2)^2-4g^2\nu^4}} \end{bmatrix}.$$

Since λ_2 is larger than λ_1, λ_1^t contributes negligibly to $\begin{bmatrix} den_t \\ num_t \end{bmatrix}$ for arbitrarily large t. Hence,

$$\lim_{t\to\infty}\frac{num_t}{g^2 den_t} = \frac{2}{1+\left(1-g^2\right)\nu^2+\sqrt{\left(1+\left(1+g^2\right)\nu^2\right)^2-4g^2\nu^4}}.$$

From proposition 1, the variance of the estimator for expected cash flow is $\frac{num_t}{g^2 den_t}\sigma_e^2$. Since

$$\lim_{t\to\infty}\frac{num_t}{g^2 den_t} = \frac{2}{1+\left(1-g^2\right)\nu^2+\sqrt{\left(1+\left(1+g^2\right)\nu^2\right)^2-4g^2\nu^4}}.$$

the asymptotic variance is

$$\frac{2}{1+\left(1-g^2\right)\nu^2+\sqrt{\left(1+\left(1+g^2\right)\nu^2\right)^2-4g^2\nu^4}}\sigma_e^2.$$

This completes the asymptotic case. ■

Proposition 2. *Let $m_t = g\ m_{t-1} + e_t$, $\Sigma = D$, $\nu = \frac{\sigma_e}{\sigma_\epsilon}$, and $\phi = \frac{\sigma_e}{\sigma_\mu}$. Then, $accruals_{t-1}$, cf_t, and y_t, collectively, are sufficient statistics for evaluating the agent with incentive payments given by*

$$\gamma_t = \Delta\frac{1}{\nu^2 den_{t-1} + \phi^2 den_t}\left[\phi^2 den_t y_t + \nu^2 den_{t-1}\left(cf_t - g^{t-1}accruals_{t-1}\right)\right]$$

and variance of payments equal to

$$Var[\gamma_t^T w] = \Delta^2\frac{den_t}{\nu^2 den_{t-1} + \phi^2 den_t}\sigma_e^2$$

where $accruals_{t-1}$ and den_t are as defined in proposition 1, and $\Delta = \frac{c(a_H)-c(a_L)}{aH-aL}$.

Proof. Outline of the proof:

1. Show that the "best" linear contract is equivalent to the BLU estimator of the agent's current act rescaled by the agent's marginal cost of the act.
2. The BLU estimator is written as a recursive least squares exercise (see Strang [1986], p. 146-148).
3. The proof is completed by induction. That is, the difference equation solution is shown, by induction, to be equivalent to the recursive least squares estimator. Again, a key step involves showing that the information matrix $\Im_a$ and its inverse can be derived in recursive fashion via LDL^T factorization (i.e., $D^{-1}L^{-1}\Im_a = L^T$).

"Best" linear contracts. The program associated with the optimal a_H-inducing LEN contract written in certainty equivalent form is

$$\underset{\delta,\gamma}{Min}\quad \delta + E\left[\gamma^T w|a_H\right]$$

subject to

$$\delta + E\left[\gamma^T w|a_H\right] - \frac{r}{2}Var\left[\gamma^T w\right] - c\left(a_H\right) \geq RW \tag{IR}$$

$$\delta + E\left[\gamma^T w|a_H\right] - \frac{r}{2}Var\left[\gamma^T w\right] - c\left(a_H\right) \geq \delta + E\left[\gamma^T w|a_L\right] - \frac{r}{2}Var\left[\gamma^T w\right] - c\left(a_L\right) \tag{IC}$$

As demonstrated in Arya et al [2004], both IR and IC are binding and γ equals the BLU estimator of a based on the history w (the history of cash flows cf and other contractible information y) rescaled by the agent's marginal cost of the act $\Delta = \frac{c(a_H)-c(a_L)}{a_H-a_L}$. Since IC is binding,

$$\begin{aligned}
& \delta + E\left[\gamma^T w | a_H\right] - \frac{r}{2} Var\left[\gamma^T w\right] - \left(\delta + E\left[\gamma^T w | a_L\right] - \frac{r}{2} Var\left[\gamma^T w\right]\right) \\
= \; & c\left(a_H\right) - c\left(a_L\right)
\end{aligned}$$

$$E\left[\gamma^T w | a_H\right] - E\left[\gamma^T w | a_L\right] = c\left(a_H\right) - c\left(a_L\right)$$
$$\gamma^T \left\{E\left[w | a_H\right] - E\left[w | a_L\right]\right\} = c\left(a_H\right) - c\left(a_L\right)$$
$$\left(a_H - a_L\right) \gamma^T \iota = c\left(a_H\right) - c\left(a_L\right)$$

where

$$w = \begin{bmatrix} cf_1 - m_0 - a_1^* \\ cf_2 - m_0 - a_2^* \\ \vdots \\ cf_t - m_0 \\ y_t \end{bmatrix}$$

and ι is a vector of zeroes except the last two elements are equal to one, and

$$\gamma^T \iota = \frac{c\left(a_H\right) - c\left(a_L\right)}{a_H - a_L}.$$

Notice, the sum of the last two elements of γ equals one, $\gamma^T \iota = 1$, is simply the unbiasedness condition associated with the variance minimizing estimator of a based on design matrix H_a. Hence, γ equals the BLU estimator of a rescaled by Δ, $\gamma_t = \Delta \widehat{a}_t$. As δ is a free variable, IR can always be exactly satisfied by setting

$$\delta = RW - \left\{E\left[\gamma^T w | a_H\right] - \frac{r}{2} Var\left[\gamma^T w\right] - c\left(a_H\right)\right\}.$$

Recursive least squares. H_t remains as defined in the proof of proposition 1. Let $H_{a1} = \begin{bmatrix} -\nu & 0 \\ 1 & 1 \\ 0 & \phi \end{bmatrix}$ (a 3 by 2 matrix), $H_{a2} = \begin{bmatrix} g\nu & -\nu & 0 \\ 0 & 1 & 1 \\ 0 & 0 & \phi \end{bmatrix}$ (a 3 by 3 matrix), $H_{at} = \begin{bmatrix} 0 & \cdots & 0 & g\nu & -\nu & 0 \\ 0 & \cdots & 0 & 0 & 1 & 1 \\ 0 & \cdots & 0 & 0 & 0 & \phi \end{bmatrix}$ (a 3 by $t+1$ matrix with leading zeroes), $\widetilde{w}_1 = \begin{bmatrix} -g\nu m_0 \\ cf_1 \\ y_1 \end{bmatrix}$, $\widetilde{w}_2 = \begin{bmatrix} 0 \\ cf_2 \\ y_2 \end{bmatrix}$, and $\widetilde{w}_t = \begin{bmatrix} 0 \\ cf_t \\ y_t \end{bmatrix}$. The information matrix for a t-period cash flow and other monitoring information history is

$$\Im_{at} = \Im_{t-1}^{aa} + H_{at}^T H_{at}$$

$$= \begin{bmatrix} 1+\nu^2+g^2\nu^2 & -g\nu^2 & 0 & 0 & \cdots & 0 \\ -g\nu^2 & 1+\nu^2+g^2\nu^2 & -g\nu^2 & \ddots & \cdots & 0 \\ 0 & -g\nu^2 & \ddots & \ddots & 0 & \vdots \\ 0 & \ddots & \ddots & 1+\nu^2+g^2\nu^2 & -g\nu^2 & 0 \\ \vdots & \cdots & 0 & -g\nu^2 & 1+\nu^2 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 1+\phi^2 \end{bmatrix}$$

a symmetric tri-diagonal matrix where $\Im^{aa}_{t-1}$ is $\Im^{a}_{t-1}$ (the augmented information matrix employed to estimate the cash flow mean in proposition 1) augmented with an additional row and column of zeroes (i.e., the information matrix from proposition 1, $\Im_{t-1}$, is augmented with two columns of zeroes on the right and two rows of zeroes on the bottom). The recursive least squares estimator is

$$b_{at} = \left[b^{aa}_{t-1} + k_{at}\left(\widetilde{w}_t - H_{at}b^{aa}_{t-1}\right)\right]$$

for $t > 1$ where b^{aa}_{t-1} is b_{t-1} (the accruals estimator of m_{t-1} from proposition 1) augmented with two zeroes and $k_{at} = \Im^{-1}_{at}H^T_{at}$. The best linear unbiased estimate of the current act is the last element in the vector b_{at} and its variance is the last row-column element of $\Im^{-1}_{at}$ multiplied by σ^2_e. Notice, recursive least squares applied to the performance evaluation exercise utilizes the information matrix $\Im^{aa}_{t-1}$ (the information matrix employed in proposition 1 augmented with two trailing rows-columns of zeroes) and estimator b^{aa}_{t-1} (the accruals estimator of m_{t-1} from proposition 1 augmented with the two trailing zeroes). This accounts for the restriction on the parameters due to past actions already having been motivated in the past (i.e., past acts are at their equilibrium level a^*). Only the current portion of the design matrix H_{at} and the current observations w_t (in place of z_t) differ from the setup for accruals (in proposition 1).

Difference equations. The difference equations are

$$\begin{bmatrix} den_t \\ num_t \end{bmatrix} = \begin{bmatrix} 1+\nu^2 & \nu^2 \\ g^2 & g^2\nu^2 \end{bmatrix} \begin{bmatrix} den_{t-1} \\ num_{t-1} \end{bmatrix}$$

with $\begin{bmatrix} den_0 \\ num_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The difference equations estimator for the linear incentive payments γ is

$$\begin{aligned} \gamma_t &= \Delta\frac{1}{\nu^2 den_{t-1} + \phi^2 den_t}\left[\phi^2 den_t y_t + \nu^2 den_{t-1}\left(cf_t - g\ \widehat{m}_{t-1}\right)\right] \\ &= \Delta\frac{1}{\nu^2 den_{t-1} + \phi^2 den_t}\left[\phi^2 den_t y_t + \nu^2 den_{t-1}\left(cf_t - g^{t-1}\ accruals_{t-1}\right)\right] \end{aligned}$$

and the variance of payments is

$$Var\left[\gamma^T w\right] = \Delta^2\frac{den_t}{\nu^2 den_{t-1} + \phi^2 den_t}\sigma^2_e.$$

Induction steps. Assume

$$\begin{aligned}
\gamma_t w &= \Delta \frac{1}{\nu^2 den_{t-1} + \phi^2 den_t} \left[\phi^2 den_t y_t + \nu^2 den_{t-1} \left(cf_t - g\ \widehat{m}_{t-1}\right)\right] \\
&= \Delta \frac{1}{\nu^2 den_{t-1} + \phi^2 den_t} \left[\phi^2 den_t y_t + \nu^2 den_{t-1} \left(cf_t - g^{t-1}\ accruals_{t-1}\right)\right] \\
&= \Delta \left[b^a_{t-1} + k_{at}\left(w_t - H_{at} b^a_{t-1}\right)\right][t+1]
\end{aligned}$$

and

$$Var\left[\gamma^T w\right] = \Delta^2 Var\left[\widehat{a}_t\right][t+1, t+1]$$

where $[t+1]$ ($[t+1, t+1]$) refers to element $t+1$ ($t+1, t+1$) in the vector (matrix). The above is clearly true for the base case, $t = 1$ and $t = 2$. Now, show

$$\begin{aligned}
&\Delta \frac{1}{\nu^2 den_t + \phi^2 den_{t+1}} \left[\phi^2 den_{t+1} y_{t+1} + \nu^2 den_t \left(cf_{t+1} - g\ \widehat{m}_t\right)\right] \\
= &\Delta \frac{1}{\nu^2 den_t + \phi^2 den_{t+1}} \left[\phi^2 den_{t+1} y_{t+1} + \nu^2 den_t \left(cf_{t+1} - g^t\ accruals_t\right)\right] \\
= &\Delta \left[b^a_t + k_{at+1}\left(\widetilde{w}_{t+1} - H_{at+1} b^a_t\right)\right][t+2].
\end{aligned}$$

Recall $\widetilde{w}_{t+1} = \begin{bmatrix} 0 \\ cf_{t+1} \\ \phi y_{t+1} \end{bmatrix}$ and $H_{at+1} = \begin{bmatrix} 0 & \cdots & 0 & g\nu & \nu & 0 \\ 0 & \cdots & 0 & 0 & 1 & 1 \\ 0 & \cdots & 0 & 0 & 0 & \phi \end{bmatrix}$. From LDL^T factorization of $\Im_{at+1}$ (recall $L^T = D^{-1} L^{-1} \Im_a$ where L^{-1} is simply products of matrices reflecting successive row eliminations - no row exchanges are involved due to the tri-diagonal structure and D^{-1} is the reciprocal of the remaining elements remaining after eliminations) the last row of $\Im^{-1}_{at+1}$ is

$$\frac{1}{\nu^2 den_t + \phi^2 den_{t+1}} \begin{bmatrix} -g^{t-1}\nu^{2(t-1)} den_1 \\ \vdots \\ -g\nu^2\left(den_{t-1} + \nu^2 num_{t-1}\right) \\ -\left(den_t + \nu^2 num_t\right) \\ den_{t+1} \end{bmatrix}^T.$$ [1]

This immediately identifies the variance associated with the estimator as the last term in $\Im^{-1}_{at+1}$ multiplied by the product of the agent's marginal cost of the act squared and the variance of cash flows, $\Delta^2 \frac{den_{t+1}}{\nu^2 den_t + \phi^2 den_{t+1}} \sigma^2_e$. Hence, the difference equation and the recursive least squares variance of payments estimators are equivalent.

Since $H^T_{at+1}\widetilde{w}_{t+1} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ cf_{t+1} \\ cf_{t+1} + y_{t+1} \end{bmatrix}$ and the difference equation implies

[1] Transposed due to space limitations.

$den_{t+1} = \left(1+\nu^2\right) den_t + \nu^2 num_t$, the lead term on the RHS is

$$
\begin{aligned}
& \frac{den_{t+1}}{\nu^2 den_t + \phi^2 den_{t+1}} \left(y_{t+1} + cf_{t+1}\right) - \frac{den_t + \nu^2 num_t}{\nu^2 den_t + \phi^2 den_{t+1}} cf_{t+1} \\
= \quad & \frac{den_{t+1}}{\nu^2 den_t + \phi^2 den_{t+1}} y_{t+1} - \frac{\nu^2 den_t}{\nu^2 den_t + \phi^2 den_{t+1}} cf_{t+1}
\end{aligned}
$$

which equals the initial expression on the LHS of the $[t+2]$ incentive payments. Similarly, the $\widehat{m}_t = g^{t-1}\ accruals_t$ term on the RHS (recall the focus is on element $t+2$) is

$$
\begin{aligned}
& \left[\left(I - k_{at+1} H_{at+1}\right) b_t^a\right][t+2] \\
= \quad & \left[\left(I - \Im_{at+1}^{-1} \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & g^2\nu^2 & -g\nu^2 & 0 \\ 0 & \cdots & 0 & -g\nu^2 & 1+\nu^2 & 1 \\ 0 & \cdots & 0 & 0 & 1 & 1+\phi^2 \end{bmatrix} \right) b_t^a \right][t+2] \\
= \quad & -\frac{g\nu^2 den_t}{\nu^2 den_t + \phi^2 den_{t+1}} \widehat{m}_t \\
= \quad & -\frac{g^t \nu^2 den_t}{\nu^2 den_t + \phi^2 den_{t+1}} accruals_t.
\end{aligned}
$$

Combining terms and simplifying produces the result

$$
\begin{aligned}
& \frac{1}{\nu^2 den_t + \phi^2 den_{t+1}} \left[\phi^2 den_{t+1} y_{t+1} + \nu^2 den_t \left(cf_{t+1} - g\ \widehat{m}_t\right)\right] \\
= \quad & \frac{1}{\nu^2 den_t + \phi^2 den_{t+1}} \left[\phi^2 den_{t+1} y_{t+1} + \nu^2 den_t \left(cf_{t+1} - g^t\ accruals_t\right)\right].
\end{aligned}
$$

Finally, recall the estimator $\widehat{a}_t$ (the last element of b_{at}) rescaled by the agent's marginal cost of the act identifies the "best" linear incentive payments

$$
\begin{aligned}
\gamma_t w &= \Delta \widehat{a}_t \\
&= \Delta \frac{1}{\nu^2 den_{t-1} + \phi^2 den_t} \left[\phi^2 den_t y_t + \nu^2 den_{t-1} \left(cf_t - g\ \widehat{m}_{t-1}\right)\right] \\
&= \Delta \frac{1}{\nu^2 den_{t-1} + \phi^2 den_t} \left[\phi^2 den_t y_t + \nu^2 den_{t-1} \left(cf_t - g^{t-1}\ accruals_{t-1}\right)\right].
\end{aligned}
$$

This completes the induction steps. ■

Corollary 2. *For the benchmark case* $\Sigma = \sigma_e^2 I$ ($\nu = \phi = 1$) *and* $g = 1$, *accruals at time* t *are an efficient summary of past cash flow history for the* *cash flow mean if*

$$\begin{aligned}
[\widehat{m}_t | cf_1, ..., cf_t] &= accruals_t \\
&= \frac{F_{2t}}{F_{2t+1}} (cf_t - a_t^*) + \frac{F_{2t-1}}{F_{2t+1}} accruals_{t-1}
\end{aligned}$$

where $F_n = F_{n-1} + F_{n-2}$*,* $F_0 = 0$*,* $F_1 = 1$ *(the Fibonacci series), and the sequence is initialized with* $accruals_0 = m_0$ *(common knowledge mean beliefs). Then,variance of accruals equals* $Var[\widehat{m}_t] = \frac{F_{2t}}{F_{2t+1}} \sigma_e^2$*.*

Proof. Replace $g = \nu = 1$ in proposition 1. Hence,

$$\begin{bmatrix} den_t \\ num_t \end{bmatrix} = B \begin{bmatrix} den_{t-1} \\ num_{t-1} \end{bmatrix}$$

reduces to

$$\begin{bmatrix} den_t \\ num_t \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} den_{t-1} \\ num_{t-1} \end{bmatrix}.$$

Since

$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}$$

and

$$\begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix},$$

$$den_t = F_{2t+1}, num_t = F_{2t}, den_{t-1} = F_{2t-1}, \text{and } num_{t-1} = F_{2t-2}.$$

For $g = \nu = 1$, the above implies

$$\widehat{m}_t = g^{t-1} accruals_t = \frac{1}{den_t} \left(\frac{num_t}{g^2} (cf_t - a_t^*) + g^{t-1} \nu^2 den_{t-1} accruals_{t-1} \right)$$

reduces to

$$\frac{F_{2t}}{F_{2t+1}} (cf_t - a_t^*) + \frac{F_{2t-1}}{F_{2t+1}} accruals_{t-1}$$

and variance of accruals equals $\frac{F_{2t}}{F_{2t+1}} \sigma_e^2$. ■

Corollary 3. *For the benchmark case* $\Sigma = \sigma_e^2 I$ $(\nu = \phi = 1)$ *and* $g = 1$*,*$accruals_{t-1}$*,* cf_t*, and* y_t *are, collectively, sufficient statistics for evaluating the agent with incentive payments given by*

$$\gamma_t = \Delta \left\{ \frac{F_{2t+1}}{L_{2t}} y_t + \frac{F_{2t-1}}{L_{2t}} (cf_t - accruals_{t-1}) \right\}$$

and variance of payments equals $\Delta^2 \frac{F_{2t+1}}{L_{2t}} \sigma_e^2$ *where* $accruals_{t-1}$ *is as defined in corollary 2 and* $L_n = L_{n-1} + L_{n-2}$*,* $L_0 = 2$*, and* $L_1 = 1$ *(the Lucas series), and* $\Delta = \frac{c(a_H) - c(a_L)}{a_H - a_L}$*.*

Proof. Replace $g = \nu = \phi = 1$ in proposition 2. Hence,

$$\begin{bmatrix} den_t \\ num_t \end{bmatrix} = B \begin{bmatrix} den_{t-1} \\ num_{t-1} \end{bmatrix}$$

reduces to

$$\begin{bmatrix} den_t \\ num_t \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} den_{t-1} \\ num_{t-1} \end{bmatrix}.$$

Since

$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}$$

and

$$\begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}$$

$den_t = F_{2t+1}$, $num_t = F_{2t}$, $den_{t-1} = F_{2t-1}$, $num_{t-1} = F_{2t-2}$, and $L_t = F_{t+1} + F_{t-1}$. For $g = \nu = \phi = 1$, the above implies

$$\gamma_t w = \Delta \frac{1}{\nu^2 den_{t-1} \phi^2 den_t} \left[\phi^2 den_t y_t + \nu^2 den_{t-1} \left(cf_t - g^{t-1} accruals_{t-1} \right) \right]$$

reduces to

$$\Delta \left\{ \frac{F_{2t-1}}{L_{2t}} \left(cf_t - accruals_{t-1} \right) + \frac{F_{2t+1}}{L_{2t}} y_t \right\}$$

and variance of payments equals $\Delta^2 \frac{F_{2t+1}}{L_{2t}} \sigma_e^2$. ■